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Classical Solutions of Hyperbolic Equation with Translation Operators in Free Terms

Vladimir Vasilyev [1,*] and Natalya Zaitseva [2]

[1] Center of Applied Mathematics, Belgorod State National Research University, Pobedy St. 85, Belgorod 308015, Russia
[2] Faculty of Computational Mathematics and Cybernetics, Lomonosov Moscow State University, GSP-1, Leninskie Gory, Moscow 119991, Russia; zaitseva@cs.msu.ru
* Correspondence: vasilyev_v@bsu.edu.ru; Tel.: +7-4722-30-13-00

Abstract: In this paper, we study the question of constructing explicit solutions in a half-space of a hyperbolic equation containing translation operators in space variables in all coordinate directions. Such equations are a natural generalization of classical equations of hyperbolic type, and the resulting solution relates the value of the desired function at different points of the half-space where the process takes place. To construct solutions, a classical operating scheme is used, namely, the formal application of an integral transformation. A theorem is proved that the constructed solutions are classical if the real part of the symbol of the differential-difference operator in the equation is positive. Classes of equations for which this condition is satisfied are given.

Keywords: hyperbolic equation; differential-difference equation; translation operator; classical solution; operational scheme.

MSC: 35L10

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1. Introduction

The theory of problems for differential-difference equations is one of the most important sections of the modern theory of differential equations. This is explained by its numerous applications in the mechanics of a deformable solid body; in the study of elasto-plastic processes; in relativistic electrodynamics; when solving some problems related to plasma; in crystal lattice modeling; in problems of nonlinear optics; in the study of neural networks; when studying models of population dynamics in mathematical biology; in the study of environmental and economic processes; in the theory of automatic control, etc.

For the first time, a differential equation with a translation operator has occurred in the work of J. Bernoulli [1] in the problem of a weightless stretched string of finite length, along which equal and equidistant masses are distributed. With the equation described by J. Bernoulli, mathematicians met in the development of the theory of sound. After that, several hundred works were written, a review of which can be found in the book by H. Burkhardt [2]. The systematic study of differential equations with translation operators began only in the middle of the last century. This is due to their applications to the theory of automatic control. Ordinary differential-difference equations (or equations with a delayed argument) have been studied very well and for a long time, see books [3–5] and the bibliographies in them.

Much less studied are differential-difference equations with partial derivatives. At present, problems for elliptic differential-difference equations in bounded domains have been studied quite well. The development of the theory of these problems belongs to A. Skubachevskii [6,7]. According to the developed theory, non-local problems for classical elliptic equations are related to boundary value problems in a bounded domain for differential-difference equations containing shifts of arguments in higher derivatives that map boundary points into the domain. The presence of such shifts in the equation leads to the appearance of solutions with a loss of their smoothness inside the region and to fundamentally new properties of these solutions. In unbounded domains, problems for elliptic equations with translation operators were studied in the works of A. Muravnik [8–11]. Problems for parabolic equations with translation operators with respect to time or space variables are studied in [12].

Very few works are known for hyperbolic differential-difference equations. In papers [13,14], hyperbolic equations were considered with shifts in time. For the first time, two-dimensional hyperbolic differential-difference equations with translation operators in spatial variables were studied in [15–17]. Infinitely smooth solutions were found for them in the half-plane. The method of integral transformations was used to construct solutions. The question of the uniqueness of the solutions obtained was not considered. In [18], the solvability of the Cauchy problem in the strip of a two-dimensional hyperbolic equation with a shift in the space variable in the free term of the equation is studied. And in papers [19–21], classical solutions of already multidimensional hyperbolic differential-difference equations with shifts in spatial variables in a half-space were found.

The purpose of this work is to investigate the question of the existence of classical solutions of a multidimensional hyperbolic equation with translation operators in space variables in the free terms of the equation, which can be arbitrarily many. Note that shifts in these non-local terms act in arbitrary directions. The results of this work are more general than those in [21].

2. Formulation of the Problem— Materials and Methods

In the half-space $\{(x,t): x \in \mathbf{R}^n,\ t > 0\}$, we study the hyperbolic equation

$$u_{tt}(x,t) - a^2 \sum_{j=1}^{n} u_{x_j x_j}(x,t) + \sum_{j=1}^{n} b_j u(x - h_j, t) = 0, \tag{1}$$

where $a \neq 0$, $b_1, \ldots, b_n$ are given real numbers; $h_j := (h_{j1}, \ldots, h_{jn})$ are given vectors with real coordinates, wherein $h_{j1}^2 + \ldots + h_{jn}^2 \neq 0$, $j = \overline{1,n}$; the function u is real-valued.

