Perturbation of an Abstract Differential Equation Containing Fractional Riemann–Liouville Derivatives

Kh. K. Avad and A. V. Glushak

Belgorod State University, Belgorod, Russia

Abstract—We consider a Cauchy type problem in a Banach space. Under the assumption that the corresponding Cauchy type problem with the operator A is uniformly well-posed and the operator $B(t)$ is subordinate to A in some sense, we prove the unique solvability of the considered problem and its continuous dependence on initial data.

In a Banach space E, consider the Cauchy type problem

$$D^{\alpha}u(t) = Au(t) + B(t)u(t), \qquad t > 0, \tag{1}$$

$$\lim_{t\to 0} D^{\alpha-1}u(t) = u_0, \tag{2}$$

where $0 < \alpha < 1$, $D^{\alpha-1}u(t) = I^{1-\alpha}u(t) = (1/\Gamma(1-\alpha))\int_0^t (t-s)^{-\alpha}u(s)\,ds$ is the left fractional Riemann–Liouville integral of order $1-\alpha$ ($I^{1-\alpha}$ is the identity operator for $\alpha = 1$), $D^{\alpha}u(t) = (d/dt)I^{1-\alpha}u(t)$ is the left fractional Riemann–Liouville derivative of order α, $\Gamma(\cdot)$ is the gamma function, A is linear closed densely defined operator, and $B(t)$ is a linear closed densely defined, not necessarily bounded operator, which depends on t and is treated as a perturbation of the operator A.

The following results are close to the perturbation theory of semigroup generators (see [1, Chap. 9]). We study how the addition of the term containing the operator $B(t)$, which is in some sense subordinate to the operator A, influences the solvability of problem (1), (2). We indicate conditions under which the well-posedness of the problem is preserved after the perturbation of the operator A by an unbounded operator $B(t)$.

For abstract differential equations containing fractional Riemann–Liouville derivatives, results on the solvability of perturbed equations are obtained for the first time. Such problems are topical in connection with numerous applications of the theory of fractional differential equations in physics and mathematical modelling. For such applications, see [2, Chap. 8; 3, Chap. 5; 4, Chap. 8].

Along with problem (1), (2), for $1 \ge \beta \ge \alpha$, we consider the unperturbed problem

$$D^{\beta}u(t) = Au(t), \qquad t > 0, \tag{3}$$

$$\lim_{t\to 0} D^{\beta-1}u(t) = u_0. \tag{4}$$

Definition 1. A *solution* of problem (3), (4) is a continuous function $u(t)$, $t > 0$, such that $I^{1-\beta}u(t)$ is a continuously differentiable function for $t > 0$ and the function $u(t)$ ranges in $D(A)$ [$D(A)$ is the domain of the operator A] and satisfies problem (3), (4).

Definition 2. Problem (3), (4) is said to be *uniformly well posed* if there exists an operator function $T_\beta(t)$ defined on E and commuting with A and numbers $M_1 > 0$ and $\omega \in R$ such that, for any $u_0 \in D(A)$, the function $T_\beta(t)u_0$ is the unique solution of the problem and in addition,

$$\|T_\beta(t)\| \le M_1 t^{\beta-1} e^{\omega t}. \tag{5}$$

By Definition 2, problem (3), (4) is uniformly well posed if the solution of this problem exists, is unique, and, by (5), continuously depends on the initial data uniformly with respect to t in any compact set in $(0, \infty)$. In addition to these usual requirements, Definition 2 contains additional information on the behavior of the solution as $t \to 0$ and $t \to \infty$ [inequality (5)].

Let us state conditions under which we shall prove the unique solvability of problem (1), (2).

Condition 1. The operator A is an operator such that for some β satisfying the inequality $\alpha \le \beta \le 1$, problem (3), (4) is uniformly well posed for $u_0 \in D(A)$.

Note that the uniform well-posedness of problem (3), (4) for $0 < \beta < 1$ was considered in [5–7]; for $\beta = 1$, the uniform well-posedness of the Cauchy problem requires that the operator A be the generator of a C_0-semigroup.

Condition 2. (i) The operator $B(t)$ has domain D independent of t; in addition, $D(A) \subset D$.

(ii) For any $x \in D$, the function $B(t)x$ belongs to $C((0, \infty), E)$, is absolutely integrable at zero, takes values in $D(A)$, $AB(t)x \in C((0, \infty), E)$, and is absolutely integrable at zero.

(iii) For any $x \in E$, there exist constants $M_2 > 0$, $\gamma \in [0, 1)$, and $\omega \in R$ such that $T_\beta(\tau)x \in D$ (the smoothing effect) and

$$\|B(t)T_\beta(\tau)x\| \le M_2 \tau^{-\gamma} e^{\omega\tau} \|x\|, \qquad t, \tau \in (0, \infty). \tag{6}$$

Note that if the operator $-A$ is strongly positive (we use the terminology [8]), i.e.,

$$\|(\lambda I - A)^{-1}\| \le \frac{M_3}{1 + |\lambda|}, \qquad \operatorname{Re} \lambda \ge 0, \qquad M_3 > 0,$$

then in Condition 1 one can set $\beta = 1$; in this case, we have $\omega = 0$, and inequality (6) implies that the operator $B(t)$ is subordinate to the fractional power $(-A)^\gamma$ (see [8, p. 298]).

If $B(t)$ is a bounded operator and the operator A satisfies Condition 1, then inequality (6) holds for $\gamma = 1 - \beta$.

The commutativity of the operators A and $B(t)$ is not assumed. As will be proved in forthcoming considerations, Conditions 1 and 2 provide the unique solvability of problem (1), (2).

In the proof, we use the nonnegative function (see [9, p. 357])

$$f_{\tau,\nu}(t) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} \exp(tz - \tau z^\nu)\, dz & \text{if } \quad t \ge 0 \\ 0 & \text{if } \quad t < 0, \end{cases} \tag{7}$$

where $\sigma > 0$, $\tau > 0$, $0 < \nu < 1$, and the branch of the function z^ν is chosen so as to ensure that $\operatorname{Re} z^\nu > 0$ for $\operatorname{Re} z > 0$. This branch is a single-valued function on the complex z-plane with a cut along the negative part of the real axis. The convergence of the integral (7) is guaranteed by the factor $\exp(-\tau z^\nu)$.

