

Characteristic Function of Quadratic Convolution Functional on the Space of Trajectories of Gaussian Processes

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Abstract—The method for calculation of the characteristic functions $Q(-i\lambda)$ and $\lambda \in \mathbb{R}$ of random value defined by quadratic functionals $J[x]$ on the space $\mathbb{L}_2[0, T]$ of trajectories $x(t)$ of gaussian random processes is proposed. The functional representing the convolution of the function $x(t)$ on the segment $[0, T]$ is under consideration. The general formula for the Fredholm determinant corresponding such characteristic functions is found. By the generalization of the reconstruction method, the calculation of the function $Q(-i\lambda)$ connected with the convolution functional $J[x]$ is fulfilled. In a result, in the case when $\{x(t); t \in [0, T]\}$ is the Ornstein–Uhlenbeck process, the formula of $Q(-i\lambda)$ is obtained.

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1. INTRODUCTION

In problems of statistical radio-physics [1] and quantum optics [2] connected with signal registration when it is necessary to account some natural random noise, the necessity of calculation of the special probability distribution of random values arises. Such distributions connected with values of functionals $J[x + \varphi]$ where functions are represent the sum of suitable random process $\{x(t); t \in \mathbb{R}\}$ realization and the function $\varphi(t)$ being not random. Herewith, the expectation $Ex(t)$ of the process is equal ro zero. For mathematical definiteness, we will consider that both terms are measurable and locally quadratically integrated functions. Therefore, they are elements of spaces $\mathbb{L}_2[0, T]$, $T > 0$.

As a rule, functionals $J[\cdot]$, that represent an interest for problems setting in statistical radio-physics and quantum optics, are the quadratic ones. It is connected with that the registration of electromagnetic irradiation realized by its energy absorbing which depends quadratically on the radiation amplitude. Random processes $\{x(t); t \in \mathbb{R}\}$, in terms of which the problems are set in pointed out scientific areas, as a rule, are gaussian (see, for example, [3]). Modeling of the noise component by a gaussian process is caused its smallness in physical meaning.

For the first time, the described approach to solving of some problems of statistical radio-physics has been realized in [4]. At this moment, its solving may be realized in two cases. In first case, the functional $J[\cdot]$ is represented in the form

$$J[x] = \int_0^T V(x(t))dt$$

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with the piecewise continuous function $V(\cdot)$ and the markovian random process $\{x(t); t \in [0, T]\}$. Calculation of the function $Q(-i\lambda)$ is realized by the solving of the suitable parabolic equation of the Schrödinger type for a supplement function that is connected with conditional probabilities density of transition of the expanded random process $\{\langle x(t), V(x(t)) \rangle; t \in [0, T]\}$. Firstly, such an approach was proposed in M. Kac works (see, for example, [5] (see also [6])).

In second case, for which the method of solving the problem may be realized, the functional $J[\cdot]$ is quadratic and the random process is gaussian. If integral kernel in such problems belongs $\mathbb{L}_2[0, T] \times \mathbb{L}_2[0, T]$, methods of analysis in the Gilbert space are used for calculation of corresponding functional integral on the trajectories space.

Such an approach has been developed by several authors [7–9]. During of solving, it is calculated the Fredholm determinant of integral operator with the kernel $(VK)(t, s)$, where $K(t, s)$ is the correlation function of the random process $\{x(t); t \in \mathbb{R}\}$. This operator determines the characteristic function of random value $J[x]$. The modification of such a method has been proposed in [10]. Further, we will significantly use the results of this work. For the determination of the characteristic function of the total random process $\{x(t) + \varphi(t); t \in [0, T]\}$ on the finite segment $[0, T]$ it is necessary to solve an integral Fredholm equation of second kind with the kernel $(VK)(t, s)$.

In the present work, the characteristic function $Q(-i\lambda)$ is calculated for the functional $J[\cdot]$ that is the self-convolution of function from $\mathbb{L}_2[0, T]$,

$$J[x] = \int_0^T x(t)x(T-t)dt. \quad (1)$$

It corresponds to the second case of the above-mentioned ones. At the first time, in the case of the Wiener process, such a problem has been studied in [11] and [12].

As concerns applications of this problem in statistical theory of signal transmission, it may be connected with the producing theory of such devices which is named the *convolvers* [13]. At present work we pose the results of the characteristic function calculating of the functional (1) defined on trajectories of the markovian gaussian stationary process $\{x(t); t \in \mathbb{R}\}$ named the Ornstein–Uhlenbeck one [14]. The importance of problem solving just for such a random process is caused the fact that it has fundamental meaning for mathematical modelling of various natural noises acting on physical systems. In [15], it has been given communication about solving of this problem in the case when the signal function $\varphi(t)$ is absent in the case when the signal function $\varphi(t)$ is absent. Here, we give all details of calculation and mathematically rigorous justification of the used solving method that is based on the work [10]. Sections 2 and 3 are devoted general theory of characteristic functions of quadratic functionals values which are depend on trajectories of gaussian processes, Section 4 is devoted the pointed out main result of present work.

2. GAUSSIAN RANDOM PROCESSES

Let $\{x(t); t \in [0, T]\}$ be a random process in $[0, T]$ such that its expectation $\mathbf{E}x(t)$ is equal to zero. Here and after the functional $\mathbf{E}$ on trajectories of random process under consideration represent the expectation on the measure of this process. Despite the fact that the setting of the solving problem has a probabilistic origin, we, further, will try do not use concepts and symbols of the probability theory. One reason of such a presentation of the material is the fact that purely probabilistic constructions and methods of proving are practically not used when gaussian processes studying. Therefore, when we say about *the random process*, we keep in mind the functional space with defined the measurability structure on it. This structure is provided with a finite measure normalized by one. In our case, such functional space is the space $\mathbb{L}_2[0, T]$ above the numeric field $\mathbb{R}$, i.e., the space of measurable, quadratically integrated real-valued functions on $[0, T]$ with the scalar production $(x, y) = \int_0^T x(t)y(t)dt$. The measure $\mathbf{P}[\cdot]$ on this space and, correspondingly, the “functional” integral $\int_{\mathbb{L}_2[0, T]}(\cdot)\mathbf{P}[dx]$ is completely defined by characteristic functional

$$\Psi[u] = \int_{\mathbb{L}_2[0, T]} \exp(i(u, x)) \mathbf{P}[dx] \equiv \mathbf{E} \exp(i(u, x)),$$

where $u \in \mathbb{L}_2[0, T]$. Keeping in mind the aim of present work, we give here the following definition of gaussian process.