Equation (1) contains translation operators by arbitrary vectors in the free terms of the equation in all spatial coordinate directions. Such equations, which in addition to differential operators contain translation operators, are called, according to [3], *differential-difference equations*.

The process described by Equation (1) relates the value of the desired function at different points of the half-space, which distinguishes this equation from the classical hyperbolic equation (which idealizes the real physical process of oscillation). Thus, the hyperbolic equation with translation operators in space variables is a generalization of the classical wave equation.

The purpose of this paper is to obtain a multiparameter family of solutions to Equation (1) containing translation operators in spatial variables, which has not been studied before. Moreover, we will be interested in classical (smooth) solutions.

Based on the results of the works [18,22,23], consider the following functions:

$$F(x,t;\xi) := e^{t\,G_1(\xi)} \sin\left(t\,G_2(\xi) + \varphi(\xi) + x \cdot \xi\right), \tag{2}$$

and

$$H(x,t;\xi) := e^{-t\,G_1(\xi)} \sin\left(t\,G_2(\xi) - \varphi(\xi) - x \cdot \xi\right), \tag{3}$$

where $x \cdot \xi$ is the inner product.

Here, the designations

$$G_1(\xi) := \rho(\xi) \sin\varphi(\xi), \quad \text{and} \quad G_2(\xi) := \rho(\xi) \cos\varphi(\xi), \tag{4}$$

$$\varphi(\xi) := \frac{1}{2}\arctan\frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}, \tag{5}$$

$$\rho(\xi) := \left[\left(a^2|\xi|^2+\alpha(\xi)\right)^2+\beta^2(\xi)\right]^{1/4},$$

$$\alpha(\xi) := \sum_{j=1}^{n} b_j \cos\left(\sum_{k=1}^{n} h_{jk}\xi_k\right), \quad \text{and} \quad \beta(\xi) := \sum_{j=1}^{n} b_j \sin\left(\sum_{k=1}^{n} h_{jk}\xi_k\right)$$

are introduced. The variable $\xi := (\xi_1, \dots, \xi_n)$ is an arbitrary vector with real coordinates; $|\xi|^2 = \xi_1^2 + \dots + \xi_n^2$.

Let for all $\xi \in \mathbb{R}^n$ the inequality

$$a^2|\xi|^2 + \sum_{j=1}^{n} b_j \cos\left(\sum_{k=1}^{n} h_{jk}\xi_k\right) > 0 \tag{6}$$

holds. This means that the Function (5) is correctly defined for all its arguments. Therefore, Functions (2)–(4) are also correctly defined.

Inequality (6) means the condition of positiveness of the real part of the symbol of the differential-difference operator with respect to the multidimensional space variable in Equation (1). According to [7], such an operator is called *strongly elliptic*.

To find Functions (2) and (3), an operating scheme was used (see [18,22,23]). First, the integral Fourier transform with respect to a multidimensional space variable x was formally applied to the Equation (1) according to the formula

$$\widehat{u}(\xi,t) := F_x[u](\xi,t) = \int_{\mathbf{R}^n} u(x,t)e^{i\xi\cdot x}dx.$$

Taking into account the well-known formulas for the Fourier transform

$$F_x[\partial_x^\alpha \partial_t^\beta u] = (-i\xi)^\alpha \partial_t^\beta F_x[u] \quad \text{and} \quad F_x[u(x-h)] = e^{i\,h\cdot\xi}F_x[u],$$

we obtain for the function $\widehat{u}(\xi,t)$, an ordinary second-order differential equation of the form

$$\frac{d^2\widehat{u}(\xi,t)}{dt^2} + \left(a^2|\xi|^2+\alpha(\xi)+i\,\beta(\xi)\right)\widehat{u}(\xi,t) = 0, \tag{7}$$

where $\xi \in \mathbb{R}^n$.