In addition, note that the function $f_{\tau,\nu}(t)$ with $t > 0$ can be expressed via the Wright function (see [4, p. 54]),

$$f_{\tau,\nu}(t) = t^{-1}\phi(-\nu, 0; -\tau t^{-\nu}), \qquad \phi(a, b; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(ak + b)},$$

or via a more general Wright type function (see [10, Chap. 1]),

$$f_{\tau,\nu}(t) = t^{-1} e_{1,\nu}^{1,0}(-\tau t^{-\nu}), \quad e_{\alpha,\beta}^{\mu,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \mu)\Gamma(\delta - \beta k)}, \quad \alpha > \max\{0; \beta\}, \quad \mu, z \in C. \tag{8}$$

In the following theorem, we show that the uniform well-posedness of problem (3), (4) implies the uniform well-posedness of the corresponding Cauchy type problem for the equation of order α, where $0 < \alpha < \beta \le 1$.

Theorem 1. *Let $\alpha < \beta \le 1$, and let Condition 1 be satisfied. Then the problem*

$$D^\alpha u(t) = Au(t), \qquad t > 0, \tag{9}$$

$$\lim_{t\to 0} D^{\alpha-1} u(t) = u_0 \tag{10}$$

is uniformly well-posed, and its resolving operator has the form

$$T_\alpha(t)u_0 = \int_0^\infty f_{\tau,\nu}(t) T_\beta(\tau) u_0 \, d\tau, \tag{11}$$

where $\nu = \alpha/\beta$ and the function $f_{\tau,\nu}(t)$ is given by relation (7).

Proof. Note that the convergence of the integral in (11) follows from the estimate (5) and from the following inequality for the Wright function (see [10, Lemma 1.2.7]) :

$$\phi(-\nu, 0; -x) \le M_4(1 + x^n) \exp(-\varrho x^{1/(1-\nu)}),$$
$$n \in \boldsymbol{N}, \quad n \ge \frac{1}{1-\nu}, \quad \varrho = (1-\nu)\nu^{\nu/(1-\nu)}, \quad x \ge 0.$$

To prove that the function $T_\alpha(t)u_0$ given by relation (11) is a solution of problem (9), (10), one can use the relations

$$D^\alpha \int_0^\infty f_{\tau,\nu}(t) T_\beta(\tau) u_0 \, d\tau = \int_0^\infty f_{\tau,\nu}(t) D^\beta T_\beta(\tau) u_0 \, d\tau = AT_\alpha(t) u_0,$$
$$\lim_{t\to 0} D^{\alpha-1} \int_0^\infty f_{\tau,\nu}(t) T_\beta(\tau) u_0 \, d\tau = u_0,$$

which are special cases of formulas (2.2.18) and (2.2.28) proved in [10] for numerical functions. Their proof for abstract functions can be performed in a similar way.

By taking into account the relations (see [10, formulas (2.2.3), (2.2.31)])

$$\int_0^\infty f_{\tau,\nu}(t)\tau^{\beta-1} \, d\tau = \frac{\Gamma(\beta)}{\Gamma(\nu\beta)} t^{\nu\beta-1}, \qquad \int_0^\infty f_{\tau,\nu}(t) e^{\omega\tau} \, d\tau = t^{\nu-1} E_{\nu,\nu}(\omega t^\nu),$$

where $E_{\mu,\varrho}(z) = \sum_{k=0}^\infty z^k/\Gamma(\mu k + \varrho)$ is a Mittag-Leffler type function (e.g., see [2, p. 31]), we obtain

$$\|T_\alpha(t)\| \le \int_0^\infty f_{\tau,\nu}(t) \|T_\beta(\tau)\| \, d\tau \le M_1 \int_0^\infty f_{\tau,\nu}(t)\, \tau^{\beta-1} e^{\omega\tau} \, d\tau$$
$$\le M_5 \int_0^\infty f_{\tau,\nu}(t)\tau^{\beta-1} \, d\tau + M_5 \int_0^\infty f_{\tau,\nu}(t) e^{\omega\tau} \, d\tau = M_5 \left(\frac{\Gamma(\beta)}{\Gamma(\nu\beta)} t^{\alpha-1} + t^{\nu-1} E_{\nu,\nu}(\omega t^\nu) \right). \tag{12}$$

By virtue of the well-known [11, p. 134] asymptotic behavior

$$E_{\mu,\varrho}(z) = \frac{1}{\mu} z^{(1-\varrho)/\mu} \exp(z^{1/\mu}) - \sum_{j=1}^n \frac{1}{\Gamma(\varrho - \mu j) z^j} + O\left(\frac{1}{|z|^{n+1}} \right), \quad z \in R, \quad z \to +\infty, \tag{13}$$

of the Mittag-Leffler function for $0 < \mu < 2$, from (12), we derive the inequality

$$\|T_\alpha(t)\| \le M_6 t^{\alpha-1} \exp(\omega_0 t), \qquad \omega_0 > \omega^{1/\nu}. \tag{14}$$

Therefore, in accordance with Definition 2, to complete the proof of the theorem it remains to justify the uniqueness of the solution of problem (9), (10). Let $u_1(t)$ and $u_2(t)$ be two distinct solutions of problem (9), (10) satisfying inequality (5). Then

$$\lambda^\alpha L(u_1(t) - u_2(t)) = AL(u_1(t) - u_2(t)),$$

where L is the Laplace transform. We have thereby arrived at a contradiction, since if $L(u_1(t) - u_2(t)) \neq 0$, then all points λ^α such that $\operatorname{Re}\lambda > \omega$ belong to the pointwise spectrum of the operator A, which is impossible, since these points should be regular by virtue of the necessary condition for the uniform well-posedness [6]. The proof of the theorem is complete.

Remark 1. For the special case in which $\nu = \alpha/\beta = 1/2$, we have (see [9, p. 369, formula (32)])

$$f_{\tau,1/2}(t) = \frac{\tau}{2t\sqrt{\pi t}} \exp\left(-\frac{\tau^2}{4t}\right),$$

and relation (11) acquires the form

$$T_{\beta/2}(t)u_0 = \frac{1}{2t\sqrt{\pi t}} \int_0^\infty \tau \exp\left(-\frac{\tau^2}{4t}\right) T_\beta(\tau)u_0 \, d\tau. \tag{15}$$

The representation (15) can provide the smoothing effect [see Condition 2 (iii)] for the resolving operator $T_{\beta/2}(t)$ for the case in which the operator $T_\beta(t)$ does not have this property, for example, if A and B are differential operators.