Definition 1. *The random process $\{x(t); t \in [0, T]\}$ with zero expectation*

$$\mathbb{E}x(t) = \int_{\mathbb{L}_2([0, T])} x(t) \mathbb{P}[dx] = 0$$

is named the gaussian one, if its characteristic functional is defined by the formula

$$\Psi[u] = \exp \left(-\frac{1}{2} \int_0^T u(t) K(t, s) u(s) dt ds \right) = \exp \left(- (u, Ku) / 2 \right),$$

where K is the integral operator

$$(Kx)(t) = \int_0^T K(t, s) x(s) ds,$$

with the kernel $K(t, s)$, which is named the correlation function

$$K(t, s) = \mathbb{E}x(t)x(s) = \int_{\mathbb{L}_2[0, T]} x(t)x(s) \mathbb{P}[dx]. \quad (2)$$

According to its definition (1), the function $K(t, s)$ is symmetric and, moreover, it is defined by nonnegative way since the following equalities

$$\begin{aligned} \int_0^T K(t, s) u(t) u(s) dt ds &= \int_0^T \left[\int_{\mathbb{L}_2[0, T]} x(t)x(s) \mathbb{P}[dx] \right] u(t) u(s) dt ds \\ &= \int_{\mathbb{L}_2[0, T]} \left| \int_0^T x(t) u(t) dt \right|^2 \mathbb{P}[dx] \geq 0 \end{aligned}$$

are correct for any function $u(t) \in \mathbb{L}_2[0, T]$. It follows from this that the operator K , which is named the *correlation* one, is self-adjoint and nonnegative.

Further, we pose that the measure $\mathbb{P}$ is such that the continuous functions $x(t)$ form the set of full measure in $\mathbb{L}_2[0, T]$. In this case, the function $K(t, s)$ is continuous over the pair $\langle t, s \rangle$ and, consequently, it is bounded on $[0, T]^2$. Therefore, the operator K possesses a finite *trace*,

$$\text{Tr}K = \int_{\mathbb{L}_2[0, T]} \|x\|^2 \mathbb{P}[dx] \equiv \int_0^T K(t, t) dt < \infty,$$

i.e., it is the nuclear operator [16].

It means that the sequence $\langle \mu_n; n \in \mathbb{N} \rangle$ of real nonnegative eigenvalues of the operator K is summated. Therefore, it follows that this sequence is quadratically summated and, therefore the operator K is the Gilbert–Schmidt [16] one, i.e., it has the finite norm

$$\|K\|^2 = \text{Tr}KK^T = \int_0^T K^2(t, s) dt ds$$

in the space operators acting in $\mathbb{L}_2[0, T]$ and the function $K(t, s)$ belongs $\mathbb{L}_2[0, T] \times \mathbb{L}_2[0, T]$.

3. CHARACTERISTIC FUNCTIONS OF QUADRATIC FUNCTIONALS

In the work [10], the general formula for characteristic function

$$Q(-i\lambda) = \int_{\mathbb{L}_2[0,T]} \exp(i\lambda J[x]) P[dx] \quad (3)$$

of the quadratic functional $J[x] = (x, Vx)$ in $\mathbb{L}_2[0, T]$ is obtained. It is calculated over the measure of the gaussian random process $\{x(t); t \in [0, T]\}$ with zero expectation in the case when the operator V is the integral operator $(Vx)(t) = \int_0^T V(t, s)x(s)ds$ with the kernel $V(t, s)$ which is continuous over the pair $\langle t, s \rangle$ variables on $[0, T]^2$. Here we generalize this result on the case of bounded operators V of more general kind. Let V_ε be an operator-function on $\varepsilon > 0$. Its values are bounded integral operators

$$(V_\varepsilon x)(t) = \int_0^T V_\varepsilon(t, s)x(s)ds. \quad (4)$$

They act in $\mathbb{L}_2[0, T]$ and are defined by functions $V_\varepsilon(t, s)$ being continuous on $[0, T]^2$. We pose that this function tends to the bounded operator V in the space $\mathbb{L}_2[0, T]$ in the sense of operator norm at $\varepsilon \rightarrow +0$.

Therefore, there is the limit bounded bilinear functional $(x, Vy) = \lim_{\varepsilon \rightarrow +0} (x, V_\varepsilon y)$ and, consequently, there exists the limit quadratic functional $J[x] = (x, Vx) = \lim_{\varepsilon \rightarrow +0} J_\varepsilon[x]$, where $J_\varepsilon[x] = (x, V_\varepsilon x)$ [16].

Since $|\exp(i\lambda J_\varepsilon[x])| \leq 1$ at $\lambda \in \mathbb{R}$, the function $\exp(i\lambda J_\varepsilon[x])$ is bounded by the function integrated on Lebesgue. Using the Lebesgue theorem [17] about transition to the limit under the sign of abstract Lebesgue integral to the integral on the measure $P[\cdot]$, we find that there is the limit

$$Q(-i\lambda) = \int_{\mathbb{L}_2[0,T]} \exp(i\lambda J[x]) P[dx] = \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{L}_2[0,T]} \exp(i\lambda J_\varepsilon[x]) P[dx] \equiv \lim_{\varepsilon \rightarrow +0} Q_\varepsilon(-i\lambda). \quad (5)$$

Since in right-hand side of the equality, under the limit sign, the characteristic functions obtained by the measure $P[\cdot]$ are located, taking into account that the limit of characteristic functions is also the characteristic function [18], we obtain that it is fulfilled the following statement

Theorem 1. *The characteristic function $Q(-i\lambda)$ of the quadratic functional $J[x] = (x, Vx)$ on the space $\mathbb{L}_2[0, T]$, calculated by the measure $P[\cdot]$, which is defined by the bounded operator V is the limit at $\varepsilon \rightarrow +0$ in sense of the operator norm for the operator-function V_ε represented (4). It is the limit of characteristic functions $Q_\varepsilon(-i\lambda)$.*

