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ORIGINAL ARTICLE

Theory of generalized Bessel potential space and functional completion

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Abstract

The article's objective is to present norms based on weighted Dirichlet integrals in the space of generalized Bessel potentials. The weighted Dirichlet integral is first defined and then that this integral may be written using the multidimensional generalized translation of the module's degree demonstrated. We then show that a defined previously norm cannot be specified in function space of arbitrary fractional order of smoothness. We present a new norm associated with the generalized Bessel potential kernel. We demonstrate the existence of a complete function space with a perfect functional completion for the class of all indefinitely differentiable finite even functions with the norm based on the generalized Bessel potential.

Keywords Bessel operator · Perfect functional completion · Generalized Bessel potential space · Weighted Dirichlet integral

Mathematics Subject Classification 42B35 · 42B20 · 47B38 · 46E30

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1 Introduction

Function spaces of fractional smoothness and its applications to the theory of partial differential equations are both heavily connected on classical Bessel potentials [1–3].

Our goal in this article is to develop the theory of the generalized Bessel potential space denoted $\mathbf{B}_\gamma^\alpha$, constructed using the multidimensional Hankel transform $\mathbf{F}_\gamma$. Such space was first constructed in [4] using the Stein–Lizorkin approach. B-hypersingular integrals and weighted Riesz potentials, which Lyakhov had previously established, were used in [4] to create the norm in $\mathbf{B}_\gamma^\alpha$.

In this article, we use the so-called Aronszajn–Smith approach [5–7] to introduce the norm in the class of all infinitely differentiable finitely supported, even by each variable functions, $\overset{\circ}{C}{}_{ev}^\infty$. This norm is based on weighted Dirichlet integral $d_{\alpha,\gamma}$ of order $\alpha \geq 0$. However, for $\alpha \geq \frac{n+|\gamma|}{4}$, the class $\overset{\circ}{C}{}_{ev}^\infty$ with norm $\sqrt{d_{\alpha,\gamma}}$ has no functional completion. Next, we introduce the class of generalized Bessel potentials and two norms $|u|_{\alpha,\gamma}$ and $\|u\|_{\alpha,\gamma}$, which are equivalent to $\sqrt{d_{\alpha,\gamma}}$. The norm $\|u\|_{\alpha,\gamma}$ is based on generalized Bessel potential. We show that $\overset{\circ}{C}{}_{ev}^\infty$ normed by $\|u\|_{\alpha,\gamma}$ is a complete function space and has a perfect functional completion.

Generalized Bessel potential space of arbitrary order α is necessary to construct a solution to the next boundary value problem

$$Au = f \text{ in } D, \qquad B_i u = 0 \text{ on } \partial D.$$

Here, A is an elliptic operator containing Bessel differential operators

$$B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \qquad \gamma_i > 0,$$

in particular, the Laplace–Bessel operator $\Delta_\gamma = \sum_{i=1}^n B_{\gamma_i}$.

A different approach for the functional spaces connected to the Laplace–Bessel operator was devised in publications [8, 9, 11]. In [10, 12], it was shown that the Bessel potentials produced by the Bessel differential operators are bounded in weighted Lebesgue space. The Bessel potentials were described in terms of the B-Lizorkin–Triebel spaces in [13]. For perturbed differential operators B_{γ_i} transmutation operators have been studied in a number of publications, including the recent papers [14–17].

2 Definitions

Suppose that $\mathbb{R}^n$ represents the n-dimensional Euclidean space. We deal with the ortant

$$\mathbb{R}_+^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \;\; x_1 > 0, \ldots, x_n > 0\}$$

and

$$\overline{\mathbb{R}}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad x_1 \geq 0, \ldots, x_n \geq 0\}.$$

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a multi-index, where γ_i are positive fixed real numbers for $i = 1, \ldots, n$, and $|\gamma| = \gamma_1 + \cdots + \gamma_n$.

We shall indicate a part of the sphere in $\mathbb{R}^n_+$ with radius r and origin centered as $S_r + (n)$:

$$S_r^+(n) = \{x \in \overline{\mathbb{R}}^n_+ : |x| = r\} \cup \{x \in \overline{\mathbb{R}}^n_+ : x_i = 0, |x| \leq r, i = 1, \ldots, n\}.$$

The next formula from [18], p. 49, for the weighed integral over the $S +_1 (n)$:

$$|S_1^+(n)|_\gamma = \int\limits_{S_1^+(n)} x^\gamma \mathrm{d}S = \frac{\prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}, \tag{1}$$

will be used later.

Let Ω be symmetric with respect to each hyperplane $x_i{=}0$, $i = 1, \ldots, n$, finite or infinite open set in $\mathbb{R}^n$. We consider $\Omega_+ = \Omega \cap \mathbb{R}^n_+$ and $\overline{\Omega}_+ = \Omega \cap \overline{\mathbb{R}}^n_+$, where $\overline{\mathbb{R}}^n_+ {=} \{x{=}(x_1, \ldots, x_n) {\in} \mathbb{R}^n, x_1 {\geq} 0, \ldots, x_n {\geq} 0\}$. The class $C^m(\Omega_+)$ consists of m times differentiable on Ω_+ function. The subset of functions from $C^m(\Omega_+)$ such that all derivatives of these functions with respect to x_i for any $i = 1, \ldots, n$ are continuous up to $x_i{=}0$ is denoted by $C^m(\overline{\Omega}_+)$. Class $C^m_{ev}(\overline{\Omega}_+)$ consists of all functions from $C^m(\overline{\Omega}_+)$ such that $\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}}\ |_{x_i=0} = 0$ for all non-negative integer $k \leq \frac{m-1}{2}$ and for all $i = 1, \ldots, n$ (see [19] and [20], p. 21).

We shall refer to $C^m_{ev}(\overline{\mathbb{R}}^n_+)$ in the following as C^m_{ev}. We define

$$C^\infty_{ev}(\overline{\Omega}_+) = \bigcap C^m_{ev}(\overline{\Omega}_+)$$

with intersection taken for all finite m and $C^\infty_{ev}(\overline{\mathbb{R}}_+) = C^\infty_{ev}$.

The space of all functions $f {\in} C^\infty_{ev}(\overline{\Omega}_+)$ with a compact support is denoted by $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)$. We will employ notations: $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+) {=} \mathcal{D}_+(\overline{\Omega}_+)$ and $\overset{\circ}{C}{}^\infty_{ev}(\overline{\mathbb{R}}_+) = \overset{\circ}{C}{}^\infty_{ev}$.

