ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2022, Vol. 43, No. 6, pp. 1408–1416. © Pleiades Publishing, Ltd., 2022.

Multi-Dimensional Generalized Integral Transform in the Weighted Spaces of Summable Functions

S. M. Sitnik[1*], O. V. Skoromnik[2], and S. A. Shlapakov[3***]**

(Submitted by A. B. Muravnik)

[1] *Belgorod State National Research University, Belgorod, 308015 Russia*

[2] *Polotsk State University, Novopolotsk, 211440 Belarus*

[3] *Vitebsk State University, Vitebsk, 210038 Belarus*

Received January 30, 2022; revised February 22, 2022; accepted March 10, 2022

Abstract—In this paper, we study a multi-dimensional generalized integral transformation. The functional and compositional properties of the integral transformation in spaces of summable functions are investigated. The scheme of study is similar to the process of constructing the theory of the H-transformation, in which the central place is given to the questions of bounded and one-to-one action of the corresponding integral operator in spaces of integrable functions with weight concentrated at zero and at infinity. Theory of the considered integral transformation in weighted spaces of summable functions is constructed.

DOI: 10.1134/S1995080222090244

Keywords and phrases: *multi-dimensional generalized integral transform, special functions in the kernel, multidimensional Mellin transform, space of summable functions*.

1. INTRODUCTION

First introduce general multi-dimensional integral transform

$$\big(\mathrm{K}f\big)(\mathbf{x}) = \overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int\limits_{\mathbf{0}}^{\infty} \mathrm{k}[\mathbf{x}\mathbf{t}] f(\mathbf{t}) \mathrm{d}\mathbf{t} \quad (\mathbf{x} > 0); \tag{1}$$

here (see [1], Section 28.4; [2], Ch. 1; [3]) $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathrm{R}^n$; $\mathbf{t} = (t_1, t_2, ..., t_n) \in \mathrm{R}^n$, R^n be the n-dimensional Euclidean space; $\mathbf{x} \cdot t = \sum_{n=1}^{n} x_n t_n$ denotes their scalar product; in particular, $\mathbf{x} \cdot \mathbf{1} = \sum_{n=1}^{n} x_n$ for $\mathbf{1} = (1, 1, ..., 1)$. The expression $\mathbf{x} > t$ means that $x_1 > t_1, x_2 > t_2, ..., x_n > t_n$, the nonstrict inequality $\geq$ has similar meaning ; $\int_{\mathbf{0}}^{\boldsymbol{\infty}} = \int_{\mathbf{0}}^{\infty} \int_{\mathbf{0}}^{\infty} \cdots \int_{\mathbf{0}}^{\infty}$; by $\mathrm{N} = \{1, 2, ...\}$ we denote the set of positive integers, $\mathrm{N}_0 = \mathrm{N} \bigcup \{0\}$, $\mathrm{N}_0^n = \mathrm{N}_0 \times \mathrm{N}_0 \times ... \times \mathrm{N}_0$; $\mathbf{k} = (k_1, k_2, ..., k_n) \in \mathrm{N}_0^n = \mathrm{N}_0 \times ... \times \mathrm{N}_0$ ($k_i \in \mathrm{N}_0,\ i = 1, 2, ..., n$) is a multi-index with $\mathbf{k}! = k_1! \cdots k_n!$ and $|\mathbf{k}| = k_1 + k_2 + ... + k_n$;

$\mathrm{R}_+^n = \{\mathbf{x} \in \mathrm{R}^n,\ \mathbf{x} > 0\}$; for $l = (l_1, l_2, ..., l_n) \in \mathrm{R}_+^n$ $\mathbf{D}^l = \frac{\partial^{|l|}}{(\partial x_1)^{l_1} \cdots (\partial x_n)^{l_n}}$; $\mathbf{dt} = dt_1 \cdot dt_2 \cdots dt_n$;

$\mathbf{t}^l = t^{l_1} t^{l_2} \cdots t^{l_n}$; $f(\mathbf{t}) = f(t_1, t_2, ..., t_n)$. Let C^n ($n \in \mathrm{N}$) be the n-dimensional space of n complex numbers $z = (z_1, z_2, \cdots, z_n)$ ($z_j \in \mathrm{C}$, $j = 1, 2, \cdots, n$); $\overline{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathrm{C}^n$; $\overline{h} = (h_1, h_2, ..., h_n)$, $h_j \in \mathrm{R} \setminus \{0\}$, $j = 1, 2, ..., n$; $\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} = \frac{d}{dx_1 \cdot dx_2 \cdots dx_n}$. We introduce the function in the kernel $\mathrm{k}[\mathbf{x}\mathbf{t}] = \mathrm{k}[x_1 t_1] \cdot \mathrm{k}[x_2 t_2] \cdots \mathrm{k}[x_n t_n]$, which is the product of some one type special functions.

[*] E-mail: sitnik@bsu.edu.ru

[**] E-mail: skoromnik@gmail.com

[***] E-mail: kgima@vsu.by

Our paper is devoted to the study of tranform (1) Kf in the weighted spaces $\mathfrak{L}_{\overline{\nu},\overline{2}}$ summabe functions $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ on $\mathrm{R}_+^{\mathbf{n}}$, such that

$$||f||_{\overline{\nu},\overline{2}} = \left\{ \int\limits_{R_+^1} x_n^{\nu_n \cdot 2-1} \{ \cdots \left\{ \int\limits_{R_+^1} x_2^{\nu_2 \cdot 2-1} [\int\limits_{R_+^1} x_1^{\nu_1 \cdot 2-1} |f(x_1, ..., x_n)|^2 dx_1] dx_2 \right\} \cdots \} dx_n \right\}^{1/2} < \infty$$

$(\overline{2} = (2, 2, ..., 2)$, $\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in \mathrm{R}^n$, $\nu_1 = \nu_2 = ... = \nu_n)$.

In this work, to study transformations of type (1), we use the technique of the multidimensional Mellin transformation

$$(\mathfrak{M}f)(\mathbf{s}) = \int\limits_0^\infty f(\mathbf{t}) \mathbf{t}^{\mathbf{s}-1} \mathbf{dt}.$$

Note that a very important class of transforms under consideration is a class of Buschman–Erdélyi operators, they have many important properties and applications, cf. [3–8]. And topic of this paper is also in a very tight connection with transmutation theory, cf. [9–13].