Let us state the solvability theorem for the Cauchy type problem for the inhomogeneous equation.

Theorem 2. *Let $\beta < 1$, let Condition 1 be satisfied, and let the function $h(t) \in C((0,\infty), E)$ be absolutely integrable at zero and range in $D(A)$, let $Ah(t) \in C((0,\infty), E)$, and let $Ah(t)$ be absolutely integrable at zero. Then the problem*

$$D^\beta u(t) = Au(t) + h(t), \qquad t > 0, \tag{16}$$

$$\lim_{t\to 0} D^{\beta-1}u(t) = u_0 \tag{17}$$

has a unique solution, which is given by the formula

$$u(t) = T_\beta(t)u_0 + \int_0^t T_\beta(t-\xi)h(\xi)\, d\xi. \tag{18}$$

Proof. It suffices to verify that the function

$$v(t) = \int_0^t T_\beta(t-\xi)h(\xi)\, d\xi$$

satisfies Eq. (16) and the zero initial condition (17). For $t > 0$, we have

$$\begin{aligned} D^\beta v(t) &= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\tau)^{-\beta}\, d\tau \int_0^\tau T_\beta(\tau-\xi)h(\xi)\, d\xi \\ &= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t d\xi \int_\xi^t (t-\tau)^{-\beta} T_\beta(\tau-\xi)h(\xi)\, d\tau \\ &= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t d\xi \int_0^{t-\xi} (t-\xi-x)^{-\beta} T_\beta(x)h(\xi)\, dx. \end{aligned}$$

Since the integrand in the integral with respect to ξ is a continuous function of $t-\xi$, we have

$$\begin{aligned}
D^{\beta}v(t) &= \frac{1}{\Gamma(1-\beta)}\lim_{\xi\to t}\int_0^{t-\xi}(t-\xi-x)^{-\beta}T_\beta(x)h(\xi)\,dx \\
&\quad+\frac{1}{\Gamma(1-\beta)}\int_0^t d\xi\,\frac{d}{dt}\int_0^{t-\xi}(t-\xi-x)^{-\beta}T_\beta(x)h(\xi)\,dx \\
&= \lim_{t-\xi\to+0}D^{\beta-1}T_\beta(t-\xi)h(\xi)+\int_0^t D^{\beta}T_\beta(t-\xi)h(\xi)\,d\xi \\
&= h(t)+\int_0^t T_\beta(t-\xi)Ah(\xi)\,d\xi = h(t)+Av(t);
\end{aligned}$$

consequently, the function $v(t)$ satisfies Eq. (16).

Next, let us verify that the function $v(t)$ satisfies the zero initial condition (17). We have

$$\lim_{t\to+0}D^{\beta-1}v(t)=\frac{1}{\Gamma(1-\beta)}\lim_{t\to+0}\int_0^t(t-\tau)^{-\beta}\,d\tau\int_0^\tau T_\beta(\tau-\xi)h(\xi)\,d\xi.$$

Since the estimate (5) holds for $T_\beta(t)$, we have

$$\begin{aligned}
&\left\|\int_0^t(t-\tau)^{-\beta}\,d\tau\int_0^\tau T_\beta(\tau-\xi)h(\xi)\,d\xi\right\| \\
&\qquad\le M_1\int_0^t(t-\tau)^{-\beta}\,d\tau\int_0^\tau(\tau-\xi)^{\beta-1}\|h(\xi)\|\,d\xi = M_1B(\beta,1-\beta)\int_0^t\|h(\xi)\|\,d\xi
\end{aligned}$$

for $t\in[0,1]$, where $B(\cdot,\cdot)$ is the beta function. Consequently, the function $v(t)$ satisfies the zero initial condition (17). The proof of the theorem is complete.

Remark 2. A solvability theorem for the Cauchy type problem was proved in [12] for the inhomogeneous equation (16) under the assumption that the function $h(t)$ has fractional derivative $D^{\beta}h(t)$ integrable in the sense of Definition 2.4 in [2]. One can readily see that this requirement can replace Condition 2 (ii) in the assertions proved below.

Theorem 3. *Let $\alpha<\beta\le 1$, and let Conditions 1 and 2 be satisfied. Then problem (1), (2) has a unique solution, and there exist constants $M>0$ and $\omega_1>\omega^{1/\nu}$ such that the estimate*

$$\|u(t)\|\le Mt^{\alpha-1}e^{\omega_1 t}\|u_0\| \tag{19}$$

holds.

Proof. By taking into account Theorems 1 and 2, we reduce problem (1), (2) to an integral equation, which, by virtue of (11) and (18), can be written out in the form

$$u(t)=\int_0^\infty f_{\tau,\nu}(t)T_\beta(\tau)u_0\,d\tau+\int_0^t\int_0^\infty f_{\tau,\nu}(t-s)T_\beta(\tau)B(s)u(s)\,d\tau\,ds, \tag{20}$$

where $u_0, T_\beta(\tau)u_0 \in D(A) \subset D$ and $\nu = \alpha/\beta$. By setting $w(t) = B(t)u(t)$, we obtain the equation

$$w(t) = \int_0^\infty f_{\tau,\nu}(t)B(t)T_\beta(\tau)u_0\,d\tau + \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)B(t)T_\beta(\tau)w(s)\,d\tau\,ds. \tag{21}$$

To solve the integral equation (21), we use the successive approximation method with

$$w_0(t) = 0, \qquad w_1(t) = \int_0^\infty f_{\tau,\nu}(t)B(t)T_\beta(\tau)u_0\,d\tau,$$

$$w_{n+1}(t) = \int_0^\infty f_{\tau,\nu}(t)B(t)T_\beta(\tau)u_0\,d\tau + \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)B(t)T_\beta(\tau)w_n(s)\,d\tau\,ds, \qquad n \in N.$$

By using inequality (6), we estimate the norm of the difference

$$\begin{aligned}\|w_2(t) - w_1(t)\| &\le \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)\|B(t)T_\beta(\tau)w_1(s)\|\,d\tau\,ds \\ &\le M_2^2\|u_0\| \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)\tau^{-\gamma}e^{\omega\tau}\int_0^\infty f_{\xi,\nu}(s)\xi^{-\gamma}e^{\omega\xi}\,d\xi\,d\tau\,ds. \end{aligned}\tag{22}$$