Let the space of all measurable in $\mathbb{R}^n_+$ functions f, even with respect to each variable x_i, $i = 1, \ldots, n$, such that

$$\int\limits_{\mathbb{R}^n_+} |f(x)|^p x^\gamma \mathrm{d}x < \infty$$

be denoted by $L^\gamma_p(\mathbb{R}^n_+) = L^\gamma_p$, $1 {\leq} p {<} \infty$. Here and in the sequel,

$$x^\gamma = \prod_{i=1}^n x_i^{\gamma_i}.$$

The L_p^γ-norm of f for a real number $p>1$ is defined by

$$\|f\|_{L_p^\gamma(\mathbb{R}^n_+)}=\|f\|_{p,\gamma}=\left(\int\limits_{\mathbb{R}^n_+}|f(x)|^p x^\gamma \mathrm{d}x\right)^{1/p}.$$

L_p^γ is a Banach space (see [20]).

A multidimensional Hankel transform of a function $f\in L_1^\gamma(\mathbb{R}^n_+)$ is written as (see [18], p. 37):

$$\mathbf{F}_\gamma[f](\xi)=\mathbf{F}_\gamma[f(x)](\xi)=\widehat{f}(\xi)=\int\limits_{\mathbb{R}^n_+} f(x)\,\mathbf{j}_\gamma(x;\xi)x^\gamma \mathrm{d}x.$$

The kernel of $\mathbf{F}_\gamma$ is

$$\mathbf{j}_\gamma(x;\xi)=\prod_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\xi_i),\qquad \gamma_1>0,\dots,\gamma_n>0,$$

where the symbol j_ν is used for the normalized Bessel function of the first kind $j_\nu(x)=\dfrac{2^\nu\Gamma(\nu+1)}{x^\nu}\,J_\nu(x)$, J_ν is Bessel function of the first kind [21].

Let $f\in L_1^\gamma(\mathbb{R}_+)$ be of bounded variation by each variable x_i, $i=1,\dots,n$ in a neighborhood of a point x of continuity of f. Then the inversion formula (see [18], p. 38)

$$\mathbf{F}_\gamma^{-1}[\widehat{f}(\xi)](x)=f(x)=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int_{\mathbb{R}^n_+}\mathbf{j}_\gamma(x,\xi)\widehat{f}(\xi)\xi^\gamma\,\mathrm{d}\xi$$

holds.

The multidimensional Hankel transform can be written using the one-dimentional Hankel transforms:

$$\mathbf{F}_\gamma[f](\xi)=F_{\gamma_1}\circ\dots\circ F_{\gamma_n}[f](\xi_1,\dots,\xi_n),$$

where $x=(x_1,\dots,x_n)$, $\xi=(\xi_1,\dots,\xi_n)$ and

$$F_{\gamma_i}[f](\xi)=\int\limits_0^\infty f(x)\,j_{\frac{\gamma_i-1}{2}}(x_i\xi_i)x_i^{\gamma_i}\mathrm{d}x_i,\qquad i=1,\dots,n.$$

Similar to the Fourier transform, the Hankel transform reduces the Bessel differentiation operation to multiplication by the corresponding arguments (see [20]):

$$F_{\gamma_i}[(B_{\gamma_i})_{x_i}f](\xi)=-|\xi_i|^2F_{\gamma_i}[f](\xi),\tag{2}$$

where $(B_{\gamma_i})_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$ is a Bessel operator and $i = 1, \dots, n$.

In [20], p. 20, the next theorem is presented.

Theorem 1 *If $x^{\frac{\nu}{2}}\varphi \in L_2[0,\infty)$, then Hankel transform $x^{\frac{\nu}{2}}F_\nu\varphi \in L_2[0,\infty)$ and Parseval's formula*

$$\int_0^\infty |F_\nu[\varphi](\xi)|^2\xi^\nu d\xi = 2^{\nu-1}\Gamma^2\left(\frac{\nu+1}{2}\right)\int_0^\infty |\varphi(x)|^2 x^\nu dx$$

is true.

Using Theorem 1, we get Parseval's formula for the multidimensional Hankel transform. If $f \in L_2^\gamma(\mathbb{R}_+^n)$, then $\mathbf{F}_\gamma f \in L_2^\gamma(\mathbb{R}_+^n)$ and

$$\int_{\mathbb{R}_+^n} |\mathbf{F}_\gamma[f](\xi)|^2\xi^\gamma \mathrm{d}\xi = 2^{|\gamma|-n}\prod_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)\int_{\mathbb{R}_+^n} |f(x)|^2 x^\gamma \mathrm{d}x. \tag{3}$$

The equality

$$({}^\gamma\mathbf{T}_x^y f)(x) = {}^\gamma\mathbf{T}_x^y f(x) = ({}^{\gamma_1}T_{x_1}^{y_1}\dots{}^{\gamma_n}T_{x_n}^{y_n} f)(x) \tag{4}$$

defines the multidimensional generalized translation, where each of one-dimensional generalized translations ${}^{\gamma_i}T_{x_i}^{y_i}$ acts for i=1, . . ., n by the formula

$$({}^{\gamma_i}T_{x_i}^{y_i} f)(x) = \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma_i}{2}\right)} \times \int_0^\pi f\left(x_1,\dots,x_{i-1},\sqrt{x_i^2+y_i^2-2x_iy_i\cos\varphi_i},x_{i+1},\dots,x_n\right)\,\sin^{\gamma_i-1}\varphi_i\,d\varphi_i.$$

Next, we will employ the notation:

$$C(\gamma) = \pi^{-\frac{n}{2}}\prod_{i=1}^n \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

Multidimensional generalized translation ${}^\gamma\mathbf{T}_x^y$ produces a generalized convolution of the form

$$(f*g)_\gamma(x) = (f*g)_\gamma = \int_{\mathbb{R}_+^n} f(y)({}^\gamma\mathbf{T}_x^y g)(x)y^\gamma\,\mathrm{d}y. \tag{5}$$

The first and second kinds of modified Bessel functions, $I_\alpha(x)$ and $K_\alpha(x)$, often known as the hyperbolic Bessel functions, are defined as follows (see [21]):

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha},$$
$$K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)}. \tag{6}$$

In (2) and (6),α is not an integer. By taking the limit in the aforementioned equations, the functions I_α and K_α are defined for integer values of the parameter α. It is clear that the function K_α is an even function because the connection $K_\alpha(x) = K_{-\alpha}(x)$ holds true.

Next, we give some definitions from [5]. Let $\mathcal{F}$ be a linear functional class. The basic set of $\mathcal{F}$ is the abstract set $\mathcal{E}$ in which the functions of a $\mathcal{F}$ are defined. We start from the exceptional set.