Calculation of the characteristic function $Q(-i\lambda)$ of the functional (1) is greatly simplified by the complexification of the random process $\{x(t); t \in [0, T]\}$, i.e., by means of its replace with the complex-valued random process $\{z(t) = x(t) + iy(t); t \in [0, T]\}$ with processes $\{x(t); t \in [0, T]\}$ and $\{y(t); t \in [0, T]\}$, which are independent and equivalent [10]. It means that the measure $P[dz]$ is equal to the production of two equivalent measures, $P[dz] = P[dx]P[dy]$ in the Gilbert space $\mathbb{L}_2^{(c)}[0, T]$ with numeric field $\mathbb{C}$. In this case, the integral in (3) is replaced by the integral

$$\int_{\mathbb{L}_2^{(c)}[0,T]} \exp(i\lambda J[z]) P[dz], \quad J[z] = (z, Vz) = \int_0^T V(t, s)z(t)z^*(s)dtds \quad (6)$$

with real-valued continuous on $[0, T]^2$ function $V(t, s)$. In the work [10], it is proved that this integral is equal $Q^2(-i\lambda)$. And in the case when the integral operator V is the self-adjoint one, the integral (6) is the characteristic function of random value, defined by the functional $J[z]$ at $\lambda \in \mathbb{R}$, since it is real on the set of full measure in $\mathbb{L}_2^{(c)}[0, T]$. In this connection, we generalize this assertion for the function on λ defined by the integral (6).

Theorem 2. Let the bounded operator V on the space $\mathbb{L}_2^{(c)}[0, T]$ is represented in the form of the limit $V = \lim_{\varepsilon \rightarrow +0} V_\varepsilon$ in sense of the operator norm for operator-function

$$(V_\varepsilon z)(t) = \int_0^T V_\varepsilon(t, s) z(s) ds,$$

defined by bounded self-adjoint integral operators. Then, there is the limit function

$$\lim_{\varepsilon \rightarrow +0} \int_{\mathbb{L}_2^{(c)}[0, T]} \exp(i\lambda J_\varepsilon[z]) P[dz] = Q^2(-i\lambda), \quad (7)$$

where $J[z] = (z, Vz)$ is the quadratic functional on $\mathbb{L}_2^{(c)}[0, T]$ with values in $\mathbb{R}$.

The limiting function is the characteristic one for random variable defined by functional $J[z]$, $z(t) \in \mathbb{L}_2^{(c)}[0, T]$ on the measure $P[\cdot]$ at $\lambda \in \mathbb{R}$.

Proof. The functional $J[z]$ is defined everywhere on $\mathbb{L}_2^{(c)}[0, T]$ as it is in the proof of Theorem 1. Therefore, the functionals $J_\varepsilon[z]$ take values on the set of the full measure in $\mathbb{L}_2^{(c)}[0, T]$.

Replacement of variables in integrals (7), namely, the changing places of the processes $\{x(t); t \in [0, T]\}$ and $\{y(t); t \in [0, T]\}$ does not lead to the change of measure and the integration domain, since the processes $\{x(t); t \in [0, T]\}$ and $\{y(t); t \in [0, T]\}$ are independent and equivalent. At the same time, the operators V_ε are self-adjoint and such a replacements does not lead to the change of their values, because $\operatorname{Re} J_\varepsilon[z]$ does not changed and $\operatorname{Im} J_\varepsilon[z]$ transforms into $-\operatorname{Im} J_\varepsilon[z]$. Thus,

$$\int_{\mathbb{L}_2^{(c)}[0, T]} \exp(i\lambda J_\varepsilon[z]) P[dz] = \int_{\mathbb{L}_2^{(c)}[0, T]} \exp(i\lambda \operatorname{Re} J_\varepsilon[z]) P[dz].$$

Due to this reason, the subintegral function is modulo bounded by the unit, and at the same time the integral is bounded at any $\varepsilon > 0$,

$$\left| \int_{\mathbb{L}_2^{(c)}[0, T]} \exp(i\lambda J_\varepsilon[z]) P[dz] \right| \leq 1.$$

Therefore, it is permitted the transition to the limit $\varepsilon \rightarrow +0$ under the sign of integral as in (5). At last, the equivalence of the limit expression of the function $Q^2(-i\lambda)$ follows due to

$$\int_{\mathbb{L}_2^{(c)}[0, T]} \exp(i\lambda J_\varepsilon[z]) P[dz] = Q_\varepsilon^2(-i\lambda),$$

according to Lemma 5.1 in the work [10]. $\square$

In the work [10] (Theorem 5.2), it is proved that, in the case when the operator V is the integral one with continuous kernel, the function $Q^2(-i\lambda)$ is meromorphic. Its poles are the imaginary ones. At the same time, it represents the Fredholm determinant $Q^{-2}(-i\lambda) = \det(1 - 2i\lambda VK)$ of the integral operator VK . Because of the nuclear property of the operator VK , the series $\sum_{n=1}^{\infty} |\mu_n|$ of their eigenvalues $\{\mu_n; n \in \mathbb{N}\}$ is summated, i.e., this operator is the nuclear one [16]. In this case, if $\operatorname{Ker} VK = \{0\}$, the Hadamard decomposition

$$Q^{-2}(-i\lambda) = \prod_n (1 - 2i\lambda\mu_n) \quad (8)$$

takes place for the entire function $Q^{-2}(-i\lambda)$. It means (see, for example, [19]), that this function has the order less than 1.

Now, we show that such a situation is retained for self-adjoint bounded V which commute with K and may be approximated as accurate as you like by bounded integral operators with continuous kernels. It is correct the following assertion.

Theorem 3. *Let V be a bounded self-adjoint operator in the space $\mathbb{L}_2^{(c)}[0, T]$, which is the limit one, in the sense of operator's norm, for the operator-function V_ε , the images of which belong to a set of self-adjoint integral operators with continuous kernels. At this time, it is fulfilled $\text{Ker } VK = \{0\}$ and $KV_\varepsilon = V_\varepsilon K$. Further, let $J[z] = (z, Vz)$ be the quadratic functional on $\mathbb{L}_2^{(c)}[0, T]$, $z(t) \in \mathbb{L}_2^{(c)}[0, T]$ and $Q^2(-i\lambda)$ is its characteristic function, $\lambda \in \mathbb{C}$ on the measure $P[dz] = P[dx]P[dy]$ of gaussian random process $\{z(t) = x(t) + iy(t); t \in [0, T]\}$, $z(t) \in \mathbb{L}_2^{(c)}[0, T]$ with processes $\{x(t); t \in [0, T]\}$, $\{y(t); t \in [0, T]\}$ being independent and equivalent to each other. The correlation integral operator K with continuous kernel on $[0, T]^2$ corresponds to each of them. The function $Q^2(-i\lambda)$ is meromorphic on $\mathbb{C}$. At this time, an entire function $Q^{-2}(-i\lambda)$ has the Hadamard decomposition (8) with imaginary zeroes $(-i/2\mu_n)$, where μ_n are eigenvalues of the operator VK .*