The characteristic roots for this equation are determined by the formulas

$$k_{1,2} = \pm\sqrt{-(a^2|\xi|^2+\alpha(\xi)+i\,\beta(\xi))} = \pm\, i\sqrt{a^2|\xi|^2+\alpha(\xi)+i\,\beta(\xi)} = \pm\, i\rho(\xi)e^{i\,\varphi(\xi)},$$

and the general solution for Equation (7) has the form

$$\widehat{u}(\xi,t) = C_1(\xi)e^{i\,t\,\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]} + C_2(\xi)e^{-i\,t\,\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]},$$

where $C_1(\xi)$, and $C_2(\xi)$ are arbitrary constants depending on the parameter ξ. To find these constants, we add initial conditions to the equation:

$$\widehat{u}(\xi,0) = 0, \quad \text{and} \quad \widehat{u}_t(\xi,0) = 1,$$

where $\xi \in \mathbb{R}^n$. Substituting the function $\widehat{u}(\xi,t)$ into these initial conditions, we obtain a system of the form

$$\begin{cases} C_1(\xi) + C_2(\xi) = 0, \\ C_1(\xi) - C_2(\xi) = -i(\rho(\xi)[\cos\varphi(\xi) + i\sin\varphi(\xi)])^{-1}. \end{cases}$$

Finding solutions to the system

$$C_1(\xi) = \frac{e^{-i\,\varphi(\xi)}}{2i\,\rho(\xi)}, \quad \text{and} \quad C_2(\xi) = -\frac{e^{-i\,\varphi(\xi)}}{2i\,\rho(\xi)}.$$

Substituting the found values of the constants $C_1(\xi)$, and $C_2(\xi)$ into the function $\widehat{u}(\xi, t)$, we obtain its final form

$$\widehat{u}(\xi,t) = \frac{e^{-i\,\varphi(\xi)}}{2i\,\rho(\xi)}\Big[e^{i\,t\,\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]} - e^{-i\,t\,\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]}\Big]$$

$$= \frac{e^{-i\,\varphi(\xi)}}{2i\,\rho(\xi)}\Big[e^{-t\,\rho(\xi)\sin\varphi(\xi)}e^{i\,t\,\rho(\xi)\cos\varphi(\xi)} - e^{t\,\rho(\xi)\sin\varphi(\xi)}e^{-i\,t\,\rho(\xi)\cos\varphi(\xi)}\Big]$$

$$= \frac{1}{2i\,\rho(\xi)}\Big[e^{-t\,\rho(\xi)\sin\varphi(\xi)}e^{i(t\,\rho(\xi)\cos\varphi(\xi)-\varphi(\xi))} - e^{t\,\rho(\xi)\sin\varphi(\xi)}e^{-i(t\,\rho(\xi)\cos\varphi(\xi)+\varphi(\xi))}\Big]$$

$$= \frac{1}{2i\,\rho(\xi)}\Big[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi))} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi))}\Big].$$

We now apply the inverse Fourier transform to the resulting expression by the formula

$$F_{\xi}^{-1}[\widehat{u}](x,t) = \frac{1}{(2\pi)^n}\int\limits_{\mathbf{R}^n} \widehat{u}(\xi,t)e^{-i\xi\cdot x}d\xi.$$

And we obtain the integral expression

$$\frac{1}{(2\pi)^n}\int\limits_{\mathbf{R}^n}\frac{1}{2i\,\rho(\xi)}\Big[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi))} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi))}\Big]e^{-ix\cdot\xi}d\xi$$

$$= \frac{1}{2i(2\pi)^n}\int\limits_{\mathbf{R}^n}\frac{1}{\rho(\xi)}\Big[e^{-t\,G_1(\xi)}e^{i(t\,G_2(\xi)-\varphi(\xi)-x\cdot\xi)} - e^{t\,G_1(\xi)}e^{-i(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi)}\Big]d\xi.$$

In the integrand, by analogy [18], we choose functions containing a sine. Thus, Functions (2) and (3) have been found, which will be used in the next section of the work.

3. Results

Based on the obtained Functions (2) and (3), we construct a family of solutions to Equation (1). The main result of the paper is formulated in the following theorem.

Theorem 1. *Under Condition (6) for all $\xi \in \mathbb{R}^n$, the family of functions*

$$G(x,t;C_1,C_2,\xi) := C_1\,F(x,t;\xi) + C_2\,H(x,t;\xi), \tag{8}$$

where F and H are determined by Formulas (2) and (3), satisfies Equation (1) in the classical sense for any parameters $C_{1,2} \in \mathbb{R}^1$ and $\xi \in \mathbb{R}^n$.