"Exceptional" sets are often used in solving various mathematical problems, since the situation when a certain property is not true for all values of a certain set is quite common. The most well-known example of the exceptional set is the set of Lebesgue measure zero. Exceptional sets have a very diverse history. Additionally, there are substantial differences between the approaches taken to study the various classes of exceptional sets. Here, following Aronszajn and Smith, we give a general descriptive definition of such systems (exceptional class), independent of the method of construction.

An exceptional class or *a system of exceptional sets* in the basic set $\mathcal{E}$ is a class $\mathcal{A}$ of subsets of $\mathcal{E}$ which is

- hereditary: if $A \in \mathcal{A}$ and $B \subset A$, then $B \in \mathcal{A}$,
- σ-additive: if $A_n \in \mathcal{A}$, $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Moreover, all sets in the class $\mathcal{A}$ are "small" or "thin" in one sense or another.

If the set of points under which a statement is false belongs to the exceptional class $\mathcal{A}$, then we say that the sentence is true, exc. $\mathcal{A}$.

If $\mathcal{A}$ is an exceptional class that contains the exceptional set of each f in $\mathcal{F}$, then $\mathcal{F}$ is a *linear functional class relative to* $\mathcal{A}$. We will write $\mathcal{F}$ rel. $\mathcal{A}$ if $\mathcal{F}$ is a functional class relative to $\mathcal{A}$. If $\mathcal{F}$ rel. $\mathcal{A}$, then $\mathcal{A}$ is called an *exceptional class for* $\mathcal{F}$, and the sets in $\mathcal{A}$ are called *exceptional sets*.

Let $\mathcal{F}$ be a functional class rel. $\mathcal{A}$. The *saturated extension* of $\mathcal{F}$ rel. $\mathcal{A}$ is the class $\mathcal{F}'$ of all functions defined exc. $\mathcal{A}$ and equal exc. $\mathcal{A}$ to some function in $\mathcal{F}$. If $\mathcal{F}$ coincides with its saturated extension, then it is called *saturated* rel. $\mathcal{A}$.

If $\mathcal{F} \subset \mathcal{F}'$, $\mathcal{A} \subset \mathcal{A}'$, and the norm of each function in $\mathcal{F}$ is the same as its norm as a function in $\mathcal{F}'$, then a normed functional class $\mathcal{F}$ rel. $\mathcal{A}$ is *embedded* in a normed functional class $\mathcal{F}'$ rel. $\mathcal{A}'$.

If each f in $\mathcal{F}$ is a limit of a sequence f_n in $\mathcal{D}$, then a subset $\mathcal{D}$ of a normed functional class $\mathcal{F}$ (or of any functional class with a pseudo-norm) is said to be *dense* in $\mathcal{F}$.

If $\mathcal{F}$ is embedded in $\mathcal{F}'$ and is a dense subset of $\mathcal{F}'$, then a functional space $\mathcal{F}'$ rel. $\mathcal{A}'$ is a *functional completion* of a normed functional class $\mathcal{F}$ rel. $\mathcal{A}$.

The saturated completion rel. $\mathcal{A}$ is known as the *perfect completion* of $\mathcal{F}$ if there is the smallest exceptional class $\mathcal{A}$ relative to which a given $\mathcal{F}$ has a functional completion.

3 Weighted Dirichlet integral

In this section, we deal with the functions from $\overset{\circ}{C}{}^{\infty}_{ev}$ and introduce the weighed Dirichlet integral $d_{\alpha,\gamma}$ of order $\alpha \geq 0$ with power weight $\xi^{\gamma} = \prod_{i=1}^{n} \xi_i^{\gamma_i}$. At first, we define $d_{\alpha,\gamma}$ by using multidimensional Hankel transforms, after which a direct form for $d_{\alpha,\gamma}$ is given in terms of Bessel operators. Next, we prove that the space $\overset{\circ}{C}{}^{\infty}_{ev}$ normed by $\sqrt{d_{\alpha,\gamma}}$ is not a functional space relative to any exceptional class when $\alpha \geq \frac{n+|\gamma|}{4}$.

Let $\mathbf{i} = (i_1, \ldots, i_m)$ be a multi-index consisting of integers between 1 and n, $d(\mathbf{i}) = m$, $\xi^{\mathbf{i}} = \prod_{k=1}^{m} \xi_{i_k}$, $\xi = (\xi_1, \ldots, \xi_n)$ and

$$\mathbb{B}_{\mathbf{i}} = (B_{\gamma_{i_m}})_{x_{i_m}} \cdots (B_{\gamma_{i_1}})_{x_{i_1}},$$

where $(B_{\gamma_{i_k}})_{x_{i_k}} = \frac{\partial^2}{\partial x_{i_k}^2} + \frac{\gamma_{i_k}}{x_{i_k}} \frac{\partial}{\partial x_{i_k}}$ is a Bessel operator, $k = 1, \ldots, m$.

For an integer $\alpha \geq 0$, the formula defines a weighted Dirichlet integral of order α as

$$d_{\alpha,\gamma}(u) = \sum_{d(\mathbf{i})=\alpha} \int_{\mathbb{R}^n_+} |\mathbb{B}_{\mathbf{i}} u|^2 x^{\gamma} \mathrm{d}x.$$

If $\alpha = 1$, then $d(\mathbf{i}) = 1$ and $\mathbf{i}$ consists only of one element $\mathbf{i} = (i_1)$, which has values from 1 to n. In this case,

$$d_{1,\gamma}(u) = \sum_{j=1}^{n} \int_{\mathbb{R}^n_+} |(B_{\gamma_j})_{x_j} u|^2 x^{\gamma} \mathrm{d}x.$$

If $\alpha = 2$, then $d(\mathbf{i}) = 2$ and $\mathbf{i}$ consists only of two elements $\mathbf{i} = (i_1, i_2)$ and each has value from 1 to n. In this case,

$$d_{2,\gamma}(u) = \sum_{k,j=1}^{n} \int_{\mathbb{R}^n_+} |(B_{\gamma_k})_{x_k} (B_{\gamma_j})_{x_j} u|^2 x^{\gamma} \mathrm{d}x.$$

In Hankel images using (2), we get

$$d_{\alpha,\gamma}(u) = \int_{\mathbb{R}^n_+} |\xi|^{4\alpha} |\mathbf{F}_{\gamma}[u](\xi)|^2 \xi^{\gamma} \mathrm{d}\xi. \tag{7}$$

Formula (7) can be used to defined the *weighted Dirichlet integral* $d_{\alpha,\gamma}$ for arbitrary $\alpha \geq 0$.