Proof. The operator K is the nuclear and nonnegative one. The operator VK is also the nuclear one, since [20] there is the orthogonal basis $\{\mathbf{e}_n; n \in \mathbb{N}\}$ of eigenvectors of the operator K , where, due to its nonnegativity, the following series absolutely converges

$$\text{Tr}VK = \sum_n (\mathbf{e}_n, VK\mathbf{e}_n) = \sum_n \kappa_n(\mathbf{e}_n, V\mathbf{e}_n) \leq \|V\| \sum_n \kappa_n = \|V\| \text{Tr}K. \quad (9)$$

It means that the series $\sum_n |\mu_n|$ is summated. In turns, it leads to convergence of infinite production in (8) to the entire analytic function on $\lambda \in \mathbb{C}$. This infinite production represents its Hadamard decomposition.

Since for the spectrum $\{\mu_n(\varepsilon), n \in \mathbb{N}\}$ of each operator V_ε , the infinite production $\prod_n (1 - 2i\lambda\mu_n(\varepsilon))$ converges at all $\lambda \in \mathbb{C}$, there exists the entire analytic function

$$Q_\varepsilon^{-2}(-i\lambda) = \prod_n (1 - 2i\lambda\mu_n(\varepsilon)). \quad (10)$$

Therefore, for proof of the theorem, it is necessary to prove that the sequence (10) of the entire analytic functions $Q_\varepsilon^{-2}(-i\lambda)$ tends to the entire analytic function $Q^{-2}(-i\lambda)$ at $\varepsilon \rightarrow +0$, for which the decomposition (8) takes place, if $V_\varepsilon \rightarrow V$. At this time, for each $\varepsilon > 0$ the collection $\{\mu_n(\varepsilon); n \in \mathbb{N}\}$ represents eigenvalues of the operator V_ε . We will divide the proof of this fact into two parts.

I. Since the operators V_ε and K commute, they have the common orthogonal basis of eigenvectors $\langle \mathbf{e}_n(\varepsilon); n \in \mathbb{N} \rangle$ with eigenvalues $\langle \mu_n(\varepsilon); n \in \mathbb{N} \rangle$.

This basis composes the complete collection of eigenvectors for the operator VK with the collection of eigenvalues $\langle \mu_n; n \in \mathbb{N} \rangle$. At this time, the following limit relationship $\mu_n = \lim_{\varepsilon \rightarrow +0} \mu_n(\varepsilon)$ takes place. This is established by passing to a strong limit in the sense of a norm in the space $\mathbb{L}_2^{(c)}[0, T]$. The last one, in turns, is possible due to the limit relationship $\lim_{\varepsilon \rightarrow +0} V_\varepsilon = V$ on the operator norm. The eigenvalues μ_n and $\mu_n(\varepsilon)$, $n \in \mathbb{N}$ are real-valued.

The following estimate of the sum takes place

$$\sum_{n>N} |\mu_n(\varepsilon)| = \sum_{n>N} |(\mathbf{e}_n, V_\varepsilon K \mathbf{e}_n(\varepsilon))| \leq \sum_{n>N} \kappa_n(\mathbf{e}_n, V_\varepsilon \mathbf{e}_n) \leq \|V_\varepsilon\| \sum_{n>N} \kappa_n, \quad (11)$$

and also, passing to the limit $\varepsilon \rightarrow +0$, it takes place the analogous estimate

$$\sum_{n>N} |\mu_n| \leq \|V\| \sum_{n>N} \kappa_n.$$

On the basis of these estimates we find the restriction from above for the the infinite production

$$|Q_{\varepsilon, N}^{-2}(-i\lambda)| = \prod_{n>N} |(1 - 2i\lambda\mu_n(\varepsilon))| \leq \prod_{n>N} (1 + 2|\lambda||\mu_n(\varepsilon)|) \leq \exp \left(2|\lambda| \sum_{n>N} |\mu_n(\varepsilon)| \right),$$

because for any $\xi > 0$ the inequality $1 + \xi < e^\xi$. Exactly the same, the restriction from below

$$|Q_N^{-2}(-i\lambda)| = \prod_{n>N} \left| (1 - 2i\lambda\mu_n) \right| \geq \prod_{n>N} (1 - 2|\lambda||\mu_n|) \geq \exp \left(-2|\lambda| \sum_{n>N} |\mu_n| \right)$$

takes place at $2|\lambda||\mu_n| < \gamma$, where γ is the unique solution of the equation $1 - \gamma = e^{-\gamma}$, $\gamma \in (0, 1)$, due to convexity of the function $e^{-\gamma}$.

II. Now, we will show that the limit of the ratio $Q_\varepsilon^{-2}(-i\lambda)/Q^{-2}(-i\lambda)$ is equal to 1 at $\varepsilon \rightarrow +0$. Consider this ratio in closed circle on the plane $\mathbb{C}$ with an arbitrary radius M . We choose the sufficiently large number N that $2M \max\{|\mu_n(\varepsilon)|, |\mu|\} < \gamma$ at all $n > N$ and, moreover, in order that it is fulfilled

$$\exp \left(2M \sum_{n>N} [|\mu_n| + |\mu_n(\varepsilon)|] \right) < \exp \left(2N [\|\mathbf{V}\| + \|\mathbf{V}_\varepsilon\|] \sum_{n>N} \varkappa_n \right) - 1 < \delta_N.$$

Therefore,

$$\begin{aligned} \left| \frac{Q_\varepsilon^{-2}(-i\lambda)}{Q^{-2}(-i\lambda)} - \frac{\prod_{n=1}^N (1 - 2i\lambda\mu_n(\varepsilon))}{\prod_{n=1}^N (1 - 2i\lambda\mu_n)} \right| &= \left| \frac{\prod_{n=1}^N (1 - 2i\lambda\mu_n(\varepsilon))}{\prod_{n=1}^N (1 - 2i\lambda\mu_n)} \right| \left| \frac{\prod_{n>N} (1 - 2i\lambda\mu_n(\varepsilon))}{\prod_{n>N} (1 - 2i\lambda\mu_n)} - 1 \right| \\ &< \delta_N \left| \frac{\prod_{n=1}^N (1 - 2i\lambda\mu_n(\varepsilon))}{\prod_{n=1}^N (1 - 2i\lambda\mu_n)} \right|. \end{aligned}$$