By replacing β in inequalities (12) and (14) by $1-\gamma$, we obtain the estimate

$$\int_0^\infty f_{\xi,\nu}(s)\xi^{-\gamma}e^{\omega\xi}\,d\xi \le Cs^{\nu(1-\gamma)-1}e^{\omega_2 s}, \quad \omega_2 > \omega^{1/\nu}. \tag{23}$$

By applying the estimate (23) twice in inequality (22) and by computing the resulting integral [13, 2.2.5.1], we obtain

$$\begin{aligned}\|w_2(t) - w_1(t)\| &\le CM_2^2\|u_0\| \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)\tau^{-\gamma}e^{\omega\tau}s^{\nu(1-\gamma)-1}e^{\omega_2 s}\,d\tau\,ds \\ &\le C^2M_2^2\|u_0\|\int_0^t (t-s)^{\nu(1-\gamma)-1}e^{\omega_2(t-s)}s^{\nu(1-\gamma)-1}e^{\omega_2 s}\,ds \\ &= C^2M_2^2e^{\omega_2 t}\|u_0\|\int_0^t (t-s)^{\nu(1-\gamma)-1}s^{\nu(1-\gamma)-1}\,ds \\ &= \frac{C^2M_2^2\Gamma^2(\nu(1-\gamma))}{\Gamma(2\nu(1-\gamma))}\,t^{2\nu(1-\gamma)-1}e^{\omega_2 t}\|u_0\|. \end{aligned}\tag{24}$$

By using (24), by induction we obtain the inequality

$$\|w_n(t) - w_{n-1}(t)\| \le \frac{C^nM_2^n\Gamma^n(\nu(1-\gamma))}{\Gamma(n\nu(1-\gamma))}\,t^{n\nu(1-\gamma)-1}e^{\omega_2 t}\|u_0\|, \qquad n \in N. \tag{25}$$

Indeed, let formula (25) hold for $n = m$. Then from inequalities (6) and (23) and by the induction assumption, we obtain

$$\begin{aligned}\|w_{m+1}(t) - w_m(t)\| &\le \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)\|B(t)T_\beta(\tau)(w_m(s) - w_{m-1}(s))\|\,d\tau\,ds \\ &\le \frac{C^m M_2^{m+1}\Gamma^m(\nu(1-\gamma))\|u_0\|}{\Gamma(m\nu(1-\gamma))}\int_0^t\int_0^\infty f_{\tau,\nu}(t-s)\tau^{-\gamma}s^{m\nu(1-\gamma)-1}e^{\omega_2 s}\,d\tau\,ds \\ &\le \frac{C^{m+1} M_2^{m+1}\Gamma^m(\nu(1-\gamma))\|u_0\|}{\Gamma(m\nu(1-\gamma))}\int_0^t (t-s)^{\nu(1-\gamma)-1}e^{\omega_2(t-s)}s^{m\nu(1-\gamma)-1}e^{\omega_2 s}\,ds \\ &= \frac{C^{m+1} M_2^{m+1}\Gamma^{m+1}(\nu(1-\gamma))}{\Gamma((m+1)\nu(1-\gamma))}t^{(m+1)\nu(1-\gamma)-1}e^{\omega_2 t}\|u_0\|,\end{aligned}$$

which completes the proof of formula (25).

Consequently, the series

$$\sum_{n=1}^{\infty}(w_n(t) - w_{n-1}(t))$$

is convergent uniformly on any interval $[t_0, t_1]$, $0 < t_0 < t_1$. Therefore, on the same interval, $w_n(t)$ uniformly converges to a continuous function $w(t)$ on the interval $[t_0, t_1]$, which satisfies the integral equation (21). By virtue of (25), this function satisfies the estimate

$$\begin{aligned}\|w(t)\| &\le \sum_{n=1}^{\infty}\|w_n(t) - w_{n-1}(t)\| \le \sum_{k=0}^{\infty}\frac{C^{k+1}M_2^{k+1}\Gamma^{k+1}(\nu(1-\gamma))t^{(k+1)\nu(1-\gamma)-1}e^{\omega_2 t}\|u_0\|}{\Gamma((k+1)\nu(1-\gamma))} \\ &\le CM_2\Gamma(\nu(1-\gamma))t^{\nu(1-\gamma)-1}e^{\omega_2 t}\sum_{k=0}^{\infty}\frac{C^k M_2^k\Gamma^k(\nu(1-\gamma))t^{k\nu(1-\gamma)}\|u_0\|}{\Gamma(k\nu(1-\gamma)+\nu(1-\gamma))} \\ &= CM_2\Gamma(\nu(1-\gamma))t^{\nu(1-\gamma)-1}e^{\omega_2 t}E_{\nu(1-\gamma),\nu(1-\gamma)}(CM_2\Gamma(\nu(1-\gamma))t^{\nu(1-\gamma)})\|u_0\|,\end{aligned}$$

where $E_{\mu,\varrho}(\cdot)$ is the Mittag-Leffler function, $t \in [t_0, t_1]$, $0 < t_0 < t_1$.

By taking into account the asymptotic behavior of the Mittag-Leffler function (13), one can claim that there exist constants $C_1 > 0$ and $\omega_1 > \omega_2$ such that

$$\|w(t)\| \le C_1 t^{\delta-1}e^{\omega_1 t}\|u_0\|, \qquad \delta = \nu(1-\gamma) < 1. \tag{26}$$

Since $[t_0, t_1]$ is an arbitrary interval, it follows that the function $w(t)$ is a continuous solution of Eq. (21) on $(0, \infty)$ satisfying inequality (26) on $(0, \infty)$, i.e., is absolutely integrable at zero. Moreover, from relation (21) and Condition 2 (ii), we find that $w(t) \in D(A)$, $Aw(t) \in C((0,\infty), E)$, and $Aw(t)$ is absolutely integrable at zero.