For transformations of type (1) there is an analogue of the multidimensional Parseval equality in the form

$$\int\limits_{\mathbf{0}}^{\infty} \mathrm{k}[\mathbf{xt}] f(\mathbf{t}) \mathrm{d}\mathbf{t} = \frac{1}{(2\pi i)^n} \int\limits_{c_1 - i\infty}^{c_1 + i\infty} \int\limits_{c_2 - i\infty}^{c_2 + i\infty} \cdots \int\limits_{c_n - i\infty}^{c_n + i\infty} (\mathfrak{M}\mathrm{k})(\mathbf{s}) (\mathfrak{M}f)(1 - \mathbf{s}) \mathbf{x}^{-\mathbf{s}} \mathrm{d}\mathbf{s}, \tag{2}$$

where infinite integration contours $(c_k - i\infty, c_k + i\infty)$ $(k = 1, 2, ..., n)$ start at points $c_k - i\infty$ $(k = 1, 2, ..., n)$ and end at points $c_k + i\infty$ $(k = 1, 2, ..., n)$, respectively, with some real $c_k \in \mathrm{R}$ $(k = 1, 2, ..., n)$. The Mellin transform of kernels of hypergeometric type is the ratio of the products of the Euler gamma functions ц(z), the asymptotics of which, in accordance with the Stirling formula, has a power-exponential character. This allows us to study the given class of integral transformations in the weighted spaces of summable functions and obtain inversion formulas directly from equality (2) and the convolution structure of the class of transformations (1) [14, 15].

The results obtained generalize those obtained earlier for the corresponding one-dimensional transformation (see [16], Ch. 3).

2. PRELIMINARIES

Denote by $[X, Y]$ a set of bounded linear operators acting from a Banach space X into a Banach space Y. For $\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in \mathrm{R}^n$, $\overline{r} = (r_1, r_2, ..., r_n) \in \mathrm{R}^n$ $(1 < \overline{r} < \infty)$ by $\mathfrak{L}_{\overline{\nu},\overline{r}}$ denote the weighted space of integrable functions $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ on $\mathrm{R}_+^{\mathbf{n}}$, for which

$$||f||_{\overline{\nu},\overline{r}} = \left\{ \int\limits_{R_+^1} x_n^{\nu_n \cdot r_n - 1} \{ \cdots \left\{ \int\limits_{R_+^1} x_2^{\nu_2 \cdot r_2 - 1} \right.\right.$$

$$\left.\left. \times \left[\int\limits_{R_+^1} x_1^{\nu_1 \cdot r_1 - 1} |f(x_1, ..., x_n)|^{r_1} dx_1 \right]^{r_2/r_1} dx_2 \right\}^{r_3/r_2} \cdots \}^{r_n/r_{n-1}} dx_n \right\}^{1/r_n} < \infty.$$

For $f(\mathbf{x}) = f(x_1, x_2, ..., x_n) \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ $(\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in \mathrm{R}^n, \nu_1 = \nu_2 = ... = \nu_n, 1 < \overline{r} < 2)$ the n-dimensional Mellin transform $(\mathfrak{M}f)(\mathbf{s})$ is defined by

$$(\mathfrak{M}f)(\mathbf{s}) = \int\limits_{\mathrm{R}^n} f(e^{\boldsymbol{\tau}}) e^{\mathbf{s}\tau} \mathbf{d}\tau, \tag{3}$$

$\mathbf{s} = \overline{\nu} + i\mathbf{t}, \overline{\nu} = (\nu_1, \nu_2, ..., \nu_n), \mathbf{t} = (t_1, t_2, ..., t_n) \in \mathrm{R}^n$.

If $f \in \mathfrak{L}_{\overline{\nu},\overline{r}} \bigcap \mathfrak{L}_{\overline{\nu},1}$ then (3) coincides with the classical multidimensional Mellin transform of the function $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ ($\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathrm{R}^n_+$) defined by the formula ([2], formula 1.4.42):

$$(\mathfrak{M}f)(\mathbf{s}) = \int_0^\infty f(\mathbf{t})\mathbf{t}^{\mathbf{s}-1}\mathbf{dt}, \quad Re(\mathbf{s}) = \overline{\nu},$$

$\mathbf{s} = (s_1, s_2, ..., s_n)$, $s_j \in \mathrm{C}^n$ $(j = 1, 2, ...n)$.

The inverse Mellin transform is given for $\mathbf{x} \in \mathrm{R}^n_+$ by the formula ([2], formula 1.4.43)

$$(\mathfrak{M}^{-1}g)(\mathbf{x}) = \mathfrak{M}^{-1}[g(\mathbf{s})](\mathbf{x}) = \frac{1}{(2\pi i)^n} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} \cdots \int_{\gamma_n - i\infty}^{\gamma_n + i\infty} \mathbf{x}^{-\mathbf{s}} g(\mathbf{s}) \mathrm{d}\mathbf{s},$$

with $\gamma_j = Re(s_j)$ $(j = 1, \cdots, n)$. The theory for these multidimensional Mellin transforms appears in the book by Brychkov [17], see also ([2], Ch. 1).

We need the following spaces. By $\mathrm{L}_{\overline{p}}(\mathrm{R}^n)$, as usual, we denote the space of functions $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$, for which

$$||f||_{\overline{p}} = \left\{ \int_{\mathrm{R}^n} |f(\mathbf{x})|^{\overline{p}} \mathrm{d}\mathbf{x} \right\}^{1/\overline{p}} < \infty, \quad \overline{p} = (p_1, p_2, ..., p_n), \quad 1 \leq \overline{p} < \infty.$$

For $\overline{p} = \infty$, the space $\mathrm{L}_\infty(\mathrm{R}^n)$ is introduced as the collection of all measurable functions with a finite norm

$$||f||_{\mathrm{L}_\infty(\mathrm{R}^n)} = esssup|f(\mathbf{x})|, \tag{4}$$

where $esssup|f(\mathbf{x})|$ is the essential supremum of the function $|f(\mathbf{x})|$ [18].

Based on Statement 3.1 (see [16]) directly verify the validity of the following properties of the Mellin transform (3).

Lemma 1. *The following properties of the Mellin transform (3) are valid:*

(a) Transformation (3) is a unitary mapping of the space $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *(*$\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in R^n$, $\nu_1 = \nu_2 = ... = \nu_n$*) onto the space* $L_{\overline{2}}(R^n)$.