Simple calculations give

$$\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+} \frac{|{}^\gamma \mathbf{T}^y_x u(x) - u(x)|^2}{|y|^{n+|\gamma|+4\alpha}} x^\gamma y^\gamma \mathrm{d}x\mathrm{d}y$$
$$= \int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+} |u(y) - u(x)|^2 \left({}^\gamma \mathbf{T}^x_y \frac{1}{|y|^{n+|\gamma|+4\alpha}} \right) x^\gamma y^\gamma \mathrm{d}x\mathrm{d}y. \tag{8}$$

Applying Theorem 1 to equality (8) and doing simple calculations, we arrive at the following statement.

Proposition 2 *For* $0 < \alpha < 1/2$ *and function* $u \in \overset{\circ}{C}{}^\infty_{ev}$, *we have the equality*

$$d_{\alpha,\gamma}(u) = \int\limits_{\mathbb{R}^n_+} |\xi|^{4\alpha} \mid \mathbf{F}_\gamma[u](\xi) \mid^2 \xi^\gamma d\xi$$
$$= \frac{1}{C(n,\gamma,\alpha)} \int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+} \mid u(y) - u(x) \mid^2 \left({}^\gamma \mathbf{T}^x_y \frac{1}{|y|^{n+|\gamma|+4\alpha}} \right) x^\gamma y^\gamma dxdy, \tag{9}$$

where

$$C(n,\gamma,\alpha) = \frac{2^{1-|\gamma|-4\alpha}\pi}{\sin(2\alpha\pi)\Gamma(2\alpha+1)\Gamma\left(\frac{n+|\gamma|}{2}+2\alpha\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}.$$

The constant $C(n,\gamma,\alpha)$ has the following properties:

$$\lim_{\alpha\to 0+} \frac{1}{C(n,\gamma,\alpha)} = 0, \qquad \lim_{\alpha\to 1/2-0} \frac{1}{C(n,\gamma,\alpha)} = 0.$$

Thus, if the greatest integer that is strictly less than α is denoted by l, then

$$d_{\alpha,\gamma}(u) = \begin{cases} \sum\limits_{|i|=\alpha} \int\limits_{\mathbb{R}^n_+} |\mathbb{B}_i u|^2 x^\gamma \mathrm{d}x, & \text{if } \alpha \text{ is integer;} \\ \sum\limits_{|i|=l} \int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+} |\mathbb{B}_i u(y) - \mathbb{B}_i u(x)|^2 T_\alpha(x,y,\gamma) x^\gamma y^\gamma \mathrm{d}x\mathrm{d}y, & \text{otherwise,} \end{cases}$$

where

$$T_\alpha(x,y,\gamma) = \frac{1}{C(n,\gamma,\alpha-l)} \left({}^\gamma \mathbf{T}^x_y \frac{1}{|y|^{n+|\gamma|+4\alpha}} \right).$$

The weighted Dirichlet integral $d_{\alpha,\gamma}$ is continuous in α and independent of the orthogonal coordinates which are used in $\mathbb{R}^n_+$.

From a practical and theoretical point of view, it is important to establish in what the space and with what norm the set $\overset{\circ}{C}{}^\infty_{ev}$ is dense.

In [22], the space of functions related to multiplication by $|x|^{-\alpha}$ in the images of Hankel transform was introduced and considered. This space is called the Riesz B-potential space. Recall that in the B-potential theory [23], Riesz B-potential has the form:

$$(U^\alpha_\gamma f)(x) = u(x) = C_{n,\gamma} \int\limits_{\mathbb{R}^n_+} f(y)({}^\gamma\mathbf{T}^y_x |x|^{\alpha-n-|\gamma|}) y^\gamma \mathrm{d}y, \qquad \alpha > 0$$

with normalized constant $C_{n,\gamma}$. For U^α_γ, the analog of Sobolev theorem is valid (see Theorem 1 in [23]). Namely, for $0<\alpha<\frac{n+|\gamma|}{p}$, $p>1$, operator U^α_γ acting on a function $f \in L^\gamma_p$ is bounded from L^γ_p to L^γ_q, where $\dfrac{1}{q}=\dfrac{1}{p}-\dfrac{\alpha}{n+|\gamma|}$. For $\alpha \geq \frac{n+|\gamma|}{p}$ potentials, U^α_γ can be determined in the sense of weighted generalized functions. As a consequence of this fact in [22] (see Theorem 5), it was proved that $\overset{\circ}{C}{}^\infty_{ev}$ is dense in the space of Riesz B-potentials only for $0<\alpha<\frac{n+|\gamma|}{p}$. So the norm based on Riesz B-potential is not convenient for using in the differential problems, because for these problems we need potentials of arbitrarily high order.

Next, we prove that in a reasonable sense the space $\overset{\circ}{C}{}^\infty_{ev}$ normed by $\sqrt{d_{\alpha,\gamma}}$ is not a functional space when $\alpha \geq \frac{n+|\gamma|}{4}$, since convergence in norm does not imply pointwise convergence of a subsequence.

Theorem 3 *The space $\overset{\circ}{C}{}^\infty_{ev}$ normed by $\sqrt{d_{\alpha,\gamma}}$ is not a functional space relative to any exceptional class if $\alpha \geq \frac{n+|\gamma|}{4}$.*

Proof Let $u \in \overset{\circ}{C}{}^\infty_{ev}$ and u be identically 1 in a neighborhood of the origin belonging to $\mathbb{R}^n_+$. Let $u_\rho = u(x/\rho)$. Changing variables by the formula $\xi/\rho = y$ gives

$$d_{\alpha,\gamma}(u_\rho) = \rho^{n+|\gamma|-4\alpha} \int\limits_{\mathbb{R}^n_+} |y|^{4\alpha} |\mathbf{F}_\gamma[u](y)|^2 y^\gamma \mathrm{d}y = \rho^{n+|\gamma|-4\alpha} d_{\alpha,\gamma}(u).$$

So for $\alpha > \frac{n+|\gamma|}{4}$, we obtain

$$\lim_{\rho\to\infty} d_{\alpha,\gamma}(u_\rho) = 0, \qquad \text{but} \qquad \lim_{\rho\to\infty} u_\rho(x) = 1.$$

Because the whole $\mathbb{R}^n_+$ would not have to be an exceptional set, this demonstrates that the space in question cannot be a functional space.