Finally, by using Theorem 2, from relation (20), we obtain a solution $u(t)$ of problem (1), (2) in the form

$$u(t) = \int_0^\infty f_{\tau,\nu}(t)T_\beta(\tau)u_0\,d\tau + \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)T_\beta(\tau)w(s)\,d\tau\,ds,$$

which, by virtue of (5), (26), and (23), satisfies the estimate

$$\|u(t)\| \le M_1\|u_0\| \int\limits_0^\infty f_{\tau,\nu}(t)\tau^{\beta-1}e^{\omega t}\,d\tau + C_1M_1\|u_0\| \int\limits_0^t\int\limits_0^\infty f_{\tau,\nu}(t-s)\tau^{\beta-1}e^{\omega\tau}s^{\delta-1}e^{\omega_1 s}\,d\tau\,ds$$

$$\le CM_1t^{\alpha-1}e^{\omega_2 t}\|u_0\| + CC_1M_1\|u_0\| \int\limits_0^t (t-s)^{\alpha-1}e^{\omega_2(t-s)}s^{\delta-1}e^{\omega_1 s}\,ds$$

$$\le CM_1t^{\alpha-1}e^{\omega_2 t}\|u_0\| + CC_1M_1e^{\omega_1 t}\|u_0\| \int\limits_0^t (t-s)^{\alpha-1}s^{\delta-1}\,ds \le Mt^{\alpha-1}e^{\omega_1 t}.$$

Next, let us prove the uniqueness of the solution of problem (1), (2). Suppose that there exists a different solution, $U(t)$. Then, by Theorems 1 and 2,

$$U(t) = \int\limits_0^\infty f_{\tau,\nu}(t)T_\beta(\tau)u_0\,d\tau + \int\limits_0^t\int\limits_0^\infty f_{\tau,\nu}(t-s)T_\beta(\tau)W(s)\,d\tau\,ds,$$

where $W(t)$ is a solution of the integral equation (21).

Let us prove the uniqueness of a solution of the integral equation (21) in the class of functions that are continuous on $(0,\infty)$ and admit the estimate (26).

Let $b>0$ and $t\in(0,b]$. Since we consider the class of functions satisfying inequality (26), it follows that one can set

$$m = \sup_{t\in[0,b]} (t^{1-\delta}e^{-\omega_1 t}\|W(t)-w(t)\|).$$

The difference $W(t)-w(t)$ satisfies Eq. (21) for $u_0=0$; therefore, by using inequality (23), we obtain

$$\|W(t)-w(t)\| \le CM_2 \int\limits_0^t (t-s)^{\delta-1}e^{\omega_2(t-s)}\|W(s)-w(s)\|\,ds. \tag{27}$$

Consequently,

$$\|W(t)-w(t)\| \le CM_2me^{\omega_1 t} \int\limits_0^t (t-s)^{\delta-1}s^{\delta-1}\,ds = CM_2\Gamma(\delta)me^{\omega_1 t}I^\delta(t^{\delta-1}). \tag{28}$$

By substituting (28) into (27), we obtain the inequality

$$\|W(t)-w(t)\| \le C^2M_2^2\Gamma^2(\delta)me^{\omega_1 t}I^{2\delta}(t^{\delta-1}).$$

By continuing this process, we arrive at the inequality

$$\begin{aligned}\|W(t)-w(t)\| &\le C^kM_2^k\Gamma^k(\delta)me^{\omega_1 t}I^{k\delta}(t^{\delta-1})\\ &= \frac{C^kM_2^k\Gamma^{k+1}(\delta)}{\Gamma((k+1)\delta)}\,t^{(k+1)\delta-1}e^{\omega_1 t}m \qquad \forall\, k\in N,\end{aligned} \tag{29}$$

which, after passage to the supremum, implies the inequality

$$m \le \frac{C^kM_2^k\Gamma^{k+1}(\delta)}{\Gamma((k+1)\delta)}\,b^{k\delta}m.$$

The factor

$$\frac{C^k M_2^k \Gamma^{k+1}(\delta)}{\Gamma((k+1)\delta)}\, b^{k\delta}$$

tends to zero as $k \to \infty$ by virtue of the asymptotics of the gamma function [2, p. 30]

$$\Gamma(x+1) = \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x \left(1 + O\left(\frac{1}{x}\right)\right), \qquad x \to \infty.$$

Consequently,

$$m = \sup_{t\in[0,b]} (t^{1-\delta} e^{-\omega_1 t}\|W(t) - w(t)\|) = 0,$$

which, together with the arbitrary choice of $b > 0$, implies the identity $W(t) \equiv w(t)$ for $t > 0$ and completes the proof of the uniqueness.

In conclusion, note that an inequality similar to (29) will be used in the proof of Theorem 5, and the uniqueness of the solution of the integral equation (21) also takes place in a wider class of functions, namely, in the class of functions that are continuous on $(0, \infty)$ and integrable for $t = 0$. Indeed, let $W(t)$ be a solution of the integral equation (21) for $u_0 = 0$. Then, by taking into account (6), for $t \in [0, b]$, we obtain the inequality

$$\begin{aligned} \|W(t)\| &\le M_2 \int_0^t \|W(s)\| \int_0^\infty f_{\tau,\nu}(t-s)\tau^{-\gamma} e^{\omega\tau}\, d\tau\, ds \\ &= \int_0^t (t-s)^{\delta-1}\Upsilon(t-s)\|W(s)\|\, ds \le m_1 I^\delta \|W(t)\|, \end{aligned}$$

where

$$\Upsilon(t) = M_2 \int_0^\infty \phi(-\nu, 0; -\xi)\xi^{-\gamma}\exp(\omega\xi t^\nu)\, d\xi$$

is a continuous function for $t \ge 0$ and $m_1 = \Gamma(\delta)\max_{t\in[0,b]}\Upsilon(t)$.

By applying this estimate k times, we obtain the inequality

$$\|W(t)\| \le m_1^k I^{k\delta}\|W(t)\| = \frac{m_1^k}{\Gamma(k\delta)}\int_0^t (t-s)^{k\delta-1}\|W(s)\|\, ds,$$

and consequently,

$$\begin{aligned} \int_0^b \|W(t)\|\, dt &\le \frac{m_1^k}{\Gamma(k\delta)}\int_0^b\int_0^t (t-s)^{k\delta-1}\|W(s)\|\, ds\, dt = \frac{m_1^k}{\Gamma(k\delta)}\int_0^b \|W(s)\|\int_s^b (t-s)^{k\delta-1}\, dt\, ds \\ &\le \frac{m_1^k b^{k\delta}}{\Gamma(k\delta+1)}\int_0^b \|W(s)\|\, ds. \end{aligned}$$

Therefore, $\int_0^b \|W(s)\|\, ds = 0$ and $\|W(t)\| = 0$ for $t \ge 0$, since $b > 0$ is arbitrary. This completes the proof of Theorem 3.

Theorem 3 permits one to prove the solvability of problem (1), (2) for any α such that $0 < \alpha < \beta \le 1$ for the case in which Conditions 1 and 2 are satisfied. Let us show that similar results remain valid if $0 < \alpha = \beta < 1$.