Firstly, passing to the limit $\varepsilon \rightarrow +0$ in both parts of the inequality

$$\left| \lim_{\varepsilon \rightarrow +0} \frac{Q_\varepsilon^{-2}(-i\lambda)}{Q^{-2}(-i\lambda)} - 1 \right| < \delta_N,$$

and after that passing $N \rightarrow \infty$, we get the required. Due to the arbitrariness of the radius M of circle such a limit relationship is fulfilled everywhere on $\mathbb{C}$. $\square$

Corollary 1. *If the integral operator $\mathbf{V}$ is bounded in the space $C[0, T]$ of continuous functions on $[0, T]$ and it is represented in the form of the limit of integral operators $\mathbf{V}_\varepsilon$ at $\varepsilon \rightarrow +0$, in the sense of the norm of operator in this space, the theorem assertion is correct when all other its conditions are fulfilled.*

Proof. Since the kernel $K(t, s)$ is continuous on $[0, T]^2$, its eigenfunctions $\mathbf{e}_k(t)$, $k \in \mathbb{N}$ are also continuous, that it follows from their definition

$$(\mathbf{K}\mathbf{e}_k)(t) = \int_0^T K(t, s)\mathbf{e}_k(s)ds = \varkappa_k\mathbf{e}_k(t).$$

At these conditions, the following estimate is used in (9) and (11)

$$|(\mathbf{e}_l, \mathbf{V}_\varepsilon \mathbf{e}_l)| \leq \int_0^T |\mathbf{e}_l(t)|dt \int_0^T |(\mathbf{V}_\varepsilon)(t, s)| |\mathbf{e}_l(s)|ds \leq T^2 \|\mathbf{V}_\varepsilon\|,$$

where the norm of operator is understood in the sense of the space $C[0, T]$, and, further, in such a sense, it is used, during the estimation process, when the estimate $\|\mathbf{V} - \mathbf{V}_\varepsilon\| < \varepsilon$ of the operator difference $(\mathbf{V} - \mathbf{V}_\varepsilon)$ is found. $\square$

Theorem 4. *For any nuclear operator A in the Gilbert space such that $\text{Ker } A = \{0\}$, there is the Fredholm determinant. It is the entire function on λ with the order less than 1 and it is represented in the form of the Hadamard decomposition*

$$\det(1 + \lambda A) = \prod_n (1 + \lambda \alpha_n), \quad (12)$$

where $\{\alpha_n; n \in \mathbb{N}\}$ is the eigenvalues collection of the operator A .

Proof. For any Gilbert–Schmidt operator, the biorthogonal expansion takes place [16, 17], which has the form $A = \sum_n \alpha_n P_n$, if the operator A is self-adjoint, where $\{P_n; n \in \mathbb{N}\}$ are the projectors on its eigene-subspaces and $\{\alpha_n; n \in \mathbb{N}\}$ are the corresponding eigenvalues. Therefore, such an operator (10) may be approximate by the finite dimensional operator $A_N = \sum_{n=1}^N \alpha_n P_n$ in the sense of the Gilbert–Schmidt norm $N^2(A - A_N) = \sum_{k,l=1}^\infty |((A_N - A)e_k, e_l)|^2 < \varepsilon^2$ in the operator space. In particular, such a property have *nuclear operators*. Therefore, for each such N th approximation, there is the determinant $\det(1 + \lambda A_N)$, $\lambda \in \mathbb{C}$. According to to definition of the Fredholm determinant, corresponding to the operator A , it is such an analytic function on on the parameter $\lambda \in \mathbb{C}$ which the limit $\lim_{N \rightarrow \infty} \det(1 + \lambda A_N)$.

For finite matrices, it takes place (see, for example, [22])

$$\det(1 + \lambda A_N) = \exp \left(\text{Tr} \ln(1 + \lambda A_N) \right) = \exp(\lambda \text{Tr} A_N) \exp \left(- \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \text{Tr} A_N^{n-1} \right). \quad (13)$$

For the nuclear operator A there is the limit $\lim_{N \rightarrow \infty} \text{Tr} A_N = A$ and, in the sense of the Gilbert–Schmidt norm, there is the limit of each summand of the series in (13) at $N \rightarrow \infty$. It is correct due to $\text{Tr}|A|^l = N^2(|A|^l)^{1/2} < \infty$, $l \in \mathbb{N} \setminus \{1\}$, if A is Gilbert–Schmidt operator.

At last, calculating the operator traces, which define coefficients of summands of power series (13), using the basis consisting of eigenvectors of the operator A , we find that, in the circle with sufficiently small radius on the parameter λ complex plane, there is the limit

$$\lim_{N \rightarrow \infty} \det(1 + \lambda A_N) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}}{n} \text{Tr} A^n = \exp \left(\sum_n \ln(1 + \lambda \alpha_n) \right) = \prod_n (1 + \lambda \alpha_n).$$

The proof finishes by the analytic continuation on λ over the plane $\mathbb{C}$. $\square$

As a result of the research conducted in this section, we can conclude that for the characteristic function (3) of the quadratic functional $J[x] = (x, Vx)$ of any gaussian (real-valued) process $\{x(t); t \in [0, T]\}$ with null expectation $Ez(t) = 0$ and with non-degenerate correlation operator K , in the case when the operator V is represented by the limit of integral operators V_ε with continuous kernels such that the operator VK is non-degenerate, the following formula is fulfilled

$$Q(-i\lambda) = \left(\det(1 - 2i\lambda VK) \right)^{-1/2}. \quad (14)$$

4. FREDHOLM'S DETERMINANT CALCULATION

Let the operator V on $\mathbb{L}_2^{(c)}[0, T]$ be defined by the equality $Vz(t) = z(T - t)$, $z(t) \in \mathbb{L}_2^{(c)}[0, T]$. Therefore, the equality $Vz(t) \equiv 0$ it follows that $z(T - t) \equiv 0$, $z(t) \equiv 0$, i.e., this operator has not eigenvalue equal to zero.