Proof. Consider first Function (2). Calculate the derivatives:

$$F_{x_j}(x,t;\xi) = \xi_j e^{tG_1(\xi)}\cos(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi),$$

$$F_{x_jx_j}(x,t;\xi) = -\xi_j^2 e^{tG_1(\xi)}\sin(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi),$$

$$F_t(x,t;\xi) = G_1(\xi)e^{tG_1(\xi)}\sin(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi) + G_2(\xi)e^{tG_1(\xi)}\cos(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi),$$

and

$$F_{tt}(x,t;\xi) = \Big[G_1^2(\xi) - G_2^2(\xi)\Big]e^{tG_1(\xi)}\sin(t\,G_2(\xi) + \varphi(\xi) + x\cdot\xi) +$$

$$+2G_1(\xi)G_2(\xi)e^{tG_1(\xi)}\cos\left(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\right). \tag{9}$$

From Formula (5), it follows that $|2\varphi(\xi)| < \pi/2$, which means that the inequality

$$\cos 2\varphi(\xi) > 0 \tag{10}$$

holds.

Since Conditions (6) and (10) are satisfied, relations

$$\sin 2\varphi(\xi) = \frac{\tan 2\varphi(\xi)}{\sqrt{1+\tan^2 2\varphi(\xi)}}$$

$$= \tan\left(\arctan\frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}\right)\left[1+\tan^2\left(\arctan\frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}\right)\right]^{-1/2}$$

$$= \frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}\left[1+\frac{\beta^2(\xi)}{\left(a^2|\xi|^2+\alpha(\xi)\right)^2}\right]^{-1/2} = \frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}\left[\frac{\left(a^2|\xi|^2+\alpha(\xi)\right)^2}{\left(a^2|\xi|^2+\alpha(\xi)\right)^2+\beta^2(\xi)}\right]^{1/2}$$

$$= \frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}\,\frac{|a^2|\xi|^2+\alpha(\xi)|}{\rho^2(\xi)} = \frac{\beta(\xi)}{a^2|\xi|^2+\alpha(\xi)}\,\frac{a^2|\xi|^2+\alpha(\xi)}{\rho^2(\xi)} = \frac{\beta(\xi)}{\rho^2(\xi)}$$

are satisfied.

From here and from Formula (4) follows the equality

$$2G_1(\xi)G_2(\xi) = \rho^2(\xi)\sin 2\varphi(\xi) = \beta(\xi). \tag{11}$$

Since Conditions (6) and (10) are satisfied, relations

$$G_1^2(\xi) - G_2^2(\xi) = \rho^2(\xi)\left[\sin^2\varphi(\xi) - \cos^2\varphi(\xi)\right]$$

$$= -\rho^2(\xi)\cos 2\varphi(\xi) = -\frac{\rho^2(\xi)}{\sqrt{1+\tan^2 2\varphi(\xi)}}$$

$$= -\rho^2(\xi)\left[\frac{\left(a^2|\xi|^2+\alpha(\xi)\right)^2}{\left(a^2|\xi|^2+\alpha(\xi)\right)^2+\beta^2(\xi)}\right]^{1/2} = -a^2|\xi|^2 - \alpha(\xi) \tag{12}$$

are satisfied.

Substituting Expressions (11), and (12) into Formula (9), we obtain

$$F_{tt}(x,t;\xi) = \Big[-(a^2|\xi|^2+\alpha(\xi))\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\right)$$

$$+\beta(\xi)\cos\left(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\right)\Big]e^{tG_1(\xi)}.$$

Let us now substitute derivatives F_{tt} and $F_{x_jx_j}$ into Equation (1):

$$F_{tt}(x,t;\xi) - a^2\sum_{j=1}^{n}F_{x_jx_j}(x,t;\xi) = \Big[-(a^2|\xi|^2+\alpha(\xi))\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\right)$$

$$+\beta(\xi)\cos\left(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\right) + a^2\sum_{j=1}^{n}\xi_j^2\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\right)\Big]e^{tG_1(\xi)}$$

$$= \left[-\alpha(\xi)\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\right)+\beta(\xi)\cos\left(t\,G_2(\xi)+\varphi(\xi)+x\cdot\xi\right)\right]e^{tG_1(\xi)}$$