Theorem 4. *Let Conditions* 1 *and* 2 *be satisfied, and let* $\alpha = \beta < 1$. *Then problem* (1), (2) *has a unique solution, for which the estimate* (19) *holds.*

Proof. By taking into account Theorem 2, we reduce problem (1), (2) to the integral equation

$$u(t) = T_\alpha(t)u_0 + \int_0^t T_\alpha(t-s)B(s)u(s)\,ds. \tag{30}$$

By setting $w(t) = B(t)u(t)$, we obtain the equation

$$w(t) = B(t)T_\alpha(t)u_0 + \int_0^t B(t)T_\alpha(t-s)w(s)\,ds, \tag{31}$$

which can be solved by the successive approximation method. We set

$$w_0(t) = 0, \quad w_1(t) = B(t)T_\alpha(t)u_0, \quad w_{n+1}(t) = B(t)T_\alpha(t)u_0 + \int_0^t B(t)T_\alpha(t-s)w_n(s)\,ds, \quad n \in N.$$

By using inequalities (5) and (6), we estimate the norm of the difference

$$\|w_2(t) - w_1(t)\| \le M_2 \int_0^t (t-s)^{-\gamma}e^{\omega(t-s)}\|w_1(s)\|\,ds \le M_2^2\Gamma(1-\gamma)e^{\omega t}I^{1-\gamma}(t^{-\gamma})\|u_0\|. \tag{32}$$

By taking into account (32) and by arguing as in the proof of inequality (25), by induction we obtain the inequality

$$\|w_n(t) - w_{n-1}(t)\| \le \frac{M_2^n\Gamma^n(1-\gamma)}{\Gamma(n(1-\gamma))}\,t^{n(1-\gamma)-1}e^{\omega t}\|u_0\|, \qquad n \in N.$$

Further considerations dealing with the existence of a unique solution are similar to the proof of Theorem 3; in addition, the solution $w(t)$ of the integral equation (31) admits the estimate

$$\|w(t)\| \le M_2t^{-\gamma}e^{\omega t}\|u_0\| + \sum_{k=1}^{\infty}\frac{M_2^{k+1}\Gamma^{k+1}(1-\gamma)t^{(k+1)(1-\gamma)-1}e^{\omega t}\|u_0\|}{\Gamma((k+1)(1-\gamma))} \le M_0t^{-\gamma}e^{\omega_3 t}\|u_0\| \tag{33}$$

with some constants $M_0 > 0$ and $\omega_3 > \omega$. By using (33), we obtain the estimate (19) of the solution $u(t)$ of problem (1), (2) from relation (30). The proof of the theorem is complete.

Remark 3. An assertion similar to Theorem 4 can also be stated and proved for $\alpha = \beta = 1$. In this case, Condition 2 (ii) should be replaced by the following condition: for any $x \in D$, the function $B(t)x$ belongs to $C([0,\infty), E)$, ranges in $D(A)$, and satisfies $AB(t)x \in C([0,\infty), E)$.

Now let us prove the theorem on the continuous dependence of the solution of problem (1), (2) on the initial conditions.

Theorem 5. *Let the assumptions of Theorem* 3 *hold, and let* $u_n(t)$ *be a sequence of solutions of the problem*

$$D^\alpha u_n(t) = Au_n(t) + B(t)u_n(t), \qquad t > 0, \tag{34}$$

$$\lim_{t\to 0} D^{\alpha-1}u_n(t) = g_n \in D(A). \tag{35}$$

If $g_n \to u_0 \in D(A)$, $Ag_n \to Au_0$, *and* $B(t)g_n$ *converges to* $B(t)u_0$ *uniformly with respect to* $t \in [t_0, b]$ *for arbitrary* $0 < t_0 < b$, *then the sequence* $u_n(t)$ *of solutions of problem* (34), (35) *converges to the solution* $u(t)$ *of problem* (1), (2) *uniformly with respect to* $t \in [t_0, b]$ *for arbitrary* $0 < t_0 < b$.

Proof. Consider the sequence

$$U_n(t) = u_n(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\, g_n,$$

which is a solution of the problem

$$D^{\alpha}U_n(t) = AU_n(t) + B(t)U_n(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)}(Ag_n + B(t)g_n), \tag{36}$$

$$\lim_{t\to 0} D^{\alpha-1}U_n(t) = 0. \tag{37}$$

By Theorems 1 and 2, the function $U_n(t)$ satisfies the integral equation

$$U_n(t) = \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)T_\beta(\tau)\left(B(s)U_n(s) + \frac{s^{\alpha-1}}{\Gamma(\alpha)}(Ag_n + B(s)g_n)\right) d\tau\, ds.$$

By setting $W_n(t) = B(t)U_n(t)$ just as in the proof of Theorem 3, we obtain the representation

$$U_n(t) = \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)T_\beta(\tau)\left(W_n(s) + \frac{s^{\alpha-1}}{\Gamma(\alpha)}(Ag_n + B(s)g_n)\right) d\tau\, ds, \tag{38}$$

where $W_n(t)$ satisfies the integral equation

$$W_n(t) = \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)B(t)T_\beta(\tau)\left(W_n(s) + \frac{s^{\alpha-1}}{\Gamma(\alpha)}(Ag_n + B(s)g_n)\right) d\tau\, ds. \tag{39}$$

Let n and k be sufficiently large positive integers, and let $\varepsilon > 0$. By taking into account (39) and by arguing as in the proof of inequality (29), we obtain the estimate

$$\begin{aligned}\|W_n(t) - W_k(t)\| \le{}& CM_2\int_0^t (t-s)^{\delta-1}e^{\omega_2(t-s)}\|W_n(s) - W_k(s)\|\, ds\\ &+ \frac{CM_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\delta-1}e^{\omega_2(t-s)}s^{\alpha-1}(\|Ag_n - Ag_k\| + \|B(s)g_n - B(s)g_k\|)\, ds,\\ m = {}&\sup_{t\in[0,b]}(t^{1-\delta}e^{-\omega_1 t}\|W_n(t) - W_k(t)\|) \le M_0 m + \varepsilon, \qquad M_0 < 1.\end{aligned}$$

Consequently, $m \le \varepsilon/(1 - M_0)$, and since E is a complete space, it follows that the sequence $t^{1-\delta}e^{-\omega_1 t}W_n(t)$ converges (uniformly with respect to $t \in [0, b]$) to a continuous function $t^{1-\delta}e^{-\omega_1 t}W(t)$ on $[0, b]$. Therefore, $W_n(t)$ converges, uniformly with respect to $t \in [t_0, b]$, $0 < t_0 < b$, to a function $W(t)$ that satisfies inequality (26), belongs to $D(A)$ by virtue of Condition 2 (ii), moreover, satisfies $AW(t) \in C((0,\infty), E)$, and is absolutely integrable at zero.