(b) For $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *(*$\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in R^n$, $\nu_1 = \nu_2 = ... = \nu_n$*), there holds*

$$f(\mathbf{x}) = \frac{1}{(2\pi i)^n} \lim_{\mathrm{R} \to \infty} \int_{\nu_1 - i\mathrm{R}}^{\nu_1 + i\mathrm{R}} \int_{\nu_2 - i\mathrm{R}}^{\nu_2 + i\mathrm{R}} \cdots \int_{\nu_n - i\mathrm{R}}^{\nu_n + i\mathrm{R}} (\mathfrak{M}f)(\mathbf{s}) \mathbf{x}^{-\mathbf{s}} \mathrm{d}\mathbf{s},$$

where the limit is taken in the topology of the space $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *(*$\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in R^n$, $\nu_1 = \nu_2 = ... = \nu_n$*) and where, if* $F(\overline{\nu} + i\mathbf{t}) = F_1(\nu_1 + it_1) \cdots F_n(\nu_n + it_n)$, $F_j(\nu_j + it_j) \in L_1(-R, R)$, $j = 1, ..., n$, *then*

$$\int_{\nu_1 - iR}^{\nu_1 + iR} \int_{\nu_2 - iR}^{\nu_2 + iR} \cdots \int_{\nu_n - iR}^{\nu_n + iR} F(\mathbf{s}) \mathrm{d}\mathbf{s} = i^n \int_{-R}^{R} \int_{-R}^{R} \cdots \int_{-R}^{R} F(\overline{\nu} + i\mathbf{t}) \mathrm{d}\mathbf{t}.$$

(c) For functions $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $g \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$ *(*$\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in R^n$, $\nu_1 = \nu_2 = ... = \nu_n$*) the following equality holds*

$$\int_{\mathbf{0}}^{\boldsymbol{\infty}} f(\mathbf{x}) g(\mathbf{x}) \mathrm{d}\mathbf{x} = \frac{1}{(2\pi i)^n} \int_{\overline{\nu} - i\infty}^{\overline{\nu} + i\infty} (\mathfrak{M}f)(\mathbf{s}) (\mathfrak{M}g)(1 - \mathbf{s}) \mathbf{x}^{-\mathbf{s}} \mathrm{d}\mathbf{s}. \tag{5}$$

Let $\mathbf{W}_\delta$ (see, for example, [2], formula (1.3.6); [16], formula (3.3.12)) be elementary operator

$$(\mathbf{W}_\delta f)(\mathbf{x}) = f\left(\frac{\mathbf{x}}{\delta}\right), \quad \mathbf{x} \in \mathrm{R}^n, \quad \delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathrm{R}^n_+. \tag{6}$$

It is known that the Mellin transform (3) of the transformation $\mathbf{W}_\delta$ is equal ([2], formula (1.4.47))

$$(\mathfrak{M}\mathbf{W}_\delta f)(\mathbf{s}) = \delta^{\mathbf{s}}(\mathfrak{M}f)(\mathbf{s}) \quad (\mathrm{Re}(\mathbf{s}) = \overline{\nu}). \tag{7}$$

Taking into account Lemma 3.1 [16], equality (7), Lemma 2.1 [19–21], it is directly verified that the operator $\mathbf{W}_\delta$ has the following properties.

Lemma 2. *Let $\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in R^n$ $(\nu_1 = \nu_2 = ... = \nu_n)$ and $1 \le \overline{r} < \infty$. $\mathbf{W}_\delta$ is bounded isomorphism of $\mathfrak{L}_{\overline{\nu},\overline{r}}$ onto itself, and if $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ $(1 \le \overline{r} \le 2)$, then*

$$(\mathfrak{M}\mathbf{W}_\delta f)(\mathbf{s}) = \delta^{\mathbf{s}}(\mathfrak{M}f)(\mathbf{s}) \quad (\mathrm{Re}(\mathbf{s}) = \overline{\nu}).$$

3. $\mathfrak{L}_{\overline{\nu},2}$-THEORY FOR THE MULTI-DIMENSIONAL K-TRANSFORM

In this section we consider multi-dimensional generalized integral transform (1):

$$\big(\mathrm{K}f\big)(\mathbf{x}) = \overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int\limits_{\mathbf{0}}^{\infty} \mathrm{k}[\mathbf{x}\mathbf{t}] f(\mathbf{t}) \mathrm{d}\mathbf{t} \quad (\mathbf{x} > 0),$$

where kernel $\mathrm{k} \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$, $\overline{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathrm{C}^n$ and $\overline{h} = (h_1, h_2, ..., h_n)$, $h_j \in \mathrm{R} \setminus \{0\}, j = 1, 2, ..., n$.

Theorem 1. *(a) Let the transformation operator (1) satisfy the condition $K \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$, then the kernel on the right side of (1) $k \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$. If we set for $\nu_1 \neq 1 - (Re(\lambda_1) + 1)/h_1, \nu_2 \neq 1 - (Re(\lambda_2) + 1)/h_2, ..., \nu_n \neq 1 - (Re(\lambda_n) + 1)/h_n$ $(\nu_1 = \nu_2 = ... = \nu_n)$*

$$(\mathfrak{M}\mathrm{k})(1 - \overline{\nu} + i\mathbf{t}) = \frac{\theta(\mathbf{t})}{\overline{\overline{\lambda} + 1 - (1 - \overline{\nu} + i\mathbf{t})\overline{h}}} \tag{8}$$

almost everywhere, then function $\theta \in L_\infty(R^n)$, and for $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ there holds the relation

$$(\mathfrak{M}\mathrm{K}f)(1 - \overline{\nu} + i\mathbf{t}) = \theta(\mathbf{t})(\mathfrak{M}f)(\overline{\nu} - i\mathbf{t}) \tag{9}$$

almost everywhere.

(b) Conversely, for given function $\theta \in L_\infty(R^n)$, $\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in R^n$ $(\nu_1 = \nu_2 = ... = \nu_n)$, $\overline{h} = (h_1, h_2, ..., h_n) \in R^n_+$, there is a transform $K \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$ so that the equality (9) holds for $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$. Moreover, if $\nu_1 \neq 1 - (Re(\lambda_1) + 1)/h_1, \nu_2 \neq 1 - (Re(\lambda_2) + 1)/h_2, ..., \nu_n \neq 1 - (Re(\lambda_n) + 1)/h_n$ $(\nu_1 = \nu_2 = ... = \nu_n)$, then transformation Kf is representable in the form (1) with the kernel k definite by (8).