$$= \left[-\sum_{j=1}^{n} b_j \cos\left(\sum_{k=1}^{n} h_{jk}\xi_k\right) \sin\left(t\, G_2(\xi) + \varphi(\xi) + x\cdot\xi\right)\right.$$

$$\left. + \sum_{j=1}^{n} b_j \sin\left(\sum_{k=1}^{n} h_{jk}\xi_k\right) \cos\left(t\, G_2(\xi) + \varphi(\xi) + x\cdot\xi\right)\right] e^{tG_1(\xi)}$$

$$= -\sum_{j=1}^{n} b_j \sin\left(t\, G_2(\xi) + \varphi(\xi) + x\cdot\xi - \sum_{k=1}^{n} h_{jk}\xi_k\right) e^{tG_1(\xi)}$$

$$= -\sum_{j=1}^{n} b_j \sin\left(t\, G_2(\xi) + \varphi(\xi) + x_1\xi_1 + \ldots + x_n\xi_n - h_{j1}\xi_1 - \ldots - h_{jn}\xi_n\right) e^{tG_1(\xi)}$$

$$= -\sum_{j=1}^{n} b_j \sin\left(t\, G_2(\xi) + \varphi(\xi) + (x_1 - h_{j1})\xi_1 + \ldots + (x_n - h_{jn})\xi_n\right) e^{tG_1(\xi)}$$

$$= -\sum_{j=1}^{n} b_j \sin\left(t\, G_2(\xi) + \varphi(\xi) + (x - h_j)\cdot\xi\right) e^{tG_1(\xi)} = -\sum_{j=1}^{n} b_j F(x - h_j, t; \xi).$$

Let us now check that Function (3) satisfies Equation (1). To do this, taking into account Expressions (11) and (12), we find its derivatives:

$$H_{x_j}(x,t;\xi) = -\xi_j e^{-tG_1(\xi)} \cos\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right),$$

$$H_{x_j x_j}(x,t;\xi) = -\xi_j^2 e^{-tG_1(\xi)} \sin\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right),$$

$$H_t(x,t;\xi) = -G_1(\xi) e^{-tG_1(\xi)} \sin\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right) +$$

$$+ G_2(\xi) e^{-tG_1(\xi)} \cos\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right),$$

and

$$H_{tt}(x,t;\xi) = \left[G_1^2(\xi) - G_2^2(\xi)\right] e^{-tG_1(\xi)} \sin\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right)$$

$$-2G_1(\xi)G_2(\xi) e^{-tG_1(\xi)} \cos\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right)$$

$$= \left[-(a^2|\xi|^2 + \alpha(\xi)) \sin\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right) - \beta(\xi) \cos\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right)\right] e^{-tG_1(\xi)}.$$

Let us now substitute derivatives H_{tt} and $H_{x_j x_j}$ into Equation (1):

$$H_{tt}(x,t;\xi) - a^2 \sum_{j=1}^{n} H_{x_j x_j}(x,t;\xi) = \left[-(a^2|\xi|^2 + \alpha(\xi)) \sin\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right)\right.$$

$$\left. -\beta(\xi) \cos\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right) + a^2 \sum_{j=1}^{n} \xi_j^2 \sin\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right)\right] e^{-tG_1(\xi)}$$

$$= -\left[\alpha(\xi) \sin\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right) + \beta(\xi) \cos\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right)\right] e^{tG_1(\xi)}$$

$$= -\left[\sum_{j=1}^{n} b_j \cos\left(\sum_{k=1}^{n} h_{jk}\xi_k\right) \sin\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right)\right.$$

$$\left. + \sum_{j=1}^{n} b_j \sin\left(\sum_{k=1}^{n} h_{jk}\xi_k\right) \cos\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi\right)\right] e^{tG_1(\xi)}$$

$$= -\sum_{j=1}^{n} b_j \sin\left(t\, G_2(\xi) - \varphi(\xi) - x\cdot\xi + \sum_{k=1}^{n} h_{jk}\xi_k\right) e^{-tG_1(\xi)}$$

$$= -\sum_{j=1}^{n} b_j \sin\Big(t\, G_2(\xi) - \varphi(\xi) - x_1\xi_1 - \ldots - x_n\xi_n + h_{j1}\xi_1 + \ldots + h_{jn}\xi_n\Big) e^{tG_1(\xi)}$$

$$= -\sum_{j=1}^{n} b_j \sin\Big(t\, G_2(\xi) - \varphi(\xi) - (x_1 - h_{j1})\xi_1 - \ldots - (x_n - h_{jn})\xi_n\Big) e^{tG_1(\xi)}$$

$$= -\sum_{j=1}^{n} b_j \sin\Big(t\, G_2(\xi) - \varphi(\xi) - (x - h_j)\cdot\xi\Big) e^{tG_1(\xi)} = -\sum_{j=1}^{n} b_j H(x - h_j, t; \xi).$$