It follows from (38) that $U_n(t)$ converges, uniformly with respect to $t \in [t_0, b]$, to the function

$$U(t) = \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)T_\beta(\tau)\left(W(s) + \frac{s^{\alpha-1}}{\Gamma(\alpha)}(Au_0 + B(s)u_0)\right) d\tau\, ds,$$

which is a solution of problem (36), (37). Finally, $u_n(t)$ converges, uniformly with respect to $t \in [t_0, b]$, to the function

$$u(t) = U(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha-1)}\, u_0,$$

which satisfies problem (1), (2). The proof of the theorem is complete.

Remark 4. An assertion (similar to Theorem 5) on the continuous dependence of the solution of problem (1), (2) on the initial conditions can be stated and proved for $\alpha = \beta \le 1$ as well.

In the part on the unique solvability, Theorem 4 contains Theorem 8 in [7] in the special case where the operator B is independent of t and is bounded. In this case, it was proved in [7] that if $\alpha = \beta < 1$, then the resolving operator $T_\alpha(t, A+B)$ of problem (1), (2) has the form

$$T_\alpha(t, A+B) = \sum_{n=0}^{\infty} S_n(t),$$

where $S_0(t) = T_\alpha(t, A)$ is the resolving operator of problem (3), (4) for $\beta = \alpha$ and

$$S_n(t) = \int_0^t T_\alpha(t-s, A) B S_{n-1}(s)\, ds, \qquad n = 1, 2, \dots$$

Note also the paper [14] in which the theorem on a perturbation was proved for an equation that, unlike Eq. (1), contains a fractional Caputo derivative under the assumption that the operator A is the generator of an analytic semigroup and $\beta = 1$. The following example was also considered in that paper.

Example 1. Let $E = L_2(R^n)$. On the set $D(A) = W_2^{2m}(R^n)$, we define the operator A as follows:

$$Au(t,x) = \sum_{|p|=2m} a_p(x) \frac{\partial^{p_1+\dots+p_n} u(t,x)}{\partial x_1^{p_1} \dots \partial x_n^{p_n}},$$

where

$$\sum_{|p|=2m} a_p(x) \xi^p \ge (-1)^{m+1} M_0 |\xi|^{2m}$$

for arbitrary $x, \xi \in R^n$ and the coefficients $a_p(x)$ with $|p| = 2m$ satisfy the uniform Hölder condition in R^n. As was mentioned in [14], the operator A satisfies Condition 1 for $\beta = 1$ and $\omega = 0$.

We define the operator $B(t)$ on $D = W_2^{2m-1}(R^n) \supset D(A)$ by the relation

$$B(t)u(t,x) = \sum_{|p|\le 2m-1} a_p(t,x) \frac{\partial^{p_1+\dots+p_n} u(t,x)}{\partial x_1^{p_1} \dots \partial x_n^{p_n}} + \int_\Omega \sum_{|p|\le 2m-1} b_p(t,x,\xi) \frac{\partial^{p_1+\dots+p_n} u(t,\xi)}{\partial \xi_1^{p_1} \dots \partial \xi_n^{p_n}}\, d\xi,$$

where $\Omega \subset R^n$; the coefficients $a_p(t,x)$ with $|p| \le 2m-1$ and with any $t \ge 0$ are continuous, are bounded with respect to $x \in R^n$, and satisfy the Hölder condition with exponent $\mu > \alpha$ with respect to t uniformly with respect to $x \in R^n$; the coefficients $b_p(t,x,\xi)$ are continuous and satisfy the condition

$$\int_{R^n}\int_\Omega |b_p(t,x,\xi)|^2\, d\xi\, dx < +\infty,$$

$$\int_{R^n}\int_\Omega |b_p(t_2,x,\xi) - b_p(t_1,x,\xi)|^2\, d\xi\, dx \le C|t_2 - t_1|^\mu, \qquad \mu > \alpha, \qquad C > 0.$$

As was mentioned in [14], the operator $B(t)$ satisfies Condition 2 for $\omega = 0$ and for some $\gamma \in (0,1)$.

If $u_0(x) \in W_2^{2m}(R^n)$ and $\alpha < 1$, then, by Theorems 3 and 5, problem (1), (2) (a Cauchy type problem for an integro-differential equation) is well-posed and uniquely solvable.

Before considering one more example of the use of Theorem 3, note that if E is a complex Banach space and $T(t)$ is a uniformly bounded C_0-semigroup with generator A, then one can define a positive fractional power of the operator $-A$ (e.g., see [9, p. 357]) as

$$-(-A)^\alpha h = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda I - A)^{-1} A h \, d\lambda, \tag{40}$$

where $\alpha \in (0,1)$ and $h \in D(A)$.

In addition, if $g \in E$, then the resolvent of the operator $-(-A)^\alpha$, which is denoted by A_α in what follows, admits the representation

$$(\mu I - A_\alpha)^{-1} g = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \frac{\lambda^\alpha (\lambda I - A)^{-1} g}{\mu^2 - 2\mu\lambda^\alpha \cos \alpha\pi + \lambda^{2\alpha}} \, d\lambda. \tag{41}$$

Let us show that the following Cauchy type problem with the operator A_α is uniformly well-posed:

$$D^\alpha v(t) = A_\alpha v(t), \qquad t > 0, \tag{42}$$

$$\lim_{t \to 0} D^{\alpha-1} v(t) = v_0, \qquad v_0 \in D(A). \tag{43}$$

Condition 3. The Banach space E has the Radon–Nikodym property (see [15, p. 15]); i.e., each absolutely integrable function $F: R_+ \to E$ is differentiable almost everywhere.

For example, reflexive Banach spaces have this property (see [15, Corollary 1.2.7]), and the spaces $L_1(a,b)$, $C[a,b]$, and c_0 (the space of sequences converging to zero) do not (see [15, Example 1.2.8, Assumptions 1.2.9 and 1.2.10]).