Operator V , obviously, is self-adjoint on $\mathbb{L}_2^{(c)}[0, T]$, since for any two functions $z_1(t)$ and $z_2(t)$, from this space it takes place

$$(z_1, Vz_2) = \int_0^T z_1(t) z_2^*(T - t) dt = \int_0^T z_1(T - t) z_2^*(t) dt = (z_2, Vz_1)^*.$$

The operator V is bounded $\|Vz(t)\| = \|z(t)\|$, $\|V\| = 1$, since $(Vz, Vz) = (z, z)$.

At last, one may be represent it as the integral operator with the kernel in the form of the generalized Dirac δ -function,

$$(Vz)(t) = z(T - t) = \int_0^T \delta(s + t - T) z(s) ds.$$

By matching the δ -function with the corresponding sequence of functions $\{V_{1/n}(t); n \in \mathbb{N}\}$ where $V_{1/n}(t) = (n/2\pi)^{1/2} \exp(-nt^2/2)$, we obtain the presentation of the operator V action on an arbitrary function $z(t) \in \mathbb{L}_2^{(c)}[0, T]$ in the form of the limit

$$(Vz)(t) = z(T-t) = \lim_{n \rightarrow \infty} \left(\frac{n}{2\pi}\right)^{1/2} \int_0^T \exp\left(-\frac{n(t-s)^2}{2}\right) z(s) ds$$

of the sequence of bounded integral operators V_ε with kernels $V_\varepsilon(t, s) = V_\varepsilon(T-t-s)$. At this time it is fulfilled

Lemma 1. *The estimate $\|V - V_{1/n}\| < \delta$ of the operator V in the sense of the norm in the space $C[0, T]$ takes place.*

Proof. Let $z(t)$ be a continuous function on $[0, T]$. We transform the function $(V_\varepsilon z)(t)$ by the following way

$$\begin{aligned} (V_\varepsilon z)(t) &= \left(\frac{n}{2\pi}\right)^{1/2} \int_0^T z(s) \exp\left(-n(T-t-s)^2/2\right) ds \\ &= \left(\frac{n}{2\pi}\right)^{1/2} \int_{-t}^{T-t} z(T-t-s) \exp\left(-ns^2/2\right) ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-tn^{1/2}}^{(T-t)n^{1/2}} z(T-t-n^{-1/2}s) \exp\left(-s^2/2\right) ds. \end{aligned}$$

Let us write down an estimate for the difference

$$\begin{aligned} & |(Vz)(t) - (V_\varepsilon z)(t)| \\ &= \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} z(T-t) e^{-s^2/2} ds - \int_{-tn^{1/2}}^{(T-t)n^{1/2}} z(T-t-n^{-1/2}s) \exp\left(-s^2/2\right) ds \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \left[\max_{t \in [0, T]} |z(t)| \left(\int_{(T-t)n^{1/2}}^{\infty} + \int_{-\infty}^{-tn^{1/2}} \right) e^{-s^2/2} ds \right. \\ &\quad \left. + \int_{-tn^{1/2}}^{(T-t)n^{1/2}} |z(T-t-n^{-1/2}s) - z(T-t)| e^{-s^2/2} ds \right]. \end{aligned} \quad (15)$$

We estimate second integral by two summands

$$\begin{aligned} & 2 \max_{t \in [0, T]} |z(t)| \left(\int_{(T-t)n^{1/4}}^{(T-t)n^{1/2}} + \int_{-tn^{1/2}}^{-tn^{1/4}} \right) e^{-s^2/2} ds \\ &+ \int_{-tn^{1/4}}^{(T-t)n^{1/4}} |z(T-t-n^{-1/2}s) - z(T-t)| e^{-s^2/2} ds. \end{aligned} \quad (16)$$

Since

$$\lim_{n \rightarrow \infty} \max_{t \in [-tn^{1/4}, (T-t)n^{1/4}]} |z(T-t-n^{-1/2}s) - z(T-t)| = 0,$$

the last summand in (16) is estimated

$$\int_{-tn^{1/4}}^{(T-t)n^{1/4}} |z(T-t-n^{-1/2}s) - z(T-t)| e^{-s^2/2} ds < \frac{1}{3} \delta(1/n) \max_{t \in [0, T]} |z(t)| \int_{-\infty}^{\infty} e^{-s^2/2} ds$$

with the suitable sequence $\{\delta(1/n); n \in \mathbb{N}\}$ tending to zero at $n \rightarrow \infty$. Due to the tending to zero of two supplementary integrals in (15), (16), each of them may be estimate from above by the same value. Therefore, we obtain the estimate $\max_{t \in [0, T]} |(Vz)(t) - (V_\varepsilon z)(t)| \leq \delta(1/n) \max_{t \in [0, T]} |z(t)|$ which proves the

lemma statement. $\square$

Thus, for calculation of the characteristic function $Q(-i\lambda)$ on the measure $P[\cdot]$ of any gaussian process $\{x(t); t \in [0, T]\}$ with null expectation $E x(t) = 0$, according to Corollary 1, it is valid the formula (14). For solving of such a problem, we use the principe which is named the reconstruction method in the work [10]. It is connected with determination of any entire analytic function $Q^{-2}(\lambda)$ which has the order less than unity and the collection of all its zeroes coincides with the collection $\{\lambda_n; n \in \mathbb{N}\}$ of *characteristic values* of the integral operator VK . According to this principle, it is sufficient to construct anything function satisfying these conditions. Such a function determines, according the Hadamard decomposition $C \prod_n (1 - \lambda/\lambda_n)$, the problem solution up to a constant multiplier C . Therefore, the reconstruction of “genuine” Fredholm determinant finishes on the basis of the condition $Q^{-1}(0) = 1$ which should be satisfied.

In present work, we present the calculation of the function $Q(-i\lambda)$ in the case when the process $\{x(t); t \in [0, T]\}$ is so-called the *Ornstein–Uhlenbeck one*. It has expectation equal to zero and its trajectories satisfy stochastic differential equation

$$dx(t) + \nu x(t)dt = \sigma dw(t)$$

where $\nu > 0$ and $\{w(t); t \in [0, T]\}$ is the standard Wiener process, $E w(t)w^*(t) = \min(t, t')$, which is the gaussian one and it has the expectation $E w(t) = 0$.