(c) Under the hypotheses of (a) or (b) with $\theta \neq 0$, K is one-to-one transformation from the space $\mathfrak{L}_{\overline{\nu},\overline{2}}$ into the space $\mathfrak{L}_{1-\overline{\nu},\overline{2}}$, and if in addition $1/\theta \in L_\infty(R^n)$, then K maps $\mathfrak{L}_{\overline{\nu},\overline{2}}$ onto $\mathfrak{L}_{1-\overline{\nu},\overline{2}}$, and for functions f, $g \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ the relation

$$\int\limits_{\mathbf{0}}^{\infty} f(\mathbf{x})(\mathrm{K}g)(\mathbf{x})\mathrm{d}\mathbf{x} = \int\limits_{\mathbf{0}}^{\infty} (\mathrm{K}f)(\mathbf{x}) g(\mathbf{x})\mathrm{d}\mathbf{x}$$

is valid.

Proof. (a) Let K is given by (1) and $\mathrm{K} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$, $\nu_1 \neq 1 - (\mathrm{Re}(\lambda_1) + 1)/h_1, ..., \nu_n \neq 1 - (\mathrm{Re}(\lambda_n) + 1)/h_n$ $(\nu_1 = ... = \nu_n)$.

First we consider the case $\nu_1 > 1 - (\mathrm{Re}(\lambda_1) + 1)/h_1, ..., \nu_n > 1 - (\mathrm{Re}(\lambda_n) + 1)/h_n$ $(\nu_1 = ... = \nu_n)$. For $a = (a_1, a_2, ..., a_n)$, where $a_j > 0$ $(j = 1, 2, ..., n)$ are real numbers, will determing the function

$$g_a(\mathbf{t}) = \begin{cases} \mathbf{t}^{(\overline{\lambda}+1)/\overline{h}-1}, & 0 < \mathbf{t} < a; \\ 0, & \mathbf{t} > a; \end{cases}$$

$$= \begin{cases} t_1^{(\lambda_1+1)/h_1-1}\cdots t_n^{(\lambda_n+1)/h_n-1}, & 0<t_j<a_j \quad (j=1,2,...n); \\ 0, \quad t_j>a_j \quad (j=1,2,...n). \end{cases} \tag{10}$$

Then

$$||g_a||_{\overline{v},\overline{2}} = \left\{ \int_0^{a_n} t_n^{v_n\cdot 2-1} \cdots \left\{ \int_0^{a_2} t_2^{v_2\cdot 2-1} \right.\right.$$

$$\left.\left. \times \left[\int_0^{a_1} t_1^{v_1\cdot 2-1} \left| t_1^{(\mathrm{Re}(\lambda_1)+1)/h_1-1} t_2^{(\mathrm{Re}(\lambda_2)+1)/h_2-1} \cdots t_n^{(\mathrm{Re}(\lambda_n)+1)/h_n-1} \right|^2 dt_1 \right] dt_2 \right\} \cdots \} dt_n \right\}^{1/2}$$

$$= \left\{ \int_0^{a_n} t_n^{2((\mathrm{Re}(\lambda_n)+1)/h_n+v_n-1)-1} dt_n \cdots \int_0^{a_1} t_1^{2((\mathrm{Re}(\lambda_1)+1)/h_1+v_1-1)-1} dt_1 \right\}^{1/2}$$

$$= \left\{ \int_0^{a} \mathbf{t}^{2((\mathrm{Re}(\overline{\lambda})+1)/\overline{h}+\overline{v}-1)-1} \mathrm{d}\mathbf{t} \right\}^{1/2} < \infty,$$

which means $g_a \in \mathfrak{L}_{\overline{\nu},\overline{2}}$. From here we get

$$\big(\mathrm{K}g_1\big)(\mathbf{x}) = \overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int_0^1 \int_0^1 \cdots \int_0^1 \mathrm{k}[\mathbf{x}\mathbf{t}] \mathbf{t}^{(\overline{\lambda}+1)/\overline{h}-1} \mathrm{d}\mathbf{t}$$

$$= h_1 h_2 \cdots h_n x_1^{1-(\lambda_1+1)/h_1} x_2^{1-(\lambda_2+1)/h_2} \cdots x_n^{1-(\lambda_n+1)/h_n} \frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} x_1^{(\lambda_1+1)/h_1} x_2^{(\lambda_2+1)/h_2} \cdots x_n^{(\lambda_n+1)/h_n}$$

$$\times \int_0^1 \int_0^1 \cdots \int_0^1 \mathrm{k}[x_1 t_1]\mathrm{k}[x_2 t_2]\cdots \mathrm{k}[x_n t_n] t_1^{(\lambda_1+1)/h_1-1} t_2^{(\lambda_2+1)/h_2-1} \cdots t_n^{(\lambda_n+1)/h_n-1} dt_1 dt_2 \cdots dt_n$$

$$= [x_j t_j = \tau_j \quad (j=1,2,...n)]$$

$$= \overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \int_0^{x_n} \int_0^{x_{n-1}} \cdots \int_0^{x_1} \mathrm{k}[\tau_1]\mathrm{k}[\tau_2]\cdots \mathrm{k}[\tau_n] \tau_1^{(\lambda_1+1)/h_1-1} \tau_2^{(\lambda_2+1)/h_2-1} \times \cdots$$

$$\times \tau_n^{(\lambda_n+1)/h_n-1} d\tau_1 d\tau_2 \cdots d\tau_n = \overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \int_0^{\mathbf{x}} \tau^{(\overline{\lambda}+1)/\overline{h}-1} \mathrm{k}[\tau]\mathrm{d}\tau = \overline{h}\mathrm{k}(\mathbf{x})$$

almost everywhere. Thus, $\mathrm{K}\,g_1 = \overline{h}\,\mathrm{k}$. Therefor since $\mathrm{K} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$, then we have $\mathrm{k} \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$.