Now, we consider the case $\alpha = \frac{n+|\gamma|}{4}$. Choosing ε such that $0 < \varepsilon < \alpha$ and $v \in \overset{\circ}{C}{}^{\infty}_{ev}$ for the bilinear form corresponding to the quadratic form $d_{\alpha,\gamma}(u)$, we obtain

$$\begin{aligned} d_{\alpha,\gamma}(u_\rho, v) &= \int\limits_{\mathbb{R}^n_+} |\xi|^{4\alpha}\mathbf{F}_\gamma[u_\rho](\xi)\overline{\mathbf{F}_\gamma[v](\xi)}\xi^\gamma \, d\xi \\ &= \int\limits_{\mathbb{R}^n_+} |\xi|^{2\alpha+2\varepsilon}\mathbf{F}_\gamma[u_\rho](\xi)|\xi|^{2\alpha-2\varepsilon}\overline{\mathbf{F}_\gamma[v](\xi)}\xi^\gamma \, d\xi \\ &\leq \left(\int\limits_{\mathbb{R}^n_+} |\xi|^{4\alpha+4\varepsilon} \mid \mathbf{F}_\gamma[u_\rho](\xi) \mid^2 \xi^\gamma \, d\xi\right)^{1/2} \\ &\quad \cdot \left(\int\limits_{\mathbb{R}^n_+} |\xi|^{4\alpha-4\varepsilon} \mid \mathbf{F}_\gamma[v](\xi) \mid^2 \xi^\gamma \, d\xi\right)^{1/2} \\ &= \sqrt{d_{\alpha+\varepsilon,\gamma}(u_\rho)}\sqrt{d_{\alpha-\varepsilon,\gamma}(v)}. \end{aligned}$$

Since $\lim_{\rho\to\infty} d_{\alpha+\varepsilon,\gamma}(u_\rho)=0$, then $\lim_{\rho\to\infty} d_{\alpha,\gamma}(u_\rho, v)=0$ holds for each $v \in \overset{\circ}{C}{}^{\infty}_{ev}$.

Let $\mathcal{H}$ be the Hilbert space with the distance function produced by the inner product $d_{\alpha,\gamma}(u, v)$. Then $\mathcal{H}$ is the abstract completion of $\overset{\circ}{C}{}^{\infty}_{ev}$ with the norm given by $\sqrt{d_{\alpha,\gamma}}$. So if $u_\rho \to 0$, then we obtain that $\rho \to \infty$ is weakly in this Hilbert space because $d_{\alpha,\gamma}(u_\rho)$ is bounded. It means that [24] there is a sequence $\rho_k \to \infty$ such that the arithmetic means of $\{u_{\rho_k}\}$ converges strongly to 0. But $\lim\limits_{\rho\to\infty} u_\rho(x) = 1$, and as a result the sequence of arithmetic means pointwise converges to 1 everywhere. So the space cannot be a functional space. □

One of the simplest norms on $\overset{\circ}{C}{}^{\infty}_{ev}$ such that it is equivalent to $\sqrt{d_{\alpha,\gamma}}$ is

$$|u|_{\alpha,\gamma} = \left(\int\limits_{\mathbb{R}^n_+} (1 + |\xi|^{4\alpha})|\mathbf{F}_\gamma[u](\xi)|^2\xi^\gamma \, d\xi\right)^{1/2}. \tag{10}$$

Next, we show that norm (10) is closely related with generalized Bessel potentials and can be represented using convolutional kernel generating generalized Bessel potential.

Proposition 4 *The norms $\sqrt{d_{\alpha,\gamma}}$ and $|u|_{\alpha,\gamma}$ are equivalent on $\overset{\circ}{C}{}^{\infty}_{ev}$.*

Proof For $|u|_{\alpha,\gamma}$, taking into account (3) and (7), we can write

$$
\begin{aligned}
|u|^2_{\alpha,\gamma} &= \int\limits_{\mathbb{R}^n_+} (1+|\xi|^{4\alpha})|\mathbf{F}_\gamma[u](\xi)|^2 \xi^\gamma d\xi \\
&= \int\limits_{\mathbb{R}^n_+} |\mathbf{F}_\gamma[u](\xi)|^2 \xi^\gamma d\xi + \int\limits_{\mathbb{R}^n_+} |\xi|^{4\alpha} |\mathbf{F}_\gamma[u](\xi)|^2 \xi^\gamma d\xi \\
&= d_{0,\gamma} + d_{\alpha,\gamma} = c\|u\|^2_{L_2^\gamma} + d_{\alpha,\gamma}.
\end{aligned}
$$

Hence, it follows that

$$\sqrt{d_{\alpha,\gamma}} \le |u|_{\alpha,\gamma}.$$

Besides,

$$|u|_{\alpha,\gamma} \le \sqrt{d_{0,\gamma}} + \sqrt{d_{\alpha,\gamma}} \le C_1\sqrt{d_{\alpha,\gamma}}.$$

So the norms $|u|_{\alpha,\gamma}$ and $\sqrt{d_{\alpha,\gamma}}$ are equivalent on $\overset{\circ}{C}{}^\infty_{ev}$. □

4 Class of generalized Bessel potentials

In this section, we give a definition and some basic properties of the generalized Bessel potential. Such potentials are generated by multiplication by $(1+|x|^2)^{\alpha/2}$ in images of the Hankel transforms. We would like to emphasize that we will indicate only those properties that are important for the theory of function spaces, namely the nature of the singularity at the origin, the fact that the decay at infinity is sufficiently rapid to make it integrable and semi-group property.

We consider the generalized Bessel potential is given by the relation (see [25]),

$$u = (\mathbf{G}^\alpha_\gamma \varphi)(x) = \int\limits_{\mathbb{R}^n_+} G^\gamma_\alpha(y)({}^\gamma\mathbf{T}^y_x \varphi(x)) y^\gamma dy, \tag{11}$$

where

$$G^\gamma_\alpha(x) = \mathbf{F}^{-1}_\gamma[(1+|\xi|^2)^{-\alpha/2}](x) \tag{12}$$

is the generalized Bessel kernel. In [26], two forms of an inverse operator to the (11) were constructed.

In [4], the space

$$\mathbf{B}^\alpha_\gamma(L^\gamma_p) = \{u : u = \mathbf{G}^\alpha_\gamma \varphi,\ \varphi \in L^\gamma_p\}$$

with the norm

$$\|u\|_{\mathbf{B}^\alpha_\gamma(L^\gamma_p)} = \|\varphi\|_{L^\gamma_p}$$

was introduced and also a Liouville class of fractional B-smoothness was constructed on the basis of B-hypersingular integrals.

In [25], it was shown that

$$G^{\gamma}_{\alpha}(x)=\frac{2^{\frac{n-|\gamma|-\alpha}{2}+1}}{|x|^{\frac{n+|\gamma|-\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}K_{\frac{n+|\gamma|-\alpha}{2}}(|x|),\tag{13}$$

where $K_{\frac{n+|\gamma|-\alpha}{2}}$ is the second kind of Bessel function (modified) (see (6)).