For the gaussian random process under consideration, the correlation function is defined by the formula $E x(t)x(s) = (\sigma^2/2\nu) \exp(-\nu|t-s|)$ (see, for example, [2]). Correspondingly, for the complex-valued random process $\{z(t) = x(t) + iy(t); t \in [0, T]\}$, the correlation function doubles

$$K(t, s) = E(x(t) + iy(t))(x(s) - iy(s)) = \frac{\sigma^2}{\nu} \exp(-\nu|t-s|),$$

due to independence of processes $\{x(t); t \in [0, T]\}$ and $\{y(t); t \in [0, T]\}$.

We note two circumstances which are necessary for further calculations. The operator V is the involution in $\mathbb{L}_2[0, T]$, i.e., $V^2 = 1$. Therefore, its eigenvalues are equal ± 1 . At this time, since the operator is self-adjoint, the splitting of the space $\mathbb{L}_2^{(c)}[0, T]$ occurs into the straight sum $\mathbb{L}_2^+[0, T] \oplus \mathbb{L}_2^-([0, T])$ of two mutually orthogonal subspaces $\mathbb{L}_2^+([0, T])$ and $\mathbb{L}_2^-([0, T])$. They consist, correspondingly, even $z_+(t)$ and odd $z_-(t)$ functions relative to the replacing of variable sign with the opposite one, when operation $t \rightarrow T-t$ is done, namely, $z_+(T-t) = z_+(t)$, $z_-(T-t) = -z_-(t)$.

The operator V commute with the operator K , since for any function $z(t) \in \mathbb{L}_2^{(c)}[0, T]$ it is fulfilled

$$\begin{aligned} (VKz)(t) &= V \int_0^T e^{-\nu|t-s|} z(s) ds = \frac{\sigma^2}{\nu} \int_0^T e^{-\nu|T-t-s|} z(s) ds = \frac{\sigma^2}{\nu} \int_0^T e^{-\nu|s-t|} z(T-s) ds \\ &= \frac{\sigma^2}{\nu} \int_0^T K(t, s) Vz(s) ds = (KVz)(t). \end{aligned}$$

In connection with this fact, these operators have the orthogonal basis, consisting of eigenvectors being common for them. Therefore, the Fredholm determinant $Q^{-2}(\lambda)$ may be represented in the form of production $Q^{-2}(\lambda) = Q_+^{-2}(\lambda)Q_-^{-2}(\lambda)$ where $Q_\pm^{-2}(\lambda)$ are Fredholm’s determinants of the operator VK ,

correspondingly, in spaces $\mathbb{L}_2^\pm[0, T]$. We will build functions $Q_\pm^{-2}(\lambda)$, alternately, in these spaces, using the reconstruction method.

Let us find eigenvectors of the operator $\mathbf{VK}$ in the $\mathbb{L}_2^+[0, T]$. The solving of this problem is reduced to the solving of such a problem for the case when $\mathbf{V} = 1$ (see, for example, [2]). Let $z_+(t)$ be the eigenfunction with eigenvalue $\mu = \lambda^{-1}$, $z_+(t) = \lambda(\mathbf{VK}z_+)(t)$. Let us represent it in the form $z_+(t) = p(t) + q(t)$, where functions

$$p(t) = \int_0^{T-t} z_+(s) e^{-\nu|T-t-s|} ds, \quad q(t) = \int_{T-t}^T dz_+(s) e^{-\nu|T-t-s|} ds \quad (17)$$

satisfy the system of differential equations with “inverted variable”

$$\dot{p}(t) = \nu p(t) - \omega(p(T-t) + q(T-t)), \quad \dot{q}(t) = -\nu q(t) + \omega(p(T-t) + q(T-t)), \quad (18)$$

$\omega = \lambda\sigma^2/\nu$. At the same time, since $z_+(T-t) = z_+(t)$, $p(T-t) + q(T-t) = p(t) + q(t)$, and, therefore, the system is transformed to the equivalent form

$$\begin{pmatrix} \dot{p}(t) \\ \dot{q}(t) \end{pmatrix} = \mathbf{H}(\omega) \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}$$

with 2×2 -matrix

$$\mathbf{H}(\omega) = \begin{pmatrix} \nu - \omega & -\omega \\ \omega & \omega - \nu \end{pmatrix}.$$

Functions $p(t)$ and $q(t)$ should be satisfy, according to its definition, boundary conditions $p(T) = q(0) = 0$. The general solution of the linear system has the form

$$\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \exp(t\mathbf{H}(\omega)) \begin{pmatrix} p(0) \\ q(0) \end{pmatrix},$$

where

$$\exp(t\mathbf{H}(\omega)) = \cosh(r_+t) + r_+^{-1}\mathbf{H}(\omega) \sinh(r_+t), \quad r_+ = (\nu^2 - 2\nu\omega)^{1/2}.$$

Then, taking into account the boundary conditions, we obtain the system of linear equations relative to coefficients defining the eigenfunction function $z_+(t)$,

$$0 = p(0) [\cosh(r_+T) + \frac{\nu - \omega}{r_+} \sinh(r_+T)], \quad q(T) = p(0) \frac{\omega}{r_+} \sinh(r_+T). \quad (19)$$

We notice, first of all, that $r_+ \neq 0$, since in this case $p(t) = (1 + (\nu - \omega)t)p(0)$, $q(t) = \omega t p(0)$, $z_+(t) = \text{const}$. Such a situation is impossible, as it is seeing from (17). Then, in order that $z_+(t) \not\equiv 0$, it is necessary and sufficient the fulfilment only the condition

$$\cosh(r_+T) + (\nu - \omega) \sinh(r_+T)/r_+ = 0. \quad (20)$$

Second equation in (19) is unimportant, since $r_+ \neq 0$ and at $\text{sh}(r_+T) = 0$ from first equation we obtain $\text{ch}(r_+T) = 0$, $z_+(t) = 0$. The equation (20) defines the characteristic values λ_n of the integral operator $\mathbf{VK}$ in the space $\mathbb{L}_2^+[0, T]$. Since $r_+ \neq 0$, it transform to the following

$$(\nu + r_+)^2 e^{r_+T} = (\nu - r_+)^2 e^{-r_+T},$$

where there is the extra root $r_+ = 0$. Thus, $Q_+^{-2}(\lambda) = Cr_+^{-1}((\nu + r_+)^2 e^{r_+T} - (\nu - r_+)^2 e^{-r_+T})$, where the multiplier r_+^{-1} is added in order that to prevent the existence of the root $r_+ = 0$ in the equation $Q_+^{-2}(\lambda) = 0$. The value C is found from the condition $Q_+(0) = 1$. Finally, we obtain

$$Q_+^{-2}(\lambda) = \frac{(\nu + r_+)^2 e^{r_+T} - (\nu - r_+)^2 e^{-r_+T}}{4\nu^2 e^{\nu T}}. \quad (21)$$