Since $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ and $\mathrm{k} \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$, by using the Cauchy–Bunyakovsky inequality [22]

$$\left| \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})g(\mathbf{x})\mathrm{d}\mathbf{x} \right| \le \left(\int_{\mathbf{a}}^{\mathbf{b}} |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right)^{1/2} \left(\int_{\mathbf{a}}^{\mathbf{b}} |g(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right)^{1/2} \quad (-\infty \le \mathbf{a} < \mathbf{b} \le \infty),$$

we have

$$\left| \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int_{\mathbf{0}}^{\infty} \mathrm{k}(\mathbf{x}\mathbf{t}) f(\mathbf{t}) \mathrm{d}\mathbf{t} \right| = \left| \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int_{\mathbf{0}}^{\infty} \left\{ \mathbf{t}^{1/2-\overline{\nu}} \mathrm{k}(\mathbf{x}\mathbf{t}) \right\} \left\{ \mathbf{t}^{-1/2+\overline{\nu}} f(\mathbf{t}) \right\} \mathrm{d}\mathbf{t} \right|$$

$$\le \mathbf{x}^{(\mathrm{Re}(\overline{\lambda})+1)/\overline{h}} \left\{ \int_{\mathbf{0}}^{\infty} \mathbf{t}^{2(1-\overline{\nu})-1} |\mathrm{k}(\mathbf{x}\mathbf{t})|^2 \, \mathrm{d}\mathbf{t} \right\}^{1/2} ||f||_{\overline{\nu},\overline{2}} = \mathbf{x}^{\overline{\nu}-1+(\mathrm{Re}(\overline{\lambda})+1)/\overline{h}} ||\mathrm{k}||_{1-\overline{\nu},\overline{2}} ||f||_{\overline{\nu},\overline{2}} = o(1)$$

as $x_1 \to +0, ..., x_n \to +0$. Integrating both sides of the equality (1), we obtain

$$\int\limits_{\mathbf{0}}^{\mathbf{x}} \mathbf{t}^{(\overline{\lambda}+1)/\overline{h}-1}(\mathrm{K}f)(\mathbf{t})d\mathbf{t} = \overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int\limits_{\mathbf{0}}^{\infty} \mathrm{k}(\mathbf{x}t)f(\mathbf{t})d\mathbf{t} \quad (\mathbf{x} > 0). \tag{11}$$

For $\mathbf{x} > 0$ and we $\mathrm{Re}(\mathbf{s}) + (\mathrm{Re}(\overline{\lambda}) + 1)/\overline{h} > 1$ we obtain for $g_{\mathbf{x}}(\mathbf{t})$

$$(\mathfrak{M}g_{\mathbf{x}})(\mathbf{s}) = \frac{\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}+\mathbf{s}-1}}{\overline{\lambda} + 1 - \overline{h}(1-\mathbf{s})}. \tag{12}$$

Since $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ and $g_{\mathbf{x}} \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, from (5) we have

$$\int\limits_{\mathbf{0}}^{\mathbf{x}} \mathbf{t}^{(\overline{\lambda}+1)/\overline{h}-1}(\mathrm{K}f)(\mathbf{t})d\mathbf{t} = \int\limits_{\mathbf{0}}^{\mathbf{x}} g_{\mathbf{x}}(\mathbf{t})(\mathrm{K}f)(\mathbf{t})d\mathbf{t}$$

$$= \frac{1}{(2\pi i)^n} \int\limits_{\overline{\nu}-i\infty}^{\overline{\nu}_1+i\infty} (\mathfrak{M}g_{\mathbf{x}})(\mathbf{s})(\mathfrak{M}\mathrm{K}f)(1-\mathbf{s})d\mathbf{s} = \frac{\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}}}{(2\pi i)^n} \int\limits_{1-\overline{\nu}-i\infty}^{1-\overline{\nu}_1+i\infty} \mathbf{x}^{-\mathbf{s}} \frac{(\mathfrak{M}\mathrm{K}f)(\mathbf{s})}{\overline{\lambda}+1-\mathbf{s}\overline{h}}d\mathbf{s}$$

$$= \frac{\overline{h}\mathbf{x}^{\overline{\nu}-1+(\overline{\lambda}+1)/\overline{h}}}{(2\pi)^n} \int\limits_{-\infty}^{+\infty} \mathbf{x}^{-i\mathbf{t}} \frac{(\mathfrak{M}\mathrm{K}f)(1-\overline{\nu}+i\mathbf{t})}{\overline{\lambda}+1-(1-\overline{\nu}+i\mathbf{t})\overline{h}}d\mathbf{t}. \tag{13}$$

Similarly, from (5) and (7) we find

$$\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int\limits_{\mathbf{0}}^{\infty} \mathrm{k}(\mathbf{x}t)f(\mathbf{t})d\mathbf{t} = \overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int\limits_{\mathbf{0}}^{\infty} (\mathbf{W}_{1/\mathbf{x}}\mathrm{k})(\mathbf{t})f(\mathbf{t})d\mathbf{t}$$

$$= \frac{\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}}}{(2\pi i)^n} \int\limits_{1-\overline{\nu}-i\infty}^{1-\overline{\nu}+i\infty} \mathbf{x}^{-\mathbf{s}}(\mathfrak{M}\mathrm{k})(\mathbf{s})(\mathfrak{M}f)(1-\mathbf{s})d\mathbf{s}$$

$$= \frac{\overline{h}\mathbf{x}^{\overline{\nu}-1+(\overline{\lambda}+1)/\overline{h}}}{(2\pi)^n} \int\limits_{-\infty}^{+\infty} \mathbf{x}^{-i\mathbf{t}}(\mathfrak{M}\mathrm{k})(1-\overline{\nu}+i\mathbf{t})(\mathfrak{M}f)(\overline{\nu}-i\mathbf{t})d\mathbf{t}. \tag{14}$$

Now we substitute (13) and (14) into (11), and denote by

$$\mathrm{F}(\mathbf{t}) = \frac{(\mathfrak{M}\mathrm{K}f)(1-\overline{\nu}+i\mathbf{t})}{\overline{\lambda}+1-(1-\overline{\nu}+i\mathbf{t})\overline{h}} - (\mathfrak{M}\mathrm{k})(1-\overline{\nu}+i\mathbf{t})(\mathfrak{M}f)(\overline{\nu}-i\mathbf{t}). \tag{15}$$

Let $\mathbf{x} = e^{\mathbf{y}}$, then we obtain

$$\int\limits_{-\infty}^{+\infty} e^{-i\mathbf{y}\mathbf{t}}\mathrm{F}(\mathbf{t})d\mathbf{t} = 0, \mathbf{y} \in \mathrm{R}^n.$$

According to property (a) of the multidimensional Mellin transform in Lemma 1, $\mathfrak{M} \in [\mathfrak{L}_{\sigma,\overline{2}}, \mathrm{L}_{\overline{2}}(\mathrm{R}^n)]$, $(\sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \in \mathrm{R}^n$, $\sigma_1 = \sigma_2 = ... = \sigma_n)$. We obtain, that multidimensional Mellin transforms

$$(\mathfrak{M}\mathrm{K}f)(1-\overline{\nu}+i\mathbf{t}), (\mathfrak{M}g_1)(\overline{\nu}-i\mathbf{t}) = \frac{\overline{h}}{\overline{\lambda}+1-(1-\overline{\nu}+i\mathbf{t})\overline{h}},$$

$$(\mathfrak{M}\mathrm{k})(1-\overline{\nu}+i\mathbf{t}), (\mathfrak{M}f)(\overline{\nu}-i\mathbf{t})$$

belong to the space $\mathrm{L}_{\overline{2}}(\mathrm{R}^n)$ ($\overline{2} = (2, ..., 2)$. From that follows $\mathrm{F}(\mathbf{t})$ in (15) belongs to the space $\mathrm{L}_{\overline{2}}(\mathrm{R}^n)$ too. Hence, the expression on the left side of (15) also belongs to the space $\mathrm{L}_{\overline{2}}(\mathrm{R}^n)$. Determining θ through (8), taking into account (15), we obtain (9).