Since each of the Functions (2) and (3) satisfies Equation (1) in the classical sense, Function (8) also satisfies Equation (1) in the classical sense, provided that Condition (6) of the theorem is satisfied for all parameters $\xi \in \mathbb{R}^n$ and $C_{1,2} \in \mathbb{R}^1$. The theorem has been proven. □

Fulfillment of the Theorem Condition

We write Condition (3) of the theorem in the form

$$a^2(\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2) +$$
$$+ b_1 \cos\Big(\sum_{k=1}^{n} h_{1k}\xi_k\Big) + b_2 \cos\Big(\sum_{k=1}^{n} h_{2k}\xi_k\Big) + \ldots + b_n \cos\Big(\sum_{k=1}^{n} h_{nk}\xi_k\Big) > 0.$$

It is obvious that for $\xi_j = 0$, $j = \overline{1,n}$, the condition on the coefficients of the free terms of the Equation (1)

$$\sum_{j=1}^{n} b_j > 0 \tag{13}$$

must be satisfied.

Since $a^2|\xi|^2 \geq 0$ for any $\xi \in \mathbb{R}^n$, then we study the fulfillment of the inequality

$$b_1 \cos\Big(\sum_{k=1}^{n} h_{1k}\xi_k\Big) + b_2 \cos\Big(\sum_{k=1}^{n} h_{2k}\xi_k\Big) + \ldots + b_n \cos\Big(\sum_{k=1}^{n} h_{nk}\xi_k\Big) > 0. \tag{14}$$

To fulfill Inequality (14), according to the Vietoris' theorem (see [24] (p. 65)), all shifts in one of the free terms of the Equation (1) must be zero, for example, in the first one, that is

$$h_{11} = h_{12} = \ldots = h_{1n} = 0.$$

The coefficients at free terms of Equation (1) must satisfy the conditions:

$$b_1 \geq b_2 \geq \ldots \geq b_n > 0,$$

which does not contradict Inequality (13); and the coefficients must also satisfy the condition

$$2m\, b_{2m-1} \leq (2m-1)b_{2m}, \quad j = 1, 2, \ldots, \Big[\frac{n-1}{2}\Big],$$

where $[\cdot]$ is the integer part of the number.

And besides this, $n-1$ equalities

$$\sum_{k=1}^{n} h_{2k}\xi_k = \theta, \ \sum_{k=1}^{n} h_{3k}\xi_k = 2\theta, \ \ldots, \ \sum_{k=1}^{n} h_{nk}\xi_k = (n-1)\theta, \quad 0 < \theta < \pi.$$

must be satisfied.

4. Conclusions and Discussion

1. In this paper, a three-parameter family of solutions is constructed in explicit form for a multidimensional hyperbolic equation containing shifts of space variables in the free terms of the equation in all coordinate directions. When constructing solutions, a classical operational scheme was used, namely, formally direct and inverse Fourier transforms were used.
2. It is proved that the obtained solutions are classical if the real part of the symbol of the differential-difference operator in the equation is positive.
3. Sufficient conditions are obtained on the coefficients and shifts in the equation, which guarantee the existence of classical infinitely smooth solutions.

In the future, we plan to use the family of solutions obtained explicitly to study the solvability of the Cauchy problem and model initial problems for the specified class of hyperbolic differential-difference equations. Also in the future, the authors plan to obtain solutions to Equation (1) in the class of generalized functions according to the same operational scheme described in the article. It is also planned to find generalized solutions of initial problems (with different initial data on the boundary $t = 0$) in the form of convolutions.

The results of the work were reported by the authors and discussed at scientific seminars of Lomonosov Moscow State University and Belgorod State National Research University. The results of the research were also reported at the international scientific conference on differential and functional-differential equations (DPDE-2022, Peoples' Friendship University of Russia, July 2022, Moscow, Russia).

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