Let us study the spectral problem for the operator VK in the space $\mathbb{L}_2^-[0, T]$. Its solving is reduced to already above-solved. Representing the eigenfunction $z_-(t)$ with the eigenvalue $\mu = \lambda^{-1}$ in the form $z_-(t) = p(t) + q(t)$, where summands are defined (17), we notice that the equation system (18) for $p(t)$ and $q(t)$, due to the condition $z_-(T-t) = -z_-(t)$, is transformed on the equivalent equation system

$$\begin{pmatrix} \dot{p}(t) \\ \dot{q}(t) \end{pmatrix} = H(-\omega) \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}.$$

Therefore, for calculation the Fredholm determinant $Q_-^{-2}(\lambda)$, it is necessary to change the sign of the parameter ω in all transformations at the calculating of $Q_+^{-2}(\lambda)$. In a result, we obtain

$$Q_-^{-2}(\lambda) = \frac{(\nu + r_-)^2 e^{r_- T} - (\nu - r_-)^2 e^{-r_- T}}{4\nu^2 e^{\nu T}}, \quad (22)$$

where $r_- = (\nu^2 + 2\nu\omega)^{1/2}$.

At last, for the proof completion of the formula for the Fredholm determinant $Q^{-2}(\lambda)$ according to the reconstruction method, we must make sure that both entire functions $Q_{\pm}^{-2}(\lambda)$ on $\lambda \in \mathbb{C}$ posses the order ρ less than 1. This characteristics is defined [21] by the formula

$$\rho = \limsup_{|\lambda| \rightarrow \infty} \frac{\ln \ln \max\{|Q_{\pm}^{-2}(\lambda)|; |\lambda| = q\}}{\ln q}.$$

First of all, we note that

$$\limsup_{|\lambda| \rightarrow \infty} \frac{\ln |r_{\pm}|}{\ln |\lambda|} = \frac{1}{2} \limsup_{|\lambda| \rightarrow \infty} \frac{\ln |\nu^2 \mp 2\sigma^2 \lambda|}{\ln |\lambda|} = \frac{1}{2}. \quad (23)$$

Therefore, it is sufficient to calculate the limits $\limsup_{q \rightarrow \infty} \ln \ln \max\{|Q_{\pm}(\lambda)|; |r_{\pm}| = q\} / \ln q$. We show now that they are equal to unity. It is sufficient to calculate such a limit for the function $Q_+(\lambda) = (R_+ - R_-)/4\nu^2 e^{\nu T}$ where $R_{\pm} = (\nu \pm r_+)^2 e^{\pm r_+ T}$. It is obvious that

$$\limsup_{q \rightarrow \infty} \frac{\ln \ln \max\{|Q_+|; |r_+| = q\}}{\ln q} = \limsup_{q \rightarrow \infty} \frac{\ln \ln \max\{(|R_+| + |R_-|); |r_+| = q\}}{\ln q}$$

and, at the same time,

$$\begin{aligned} \limsup_{q \rightarrow \infty} \frac{\ln \ln \max\{|R_{\pm}|; |r_{\pm}| = q\}}{\ln q} &= \limsup_{q \rightarrow \infty} \frac{\ln \ln \max\{|\nu \pm r_+|^2 e^{\pm T \operatorname{Re} r_+}; |r_{\pm}| = q\}}{\ln q} \\ &= \lim_{q \rightarrow \infty} \frac{\ln \ln q^2 e^{Tq}}{\ln q} = 1. \end{aligned}$$

Thus, due to (23), we obtain that $\rho = 1/2$.

Taking into account the connection formula of the characteristic function $Q(-i\lambda) = Q_+(-i\lambda)Q_-(-i\lambda)$ with expressions (21) and (22), we formulate the following statement.

Theorem 5. *Characteristic function $Q(-i\lambda) = E \exp(i\lambda J[x])$ of the quadratic functional $J[x] = (x, Vx)$ in the space $\mathbb{L}_2[0, T]$, which is defined by the operator $(Vx)(t) = x(T-t)$ on the set of trajectories of the gaussian Ornstein–Uhlenbeck random process $\{x(t); t \in [0, T]\}$ with zero expectation $Ex(t) = 0$ and the correlation function $K(t, s) = (\sigma^2/2\nu) \exp(-\nu|t-s|)$, $\nu > 0$, is represented by the formula*

$$Q(-i\lambda) = \frac{4\nu^2 \exp(\nu T)}{((\nu + r_+)^2 e^{r_+ T} - (\nu - r_+)^2 e^{-r_+ T})^{1/2} ((\nu + r_-)^2 e^{r_- T} - (\nu - r_-)^2 e^{-r_- T})^{1/2}},$$

where $r_{\pm} = (\nu^2 \pm 2i\lambda\sigma^2)^{1/2}$.

5. CONCLUSIONS

The main announced result of our work has been formulated in Theorem 5. However, from our opinion, it is more important that some assertions have been proved, which give the opportunity to produce correct calculations of characteristic functions of quadratic functionals on a measure of gaussian random processes in that case when the operator V is already not an integral operator with a continuous kernel $V(t, s)$ as it is done in [10]. On the contrary, we have considered the case when this operator is bounded self-adjoint of a rather general kind. Such statements have been formulated in Theorem 3 and Corollary 1.

In further, it is preferable to generalize our theory on the operators V of more wider class. It is caused by necessity of applications for some statistical problems of mathematical physics. At first, it is necessary to abandon the self-conjugate condition of the operator V . Just it is exactly needed for characteristic function calculation of quadratic “correlation” functional

$$J[x(t)] = \int_0^T x(t)x(t + \Delta)dt, \quad \Delta > 0.$$

Calculation of such a characteristic function is important in connection with the use of such electronic devices as *corelometers*.

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CONFLICT OF INTEREST

The author of this work declare that they have no conflicts of interest.

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