Show that $\theta \in \mathrm{L}_\infty(\mathrm{R}^n)$. From (9) it follows that if $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, then $\theta(\mathbf{t})(\mathfrak{M}f)(\overline{\nu} - i\mathbf{t}) \in \mathrm{L}_{\overline{2}}(\mathrm{R}^n)$. According to property (a) of Lemma 1, the Mellin transform (3) maps space $\mathfrak{L}_{\overline{\nu},\overline{2}}$ ($\nu_1 = \nu_2 = \ldots = \nu_n$) onto space $\mathrm{L}_{\overline{2}}(\mathrm{R}^n)$ and, thus, $\theta(\mathbf{t})\vartheta(\mathbf{t}) \in \mathrm{L}_{\overline{2}}(\mathrm{R}^n)$ for any function $\vartheta(\mathbf{t}) \in \mathrm{L}_{\overline{2}}(\mathrm{R}^n)$. Therefore $\theta \in \mathrm{L}_\infty(\mathrm{R}^n)$. This completes the proof of (a) for the case $\nu_1 > 1 - (\mathrm{Re}(\lambda_1) + 1)/h_1, \ldots, \nu_n > 1 - (\mathrm{Re}(\lambda_n) + 1))/h_n$ ($\nu_1 = \ldots = \nu_n$).

The case, when $\nu_1 < 1 - (\mathrm{Re}(\lambda_1) + 1)/h_1, \ldots, \nu_n < 1 - (\mathrm{Re}(\lambda_n) + 1)/h_n$ ($\nu_1 = \ldots = \nu_n$), is proved similarly after replacing the function $g_a(\mathbf{t})$ in (10) by the function $h_a(\mathbf{t})$ defined for $a = (a_1, a_2, \ldots, a_n)$, $a_j > 0$ ($j = 1, 2, \ldots, n$), by the formula

$$h_a(\mathbf{t}) = \begin{cases} 0, & 0 < \mathbf{t} < a; \\ \mathbf{t}^{(\overline{\lambda}+1)/\overline{h}-1}, & \mathbf{t} > a. \end{cases} \tag{16}$$

Let us prove the case (b). We suppose that $\theta \in \mathrm{L}_\infty(\mathrm{R}^n)$ and $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$. From Lemma 1 the Mellin transform maps space $\mathfrak{L}_{1-\overline{\nu},\overline{2}}$ onto space $\mathrm{L}_2(\mathrm{R}^n)$ unitarily, so there is a unique function $g \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$ such that

$$(\mathfrak{M}g)(1 - \overline{\nu} + i\mathbf{t}) = \theta(\mathbf{t})(\mathfrak{M}f)(\overline{\nu} - i\mathbf{t}).$$

We define K by $\mathrm{K}f = g$. Then (9) is satisfied. K is also a linear operator, namely, if $f_1 \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, $f_2 \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ ($\nu_1 = \nu_2 = \ldots = \nu_n$) and $\mathbf{c}_1 \in \mathrm{R}^n$, $\mathbf{c}_2 \in \mathrm{R}^n$, then

$$\begin{gathered}(\mathfrak{M}\,\mathrm{K}(\mathbf{c}_1 f_1 + \mathbf{c}_2 f_2))(1 - \overline{\nu} + i\mathbf{t}) = \theta(\mathbf{t})(\mathfrak{M}\,(\mathbf{c}_1 f_1 + \mathbf{c}_2 f_2))(\overline{\nu} - i\mathbf{t}) \\ = \mathbf{c}_1\,\theta(\mathbf{t})(\mathfrak{M}\,f_1)(\overline{\nu} - i\mathbf{t}) + \mathbf{c}_2\theta(\mathbf{t})(\mathfrak{M}\,f_2)(\overline{\nu} - i\mathbf{t}) \\ = \mathbf{c}_1\,(\mathfrak{M}\,f_1)(1 - \overline{\nu} + i\mathbf{t}) + \mathbf{c}_2(\mathfrak{M}\,f_2)(1 - \overline{\nu} + i\mathbf{t}) = (\mathfrak{M}\,(\mathbf{c}_1\mathrm{K}\,f_1 + \mathbf{c}_2\mathrm{K}\,f_2))(1 - \overline{\nu} + i\mathbf{t}),\end{gathered}$$

hence follows $\mathrm{K}(\mathbf{c}_1 f_1 + \mathbf{c}_2 f_2) = \mathbf{c}_1\mathrm{K}\,f_1 + \mathbf{c}_2\mathrm{K}\,f_2$.

Further, from Lemma 1 it follows that taking as $\theta^*(\mathbf{t}) = \theta(-\mathbf{t})$, we obtain

$$||\mathrm{K}\,f||_{1-\overline{\nu},\overline{2}} = ||\mathfrak{M}\mathrm{K}\,f||_{\overline{2}} = |\theta^*\mathfrak{M}\,f||_{\overline{2}} \leq ||\theta^*||_\infty ||\mathfrak{M}\,f||_{\overline{2}} = ||\theta||_\infty ||\,f||_{\overline{\nu},\overline{2}},$$

where $||\theta||_\infty$ is the norm of θ in the space (4). This means that $\mathrm{K} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$.