If the Cauchy type problem (3), (4) is uniformly well-posed and $\omega = 0$ in inequality (5), then, as was proved in [6], for $\operatorname{Re}\lambda > 0$ the number λ^β belongs to the resolvent set $\varrho(A)$ of the operator A, for any $x \in E$, the resolvent $R(\lambda^\beta) = (\lambda^\beta I - A)^{-1}$ admits the representation

$$R(\lambda^\beta) x = \int_0^{+\infty} \exp(-\lambda t) T_\beta(t) x \, dt,$$

and, in addition,

$$\left\| \frac{d^n R(\lambda^\beta)}{d\lambda^n} \right\| \le \frac{M\Gamma(n+\beta)}{(\operatorname{Re}\lambda)^{n+\beta}}, \qquad \operatorname{Re}\lambda > 0, \tag{44}$$

for all integer $n \ge 0$.

In a Banach space E that has the Radon–Nikodym property, the validity of inequalities (44) (even for real $\lambda > 0$) is also a sufficient condition for the uniform well-posedness of problem (3), (4). In this case, the resolving operator has the form (see formula (13) in [6])

$$T_\beta(t) u_0 = D^{1-\beta} \frac{1}{2\pi i} \int_{\omega_0 - i\infty}^{\omega_0 + i\infty} \lambda^{\beta-1} \exp(\lambda t) R(\lambda^\beta) u_0 \, d\lambda, \qquad \omega_0 > 0.$$

Note that Condition 3 is absent in [6] but should be imposed in the proof of the uniform well-posedness represented there. Here we bridge the gap.

Theorem 6. *Let Condition 3 be satisfied, let the operator A be the generator of a uniformly bounded C_0-semigroup, and let A_α be the operator given by relation* (40). *Then the Cauchy type problem* (42), (43) *is uniformly well-posed.*

Proof. As was mentioned above, problem (42), (43) is uniformly well-posed if the resolvent $(\mu I - A_\alpha)^{-1}$ satisfies the inequality

$$\left\| \frac{d^n(\mu^\alpha I - A_\alpha)^{-1}}{d\mu^n} \right\| \le \frac{M\Gamma(n+\alpha)}{\mu^{n+\alpha}} \tag{45}$$

for $\mu > 0$.

We set $R(\lambda) = (\lambda I - A)^{-1}$ and, by using the representation (41), prove the validity of the estimate (45). After the substitution from (41), we obtain the representation

$$(\mu^\alpha I - A_\alpha)^{-1} g = \frac{\mu^{1-\alpha}\sin\alpha\pi}{\alpha\pi} \int_0^\infty \frac{s^{1/\alpha} R(\mu s^{1/\alpha}) g}{s^2 - 2s\cos\alpha\pi + 1}\, ds = \frac{\sin\alpha\pi}{\alpha\pi} \int_0^\infty \frac{s x^{1-\alpha} R(x) g}{s^2 - 2s\cos\alpha\pi + 1}\, ds,$$

where $x = \mu s^{1/\alpha}$, and consequently,

$$\frac{d^n(\mu^\alpha I - A_\alpha)^{-1} g}{d\mu^n} = \frac{\sin\alpha\pi}{\alpha\pi} \int_0^\infty \frac{s^{n/\alpha+1}}{s^2 - 2s\cos\alpha\pi + 1} \frac{d^n}{dx^n}(x^{1-\alpha} R(x) g)\, ds. \tag{46}$$

By using the Leibniz formula and the inequality

$$\left\| \frac{d^n R(x)}{dx^n} \right\| \le \frac{M n!}{x^{n+1}}, \qquad x > 0,$$

which is valid by virtue of the Hille–Yosida theorem, we estimate the norm

$$\begin{aligned}
\left\| \frac{d^n}{dx^n}(x^{1-\alpha} R(x) g) \right\| &\le \frac{M\|g\|}{x^{n+\alpha}} \sum_{j=0}^n C_n^j |(1-\alpha)(-\alpha)(-\alpha-1)\cdots(-\alpha-n+j+2)| j! \\
&= \frac{M\|g\|\Gamma(n+\alpha-1)}{|\Gamma(\alpha-1)| x^{n+\alpha}} \sum_{j=0}^n C_n^j {C_{n+\alpha-2}^j}^{-1} \\
&= \frac{M\|g\|\Gamma(n+\alpha)}{\Gamma(\alpha) x^{n+\alpha}} \left(1 - C_n^{n+1} {C_{n+\alpha-1}^{n+1}}^{-1}\right) = \frac{M\|g\|\Gamma(n+\alpha)}{\Gamma(\alpha) x^{n+\alpha}};
\end{aligned} \tag{47}$$

here we have used formula 4.2.8.1 in [13] and the relation $C_n^{n+1} = 0$. It follows from (46) and (47) that

$$\left\| \frac{d^n(\mu^\alpha I - A_\alpha)^{-1}}{d\mu^n} \right\| \le \frac{M_7\Gamma(n+\alpha)}{\mu^{n+\alpha}} \int_0^\infty \frac{ds}{s^2 - 2s\cos\alpha\pi + 1} \le \frac{M_8\Gamma(n+\alpha)}{\mu^{n+\alpha}},$$

which completes the proof of the theorem.

Example 2. Let $0 < \alpha \le \beta \le 1$, and let $-A$ be a strongly positive operator; then the operator $-A_\beta$ given by relation (40) is also strongly positive (see [8, p. 299]). By Theorems 3 and 6, the operator A_β can be perturbed by an operator $B(t)$ subjected to a fractional power of the operator $(-A_\beta)^\gamma$, $\gamma \in (0,1)$; in this case, the well-posedness of problem (1), (2) with the operator A_β is preserved.

If the conditions imposed on the operator A are weakened, and, instead of the strong positivity, we require the validity of the estimate

$$\left\|(\lambda I - A)^{-1}\right\| \le \frac{M_3}{1+\lambda}, \qquad \lambda \ge 0, \qquad M_3 > 0,$$

then the operator $-A$ is said to be positive [8, p. 274]. Positive operators are not necessarily generators of C_0-semigroups, but their fractional powers $(-A)^\beta$ are defined and strongly positive for $0 < \beta \le 1/2$ [8, pp. 274, 299]. In this case, problem (1), (2) with the operator $A_\beta = -(-A)^\beta$, $0 < \beta \le 1/2$, can also be perturbed by the operator $B(t)$ subjected to the fractional power $(-A_\beta)^\gamma$, $\gamma \in (0, 1)$.

ACKNOWLEDGMENTS

The second author was financially supported by the Russian Foundation for Basic Research (project no. 10-01-00276).

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