The kernel $G^{\gamma}_{\alpha}(x)$ has the following properties:

1. $G^{\gamma}_{\alpha}(x)$ is infinitely differentiable beyond the origin,
2. for $|x|\to 0$, function $G^{\gamma}_{\alpha}(x)$ admits the estimate

$$G^{\gamma}_{\alpha}(x)\sim M_{\alpha}(n,\gamma)\begin{cases}\frac{\Gamma\left(\frac{n+|\gamma|-\alpha}{2}\right)}{2^{\alpha-|\gamma|}}|x|^{\alpha-n-|\gamma|}, & \text{if } 0<\alpha<n+|\gamma|;\\ -2^{1-n}\left(\ln\left(\frac{|x|}{2}\right)+\vartheta\right), & \text{if } \alpha=n+|\gamma|;\\ \frac{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}\right)}{2^{n}}, & \text{if } n+|\gamma|<\alpha,\end{cases}\tag{14}$$

$$M_{\alpha}(n,\gamma)=\frac{2^{n-|\gamma|}}{\Gamma\left(\frac{\alpha}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)},$$

3. for $|x|\to\infty$, function $G^{\gamma}_{\alpha}(x)$ admits the estimate

$$G^{\gamma}_{\alpha}(x)\sim\frac{\sqrt{\pi}2^{\frac{n-|\gamma|-\alpha+1}{2}}}{|x|^{\frac{n+|\gamma|-\alpha+1}{2}}\Gamma\left(\frac{\alpha}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}e^{-|x|},\tag{15}$$

4. $G^{\gamma}_{\alpha}(x)\in L^{\gamma}_{1}(\mathbb{R}^n_+)$, $\alpha>0$,
5. $\int\limits_{\mathbb{R}^n_+}G^{\gamma}_{\alpha}(x)x^{\gamma}\mathrm{d}x=1$,
6. $(G^{\gamma}_{\alpha}*G^{\gamma}_{\beta})_{\gamma}=G^{\gamma}_{\alpha+\beta}$, $\alpha>0$, $\beta>0$, where $(G^{\gamma}_{\alpha}*G^{\gamma}_{\beta})_{\gamma}$ is generalized convolution (5).

Here,

$$\vartheta=\lim_{n\to\infty}\left(-\ln n+\sum_{k=1}^{n}\frac{1}{k}\right)=\int\limits_{1}^{\infty}\left(-\frac{1}{x}+\frac{1}{\lfloor x\rfloor}\right)\mathrm{d}x$$

is the Euler–Mascheroni constant.

With the exception of $x = 0$, the kernel $G^{\gamma}_{\alpha}(x)$ is an analytical function of $|x|$. For $x \neq 0$, $G^{\gamma}_{\alpha}(x)$ is an entire function of α.

Since $|G^{\gamma}_{\alpha}|$ is integrable with weight x^{γ}, its Hankel transform exists for each ξ. The kernel G^{γ}_{α} is analytic for $\alpha > 0$ as a function of α. Therefore, from (12) we obtain for $\alpha > 0$ the Hankel transform of the generalized Bessel kernel by analytical continuation of

$$\mathbf{F}_{\gamma}[G^{\gamma}_{\alpha}](\xi) = (1 + |\xi|^2)^{-\alpha/2}. \tag{16}$$

Let us introduce the norm

$$\|u\|^2_{\alpha,\gamma} = \int\limits_{\mathbb{R}^n_+} (1 + |\xi|^2)^{2\alpha} |\mathbf{F}_{\gamma}[u](\xi)|^2 \xi^{\gamma} \mathrm{d}\xi. \tag{17}$$

To obtain the direct expression of (17) for $0 < \alpha < 1/2$, we first introduce the function

$$\omega_{\alpha,\gamma}(|x|) = \frac{2^{n-\alpha+2}}{\Gamma\left(\frac{\alpha}{2}\right) \prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)} \int_0^{\infty} t^{\frac{n+|\gamma|-\alpha}{2}-1} e^{-t-\frac{|x|^2}{4t}} dt.$$

We can calculate an integral and obtain

$$\omega_{\alpha,\gamma}(|x|) = \frac{2^{\frac{n-|\gamma|-\alpha}{2}+1}}{\Gamma\left(\frac{\alpha}{2}\right) \prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)} |x|^{\frac{n+|\gamma|-\alpha}{2}} K_{\frac{n+|\gamma|-\alpha}{2}}(|x|). \tag{18}$$

The generalized convolution (5) with (18) can then be used to express the generalized Bessel potential in Eq. (11):

$$(\mathbf{G}^{\alpha}_{\gamma}\varphi)(x) = \left(\frac{\omega_{\alpha,\gamma}(|x|)}{|x|^{n+|\gamma|-\alpha}} * \varphi\right)_{\gamma}, \qquad \alpha > 0.$$

Next, we need $\omega_{-4\alpha,\gamma}(|x|)$ for $0 < \alpha < 1/2$. The kernel function $\omega_{-4\alpha,\gamma}(|x|)$ exponentially decays at infinity and goes to a constant at the origin according to the asymptotic features of the modified Bessel function $K_n u$:

$$\lim_{|x|\to\infty} \omega_{-4\alpha,\gamma}(|x|) = 0, \qquad \lim_{|x|\to 0} \omega_{-4\alpha,\gamma}(|x|) = \frac{2^{n+4\alpha}\Gamma\left(\frac{n+|\gamma|+4\alpha}{2}\right)}{\Gamma(-2\alpha) \prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}. \tag{19}$$

Using $\omega_{-4\alpha,\gamma}$, we can write the direct expression of (17) for $0<\alpha<1/2$:

$$\|u\|^2_{\alpha,\gamma}=2^{|\gamma|-n+1}\prod_{i=1}^{n}\Gamma^2\left(\frac{\gamma_i+1}{2}\right)$$
$$\times\left(\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+}\frac{|{}^\gamma\mathbf{T}^y_x u(x)+u(y)|^2}{|x|^{n+|\gamma|+4\alpha}}(\omega_{-4\alpha,\gamma}(|x|)-\omega_{-4\alpha,\gamma}(0))x^\gamma\mathrm{d}x\,y^\gamma\mathrm{d}y-\right.$$
$$\left.-\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+}\frac{|{}^\gamma\mathbf{T}^y_x u(x)-u(y)|^2}{|x|^{n+|\gamma|+4\alpha}}(\omega_{-4\alpha,\gamma}(|x|)+\omega_{-4\alpha,\gamma}(0))x^\gamma\mathrm{d}x\,y^\gamma\mathrm{d}y\right). \tag{20}$$

Proposition 5 *The norms $\|u\|_{\alpha,\gamma}$ and $|u|_{\alpha,\gamma}$ are equivalent on $\overset{\circ}{C}{}^\infty_{ev}$.*