Let $\nu_1 \neq 1 - (\mathrm{Re}(\lambda_1) + 1)/h_1, \ldots, \nu_n \neq 1 - (\mathrm{Re}(\lambda_n) + 1)/h_n$ ($\nu_1 = \ldots = \nu_n$) and let the function $\mathrm{k}(\mathbf{t})$ be defined by (8). Then, based on property (c) of the Lemma 1, we obtain that $\mathrm{k} \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$, since $\frac{1}{(\overline{u}+i\mathbf{t})} \in \mathrm{L}_\infty(\mathrm{R}^n)$ for a constant vector $\overline{u} = (u_1, \ldots, u_1)$ ($u_1 \neq 0, \ldots, u_n \neq 0$). If $\nu_1 < 1 - (\mathrm{Re}(\lambda_1) + 1)/h_1, \ldots, \nu_n < 1 - (\mathrm{Re}(\lambda_n) + 1)/h_n$ ($\nu_1 = \ldots = \nu_n$) and the function $h_a(\mathbf{t})$ is given by (16), then

$$(\mathfrak{M}h_{\mathbf{x}})(\mathbf{s}) = \frac{-\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}+\mathbf{s}-1}}{\overline{\lambda} + 1 - \overline{h}(1 - \mathbf{s})}. \tag{17}$$

From (16), (5), (17), (9), (8), and (6), if $\mathbf{x} > 0$, similarly to (13), we obtain

$$\begin{gathered}\int\limits_{\mathbf{x}}^{\infty} \mathbf{t}^{(\overline{\lambda}+1)/\overline{h}-1}(\mathrm{K}f)(\mathbf{t})\mathrm{d}\mathbf{t} = \int\limits_{\mathbf{0}}^{\infty} h_{\mathbf{x}}(\mathbf{t})(\mathrm{K}f)(\mathbf{t})\mathrm{d}\mathbf{t} = \frac{1}{(2\pi i)^n}\int\limits_{\overline{\nu}-i\infty}^{\overline{\nu}+i\infty} (\mathfrak{M}h_{\mathbf{x}})(\mathbf{s})(\mathfrak{M}\mathrm{K}f)(1 - \mathbf{s})\mathrm{d}\mathbf{s} \\ = \frac{1}{(2\pi)^n}\int\limits_{-\infty}^{+\infty} \frac{-\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}+\overline{\nu}+i\mathbf{t}-1}(\mathfrak{M}\mathrm{K}f)(1 - \overline{\nu} - i\mathbf{t})}{\overline{\lambda} + 1 - (1 - \overline{\nu} - i\mathbf{t})\overline{h}}\mathrm{d}\mathbf{t} \\ = \frac{-\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}+\overline{\nu}-1}}{(2\pi)^n}\int\limits_{-\infty}^{+\infty} \mathbf{x}^{i\mathbf{t}}\frac{\theta^*(\mathbf{t})(\mathfrak{M}f)(\overline{\nu} + i\mathbf{t})}{\overline{\lambda} + 1 - (1 - \overline{\nu} - i\mathbf{t})\overline{h}}\mathrm{d}\mathbf{t} \\ = \frac{-\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}+\overline{\nu}-1}}{(2\pi)^n}\int\limits_{-\infty}^{+\infty} \mathbf{x}^{i\mathbf{t}}(\mathfrak{M}\mathrm{k})(1 - \overline{\nu} - i\mathbf{t})(\mathfrak{M}f)(\overline{\nu} + i\mathbf{t})\mathrm{d}\mathbf{t}\end{gathered}$$

$$= \frac{-\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}+\overline{\nu}-1}}{(2\pi i)^n} \int\limits_{1-\overline{\nu}-i\infty}^{1-\overline{\nu}+i\infty} \mathbf{x}^{1-\overline{\nu}-\mathbf{s}}(\mathfrak{M}\mathrm{k})(\mathbf{s})(\mathfrak{M}f)(1-\mathbf{s})\mathrm{d}\mathbf{s}$$

$$= \frac{-\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}}}{(2\pi i)^n} \int\limits_{1-\overline{\nu}-i\infty}^{1-\overline{\nu}+i\infty} (\mathfrak{M}\mathbf{W}_{1/\mathbf{x}}\mathrm{k})(\mathbf{s})(\mathfrak{M}f)(1-\mathbf{s})\mathrm{d}\mathbf{s}$$

$$= \frac{-\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}}}{(2\pi i)^n} \int\limits_{1-\overline{\nu}-i\infty}^{1-\overline{\nu}+i\infty} (\mathfrak{M}\mathrm{k}(\mathbf{x}t))(\mathbf{s})(\mathfrak{M}f)(1-\mathbf{s})\mathrm{d}\mathbf{s} = -\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int\limits_0^{\infty} \mathrm{k}(\mathbf{x}t)f(\mathbf{t})\mathrm{d}\mathbf{t}.$$

Differentiating the left and right sides of the last equality, we obtain (1). Similarly, for the case $\nu_1 > 1 - (\mathrm{Re}(\lambda_1)+1)/h_1, ..., \nu_n > 1 - (\mathrm{Re}(\lambda_n)+1)/h_n$ $(\nu_1 = ... = \nu_n)$ formula (12) for the function $g_a(\mathbf{t})$ in (10) is used.

Prove (c). Let $\theta \neq 0$ almost everywhere. Then if $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ and $\mathrm{K}f = 0$, it follows from (9) that $\theta(\mathbf{t})(\mathfrak{M}f)(\overline{\nu} - i\mathbf{t}) = 0$ almost everywhere, hence $(\mathfrak{M}f)(\overline{\nu} - i\mathbf{t}) = 0$ almost everywhere. This implies that $f(\mathbf{t}) = 0$ almost everywhere,which means that transformation $\mathrm{K}f$ is one-to-one.We suppose that $1/\theta \in \mathrm{L}_\infty(\mathrm{R}^n)$. Based on statement (b) of the Theorem 1, there is a transformation $\mathrm{T} \in [\mathfrak{L}_{1-\overline{\nu},\overline{2}}, \mathfrak{L}_{\overline{\nu},\overline{2}}]$ such that if $g \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$, then

$$(\mathfrak{M}\,\mathrm{T}\,g)(\overline{\nu} + i\mathbf{t}) = \frac{1}{\theta(-\mathbf{t})}(\mathfrak{M}g)(1 - \overline{\nu} - i\mathbf{t})$$

almost everywhere. Based on (9), we have

$$(\mathfrak{M}\mathrm{K}\,\mathrm{T}\,g)(1 - \overline{\nu} + i\mathbf{t}) = \theta(\mathbf{t})(\mathfrak{M}\,\mathrm{T}\,g)(\overline{\nu} - i\mathbf{t}) = (\mathfrak{M}g)(1 - \overline{\nu} - i\mathbf{t}).$$

Thus, for any function $g \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$, the identity $\mathrm{K}\,\mathrm{T}\,g = g$ holds, and therefore K maps the space $\mathfrak{L}_{\overline{\nu},\overline{2}}$ onto space $\mathfrak{L}_{1-\overline{\nu},\overline{2}}$.

Further, if the functions $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ and $g \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$, then from (5) and (9) we finally obtain

$$\int\limits_0^{\infty} f(\mathbf{x})(\mathrm{K}g)(\mathbf{x})\mathrm{d}\mathbf{x} = \frac{1}{(2\pi i)^n} \int\limits_{\overline{\nu}-i\infty}^{\overline{\nu}+i\infty} (\mathfrak{M}f)(\mathbf{s})(\mathfrak{M}\mathrm{K}g)(1-\mathbf{s})\mathrm{d}\mathbf{s}$$

$$= \frac{1}{(2\pi)^n} \int\limits_{-\infty}^{+\infty} (\mathfrak{M}f)(\overline{\nu} + i\mathbf{t})(\mathfrak{M}\mathrm{K}g)(1 - \overline{\nu} - i\mathbf{t})\mathrm{d}\mathbf{t}$$

$$= \frac{1}{(2\pi)^n} \int\limits_{-\infty}^{+\infty} (\mathfrak{M}f)(\overline{\nu} + i\mathbf{t})\theta(-\mathbf{t})(\mathfrak{M}g)(\overline{\nu} + i\mathbf{t})\mathrm{d}\mathbf{t}$$

$$= \frac{1}{(2\pi)^n} \int\limits_{-\infty}^{+\infty} (\mathfrak{M}f)(\overline{\nu} - i\mathbf{t})\theta(\mathbf{t})(\mathfrak{M}g)(\overline{\nu} - i\mathbf{t})\mathrm{d}\mathbf{t}$$

$$= \frac{1}{(2\pi)^n} \int\limits_{-\infty}^{+\infty} (\mathfrak{M}\mathrm{K}f)(1 - \overline{\nu} + i\mathbf{t})(\mathfrak{M}g)(1 - (1 - \overline{\nu} + i\mathbf{t}))\mathrm{d}\mathbf{t}$$

$$= \frac{1}{(2\pi i)^n} \int\limits_{1-\overline{\nu}-i\infty}^{1-\overline{\nu}+i\infty} (\mathfrak{M}\mathrm{K}f)(\mathbf{s})(\mathfrak{M}g)(1-\mathbf{s})\mathrm{d}\mathbf{s} = \int\limits_0^{\infty} (\mathrm{K}f)(\mathbf{x})g(\mathbf{x})\mathrm{d}\mathbf{x}.$$

This proves Theorem 1.

REFERENCES

1. S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach Science, London, 1993).
2. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* (Elsevier, Amsterdam, 2006).
3. S. M. Sitnik and O. V. Skoromnik, "One-dimensional and multi-dimensional integral transforms of Buschman–Erdelyi type with Legendre Functions in kernels," in *Transmutation Operators and Applications. Trends in Mathematics* (2020), pp. 293–319.
4. V. V. Katrakhov and S. M. Sitnik, "A boundary-value problem for the steady-state Schrodinger equation with a singular potential," Sov. Math. Dokl. **30**, 468–470 (1984).
5. S. M. Sitnik, "Factorization and estimates of the norms of Buschman-Erdelyi operators in weighted Lebesgue spaces," Sov. Math. Dokl. **44**, 641–646 (1992).
6. V. V. Katrakhov and S. M. Sitnik, "Composition method for constructing B-elliptic, B-hyperbolic, and B-parabolic transformation operators," Russ. Acad. Sci., Dokl. Math. **50**, 70–77 (1995).
7. S. M. Sitnik, "A short survey of recent results on Buschman–Erdelyi transmutations," J. Inequal. Spec. Funct. **8**, 140–157 (2017).
8. O. V. Skoromnik, *Integral Transforms with Gauss and Legendre Functions as Kernels and Integral Equations of the First Kind* (Polotsk Gos. Univ., Novopolotsk, Belorussia, 2019) [in Russian].
9. V. V. Katrakhov and S. M. Sitnik, "The transmutation method and boundary-value problems for singular elliptic equations," Sovrem. Mat. Fundam. Napravl. **64**, 211–426 (2018).
10. E. L. Shishkina and S. M. Sitnik, *Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics* (Elsevier, Amsterdam, 2020).
11. S. M. Sitnik and E. L. Shishkina, *Transmutation Method for Differential Equations with Bessel Operators* (Fizmatlit, Moscow, 2019) [in Russian].
12. V. V. Kravchenko and S. M. Sitnik, *Transmutation Operators and Applications, Trends in Mathematics* (Birkhauser, Springer Nature, Switzerland AG, Basel, 2020).
13. A. Fitouhi, I. Jebabli, E. L. Shishkina, and S. M. Sitnik, "Applications of integral transforms composition method to wave-type singular differential equations and index shift transmutations. Electron," J. Differ. Equat. **2018** (130), 1–27 (2018).
14. Vu Kim Tuan, O. I. Marichev, and S. B. Yakubovich, "On a boundary-value problem for elliptic equations of the second order at the sphere domain," Dokl. Akad. Nauk SSSR **286**, 786–790 (1986).
15. O. I. Marichev, *Method Evulation of Integrals of Special Functions (Theory and Formulas Tables)* (Nauka Tekhnika, Minsk, 1978) [in Russian].
16. A. A. Kilbas and M. Saigo, *H-Transforms. Theory and Applications* (Chapman and Hall, Boca Raton, 2004).
17. Yu. A. Brychkov, H.-Y. Glaeske, A. P. Prudnikov, and Vu Kim Tuan, *Multidimensional Integral Transformations* (Gordon and Breach, Philadelphia, 1992).
18. S. M. Nikolski, *Approximation of Functions of Many Variables and Embedding Theorems* (Nauka, Moscow, 1975) [in Russian].
19. P. G. Rooney, "On integral transformations with G-function kernels," Proc. R. Soc. Edinburgh, Sect. A **93**, 265–297 (1982–1983).
20. P. G. Rooney, "On the range of the integral transformation," Canad. Math. Bul. **37**, 545–548 (1994).
21. P. G. Rooney, "On the representation of functions by the Hankel and some related transformations," Proc. R. Soc. Edinburgh, Sect. A **125**, 449–463 (1995).
22. S. M. Sitnik, "Refinements and generalizations of classical inequalities," Issled. Mat. Anal., Ser.: Mat. Forum **3**, 221–266 (2009).