Proof It is obvious that $(1+|\xi|^{4\alpha})\leq(1+|\xi|^2)^{2\alpha}$, so $|u|_{\alpha,\gamma}\leq\|u\|_{\alpha,\gamma}$. Let us consider a function $y=\frac{1+x^{2\alpha}}{(1+x)^{2\alpha}}$. Clearly, this function has a positive point of minimum $x=1$; therefore,

$$(1+|\xi|^2)^{2\alpha}\leq 2^{2\alpha-1}(1+|\xi|^{4\alpha}).$$

So, $\|u\|_{\alpha,\gamma}\leq C|u|_{\alpha,\gamma}$ and the norms $|u|_{\alpha,\gamma}$ and $\|u\|_{\alpha,\gamma}$ are equivalent on $\overset{\circ}{C}{}^\infty_{ev}$. □

5 Perfect functional completion of the class $\mathcal{F}^\gamma_\alpha$

In this section, we study the normalized function class $\mathcal{F}^\gamma_\alpha$ which is $\overset{\circ}{C}{}^\infty_{ev}$ normed by $\|u\|_{\alpha,\gamma}$ of the form (17). Also we show that for the normalized function class, $\mathcal{F}^\gamma_\alpha$ perfect functional completion can be obtained. The exceptional class of this perfect completion is the class of sets where the potential $\mathbf{G}^\alpha_\gamma\varphi$, $\alpha>0$ of $\varphi\in L^\gamma_2$ may be undefined.

Now, we construct an exceptional set based on the generalized Bessel potential. we denote by $\mathcal{A}^\gamma_{2\alpha}$, $\alpha>0$ the class of all sets A such that for some function $\varphi\in L^\gamma_2$, $\varphi\geq0$ the property

$$A\subset\bigcup_x\{x\in\mathbb{R}^n_+:(\mathbf{G}^{2\alpha}_\gamma\varphi)(x)=+\infty\}$$

is valid.

Since the kernel $G^\gamma_\alpha(x)\in L^\gamma_1(\mathbb{R}^n_+)$, then for any $\varphi\in L^\gamma_2$ function $(\mathbf{G}^\alpha_\gamma\varphi)(x)$ is defined and finite almost everywhere and $(\mathbf{G}^\alpha_\gamma\varphi)(x)\in L^\gamma_2$. In particular, each set from $\mathcal{A}^{\gamma}_{2\alpha}$ has the weighted Lebesgue measure 0. In addition, the Hankel transform of $(\mathbf{G}^{\alpha}_{\gamma}\varphi)(x)$ is $\mathbf{F}_{\gamma}[\mathbf{G}^{\alpha}_{\gamma}\varphi](x) = (1+|\xi|^2)^{-\alpha/2}\mathbf{F}_{\gamma}[\varphi](x)$. Due to the previous equality and the Parseval equality (3) for the Hankel transform, we have

$$\|\mathbf{G}^{2\alpha}_{\gamma}\varphi\|_{\alpha,\gamma} = \left(\int\limits_{\mathbb{R}^n_+} |\mathbf{F}_{\gamma}[\varphi](x)|^2 \xi^{\gamma} \mathrm{d}\xi\right)^{1/2} = c\|\varphi\|_{L^{\gamma}_2}, \qquad \varphi \in L^{\gamma}_2,$$

which proves the following proposition.

Proposition 6 *For $\varphi \in L^{\gamma}_2$, the following conditions are comparable:*

1. $\varphi = 0$ *everywhere, except for sets of weighted Lebesgue measure* 0*,*
2. $\mathbf{G}^{2\alpha}_{\gamma}\varphi \equiv 0$*,*
3. $\mathbf{G}^{2\alpha}_{\gamma}\varphi = 0\,\text{exc.}\,\mathcal{A}^{\gamma}_{2\alpha}$*,*
4. $\mathbf{G}^{2\alpha}_{\gamma}\varphi = 0$ *everywhere, except for sets of weighted Lebesgue measure* 0*,*
5. $\|\mathbf{G}^{2\alpha}_{\gamma}\varphi\|_{\alpha,\gamma} = 0$*.*

Let $\mathbf{P}^{\alpha}_{\gamma}$ denote the class of all functions u defined exc. $\mathcal{A}^{\gamma}_{2\alpha}$ such that the equality

$$u(x) = (\mathbf{G}^{2\alpha}_{\gamma}\varphi)(x)\,\text{exc.}\,\mathcal{A}^{\gamma}_{2\alpha}$$

is valid for some function $\varphi \in L^2_{\gamma}$.

Henceforth, the normed class with the norm $\|u\|_{\alpha,\gamma}$ we denote by $\mathbf{P}^{\alpha}_{\gamma}$.

Proposition 7 *Class $\mathcal{A}^{\gamma}_{2\alpha}$ is an exceptional class. Class $\mathbf{P}^{\alpha}_{\gamma}$ is a complete function space with respect to $\mathcal{A}^{\gamma}_{2\alpha}$.*

Proof To prove that $\mathcal{A}^{\gamma}_{2\alpha}$ is an exceptional class, we should show that $\mathcal{A}^{\gamma}_{2\alpha}$ is hereditary and σ-additive. If $A \in \mathcal{A}^{\gamma}_{2\alpha}$ and $B \subset A$, then for some function $\varphi \in L^{\gamma}_2$, such that $\varphi \geq 0$, the following embedding

$$B \subset A \subset \bigcup_x \{x \in \mathbb{R}^n_+ : (\mathbf{G}^{2\alpha}_{\gamma}\varphi)(x) = +\infty\},$$

is valid. Therefore, $B \in \mathcal{A}^{\gamma}_{2\alpha}$. Next, let $A_n \in \mathcal{A}^{\gamma}_{2\alpha}$ and let $\varphi_n \geq 0$ a function such that

$$A_n \subset \bigcup_x \{x \in \mathbb{R}^n_+ : (\mathbf{G}^{2\alpha}_{\gamma}\varphi_n)(x) = +\infty\}, \qquad \|\varphi_n\|_{2,\gamma} \leq \frac{1}{2^{n+|\gamma|}}.$$

Then if $A = \bigcup_{n=1}^{\infty} A_n$ and $\varphi = \sum_{n=1}^{\infty} \varphi_n$, it is clear that $\varphi \geq 0$ is a function from L^{γ}_2 such that

$$A \subset \bigcup_x \{x \in \mathbb{R}^n_+ : (\mathbf{G}^{2\alpha}_{\gamma}\varphi)(x) = +\infty\},$$

so that $A \in \mathcal{A}^{\gamma}_{2\alpha}$. □

Theorem 8 *Class $\mathbf{P}^{\alpha}_{\gamma}$ is a complete function space with respect to $\mathcal{A}^{\gamma}_{2\alpha}$.*

Proof From Proposition 6, it follows that $\mathbf{P}^{\alpha}_{\gamma}$ is a normalized function class rel. $\mathcal{A}^{\gamma}_{2\alpha}$, i.e., conditions $u=0$ exc. $\mathcal{A}^{\gamma}_{2\alpha}$ and $\|u\|_{\alpha,\gamma}=0$ are equivalent. From

$$\|\mathbf{G}^{2\alpha}_{\gamma}\varphi\|_{\alpha,\gamma}=c\|\varphi\|_{L^{\gamma}_{2}},$$

$\varphi\in L^{\gamma}_{2}$, it follows that $\mathbf{P}^{\alpha}_{\gamma}$ is complete; moreover, it is saturated. It remains only to prove the function space property.

One may choose a subsequence $\{u_n\}$ from any sequence convergent to 0 such that

$$\sum_{n=1}^{\infty}\|u_n\|_{\alpha,\gamma}<\infty.$$

If $u_n=\mathbf{G}^{2\alpha}_{\gamma}\varphi_n$, except for a set $A_n\in\mathcal{A}^{\gamma}_{2\alpha}$, then let

$$\varphi(x)=\sum_{n=1}^{\infty}|\varphi_n(x)|.$$

Then $\varphi\in L^{\gamma}_{2}$ and $\mathbf{G}^{2\alpha}_{\gamma}\varphi(x)\to 0$ for all $x\notin A_0=\bigcup_x\{x\in\mathbb{R}^n_+:(\mathbf{G}^{2\alpha}_{\gamma}\varphi)(x)=+\infty\}$. Since $u_n\to 0$ for all $x\notin\bigcup_{n=1}^{\infty}A_n$, $A_n\in\mathcal{A}^{\gamma}_{2\alpha}$. This proves that $\mathbf{P}^{\alpha}_{\gamma}$ is a complete function space with respect to $\mathcal{A}^{\gamma}_{2\alpha}$. □

Theorem 9 *A class $\mathbf{P}^{\alpha}_{\gamma}$ is a perfect functional completion of the class $\mathcal{F}^{\gamma}_{\alpha}$.*

Proof Let us show that $\mathcal{F}^{\gamma}_{\alpha}\subset\mathbf{P}^{\alpha}_{\gamma}$. Let $u\in\mathcal{F}^{\gamma}_{\alpha}$ and $\mathbf{F}_{\gamma}\varphi=(1+|\xi|^2)^{\alpha}\mathbf{F}_{\gamma}u$. Function φ is the inverse Hankel transform: $\varphi=\mathbf{F}^{-1}_{\gamma}(1+|\xi|^2)^{\alpha}\mathbf{F}_{\gamma}u$. Since $u\in\overset{\circ}{C}{}^{\infty}_{ev}(\mathbb{R}^n_+)$, then $\mathbf{F}_{\gamma}\varphi\in L^{\gamma}_{1}$ and $\mathbf{F}_{\gamma}\varphi\in L^{\gamma}_{2}$. This means that $\varphi\in L^{\gamma}_{1}$, $\varphi\in L^{\gamma}_{2}$ is continuous and bounded. Hence, $\mathbf{G}^{2\alpha}_{\gamma}\varphi$ is continuous and belongs to $\mathbf{P}^{\alpha}_{\gamma}$. Inasmuch as $\mathbf{F}_{\gamma}u=\mathbf{F}_{\gamma}\mathbf{G}^{2\alpha}_{\gamma}\varphi$, then $u=\mathbf{G}^{2\alpha}_{\gamma}\varphi$ with the exception of the set of the weighted Lebesgue measure zero; nevertheless, since both functions are continuous, then $u=\mathbf{G}^{2\alpha}_{\gamma}\varphi$ everywhere, so that $u\in\mathbf{P}^{\alpha}_{\gamma}$ and $\mathcal{F}^{\gamma}_{\alpha}\subset\mathbf{P}^{\alpha}_{\gamma}$.

Denote by $\overline{\mathcal{F}^{\gamma}_{\alpha}}$ the closure of $\mathcal{F}^{\gamma}_{\alpha}$ in $\mathbf{P}^{\alpha}_{\gamma}$. Then $\overline{\mathcal{F}^{\gamma}_{\alpha}}$ is a functional completion of $\mathcal{F}^{\gamma}_{\alpha}$. We need to show that $\overline{\mathcal{F}^{\gamma}_{\alpha}}=\mathbf{P}^{\alpha}_{\gamma}$ and this completion is perfect. Since the norm $\|u\|_{\alpha,\gamma}$ is finite for every $u\in\mathbf{P}^{\alpha}_{\gamma}$, every $u\in\mathbf{P}^{\alpha}_{\gamma}$ is equal to some $v\in\overline{\mathcal{F}^{\gamma}_{\alpha}}$ everywhere except for a set of weighed Lebesgue measure 0. However, both u and v are in $\mathbf{P}^{\alpha}_{\gamma}$, so that u equals v everywhere except for a set of weighed Lebesgue measure 0 which implies that $\|u-v\|_{\alpha,\gamma}=0$, and hence $u=v$ exc. $\mathcal{A}^{\gamma}_{2\alpha}$. Therefore, $\overline{\mathcal{F}^{\gamma}_{\alpha}}=\mathbf{P}^{\alpha}_{\gamma}$. □

6 Conclusion

There are two possible definitions of the class of the generalized Bessel potentials $\mathbf{B}_\gamma^\alpha(L_p^\gamma)$ of order α in $\mathbb{R}_+^n$. The first one is that $u \in \mathbf{B}_\gamma^\alpha(L_p^\gamma)$ if u is the generalized convolution $(G_\alpha^\gamma * \varphi)_\gamma$ for some $\varphi \in L_p^\gamma(\mathbb{R}_+^n)$. This approach was presented in [4], where B-hypersingular integrals were used. The second is that $\mathbf{B}_\gamma^\alpha(L_p^\gamma)$ is based on the norm

$$\|u\|_{\alpha,\gamma}^2 = \int\limits_{\mathbb{R}_+^n} (1 + |\xi|^2)^\alpha |\mathbf{F}_\gamma[u](\xi)|^2 \xi^\gamma \, d\xi,$$

which can be written using convolutional kernel generating generalized Bessel potential. This expression shows that the quadratic interpolation between $\|u\|_{\alpha,\gamma}$ and $\|u\|_{\beta,\gamma}$ gives $\|u\|_{\delta,\gamma}$, where δ is the interpolated order $\alpha(1-t)+\beta t$. The norm $\|u\|_{\alpha,\gamma}$ is the most convenient for the study of the class of the generalized Bessel potentials in $\mathbb{R}_+^n$.

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