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Mean-Value Theorem for B-Harmonic Functions

E. L. Shishkina[1,2*]

(Submitted by A. B. Muravnik)

[1] *Voronezh State University, Voronezh, 394018 Russia*

[2] *Belgorod State National Research University (BelGU), Belgorod, 308015 Russia*

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Abstract—We establish a mean value property for the functions which is satisfied to Laplace–Bessel equation. Also results involving generalized divergence theorem and the second Green's identities relating the bulk with the boundary of a region on which differential Bessel operators act we obtained.

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1. INTRODUCTION

As is well known, the spherical mean operator has many important properties with application to classical harmonic analysis and PDEs (see [1]). B-harmonic analysis provides a mathematical theory to deal with problems connection with the singular Bessel differential operator of the form [2]

$$B_{\gamma_j} = \frac{1}{x_j^{\gamma_j}} \frac{\partial}{\partial x_j} x_j^{\gamma_j} \frac{\partial}{\partial x_j}, \quad j = 1, ..., n. \tag{1}$$

We will use notation $\triangle_\gamma = (\triangle_\gamma)_x = \sum_{k=1}^n (B_{\gamma_k})_{x_k}$. For $\triangle_\gamma$ the term *Laplace–Bessel operator* is used. A function $u = u(x) = u(x_1, ..., x_n)$ defined in a domain $\Omega \in \mathbb{R}^n$ for $x_i \geq 0$, $i = 1, ..., n$ is said to be *B-harmonic* if $u \in C^2(\Omega)$ such that $\frac{\partial u}{\partial x_i}|_{x_i=0} = 0$, $i = 1, ..., n$ and satisfies the Laplace–Bessel equation of the form $\Delta_\gamma u = 0$ at every point of the domain Ω.

Laplace–Bessel equation $\Delta_\gamma u = 0$ is a singular elliptic equation containing the Bessel operator. These equations are mathematical models of axial and multiaxial symmetry of the most diverse processes and phenomena in the nature. Difficulties in the study of such equations are connected with singularities in the coefficients. Such equations were started to be analyzed systematically by Weinstein in [3, 4]. I.A. Kipriyanov, together with V.V. Katrakhov and V.I. Kononenko (see [5, 6]) studied boundary value problems for elliptic equations, with singularities of the type of essential singularities of analytic functions at isolated boundary points. Trace theory for boundary value problems for elliptic equations with power singularities was presented in [7]. Another problems with singular differential equations with a Bessel operator were considered in [8, 9].

The first who apply the Fourier–Bessel (Hankel) transform to equations with the Bessel operator B_γ was Ya. I. Zhitomirsky [10]. This served as an impetus for the development of B-harmonic analysis and its application to the solution of a wide variety of problems associated with the Bessel operator. In this article we continue to develop B-harmonic analysis and would like to present mean-value theorem for B-harmonic functions. In order to do it we will need the second Green's formula for the Laplace–Bessel operator.

*E-mail: ilina_dico@mail.ru

2. DEFINITIONS AND B-HARMONIC FUNCTIONS

The theory of B-harmonic functions is should include generalizations of classical tools for solving problems with the Laplace–Bessel operator. We need following definitions. First of all since divergent form (1) of Bessel operator contains power function $x_j^{\gamma_j}$ we should restrict our consideration to not negative (or positive) x_j for all $j=1,...,n$. Next, all integrals by n-dimensional regions in this theory should be taken by weight measure.

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space,

$$\mathbb{R}^n_+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n,\ x_1{>}0,\ldots,x_n{>}0\},$$
$$\overline{\mathbb{R}}^n_+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n,\ x_1{\geq}0,\ldots,x_n{\geq}0\},$$

$\gamma=(\gamma_1,\ldots,\gamma_n)$ is a multi-index consisting of positive fixed real numbers γ_i, $i=1,\ldots,n$, and $|\gamma|=\gamma_1+\ldots+\gamma_n$.

Let Ω be finite or infinite open set in $\mathbb{R}^n$ symmetric with respect to each hyperplane $x_i{=}0$, $i=1,...,n$, $\Omega^+=\Omega\cap\overline{\mathbb{R}}^n_+$. We deal with the class $C^m(\Omega^+)$ consisting of m times differentiable on Ω^+ functions such that all derivatives of these functions with respect to x_i for any $i=1,...,n$ are continuous up to $x_i{=}0$. Class $C^m_{ev}(\Omega^+)$ consists of all functions from $C^m(\Omega^+)$ such that $\frac{\partial^{2k+1}f}{\partial x_i^{2k+1}}|_{x_i=0}=0$ for all non-negative integer $k\leq\frac{m-1}{2}$ (see [10] and [2], p. 21). In the following, we will denote $C^m_{ev}(\overline{\mathbb{R}}^n_+)$ by C^m_{ev}.

Part of the sphere of radius r with center at the origin belonging to $\mathbb{R}^n_+$ we will denote $S^+_r(n)$:

$$S^+_r(n)=\{x\in\overline{\mathbb{R}}^n_+:|x|=r\}\cup\{x\in\overline{\mathbb{R}}^n_+:x_i=0,|x|\leq r,i=1,\ldots,n\}.$$

For the weighed integral by the $S^+_1(n)$ we have formula ([11], formula 107, p. 49)

$$|S^+_1(n)|_\gamma=\int\limits_{S^+_1(n)}x^\gamma dS=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)},\quad x^\gamma=\prod_{i=1}^{n}x_i^{\gamma_i}. \tag{2}$$

The multidimensional generalized translation is defined by the equality

$$({}^\gamma\mathbf{T}^y_xf)(x)={}^\gamma\mathbf{T}^y_xf(x)=({}^{\gamma_1}T^{y_1}_{x_1}...{}^{\gamma_n}T^{y_n}_{x_n}f)(x), \tag{3}$$

where each of one-dimensional generalized translation ${}^{\gamma_i}T^{y_i}_{x_i}$ acts for $i{=}1,...,n$ according to [12]

$$({}^{\gamma_i}T^{y_i}_{x_i}f)(x){=}\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma_i}{2}\right)}\int\limits_0^\pi f(x_1,...,x_{i-1},\sqrt{x_i^2+\tau_i^2-2x_iy_i\cos\varphi_i},x_{i+1},...,x_n)\sin^{\gamma_i-1}\varphi_i\,d\varphi_i.$$

Next we will use notation $C(\gamma)=\pi^{-\frac{n}{2}}\prod\limits_{i=1}^{n}\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}$.

We will use notation $\triangle_\gamma=(\triangle_\gamma)_x=\sum\limits_{k=1}^{n}(B_{\gamma_k})_{x_k}$, where $B_{\gamma_j}=\frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}=\frac{\partial^2}{\partial x_j^2}+\frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}$, $j=1,...,n$ is the Bessel operator For $\triangle_\gamma$ the term *Laplace–Bessel operator* is used. A function $u=u(x)=u(x_1,...,x_n)$ defined in a domain $\Omega^+\subset\overline{\mathbb{R}}^n_+$ is said to be *B-harmonic* if $u\in C^2_{ev}(\Omega^+)$ and satisfies the Laplace–Bessel equation $\Delta_\gamma u=0$ at every point of the domain Ω^+.

Let $x\in\mathbb{R}^n_+$, $n>1$ and

$$E(x)=\begin{cases}\frac{1}{|S^+_1(n)|_\gamma}\ln|x|, & n+|\gamma|=2;\\ \frac{|x|^{2-n-|\gamma|}}{(2-n-|\gamma|)|S^+_1(n)|_\gamma}, & n+|\gamma|>2,\end{cases}$$

where $|S^+_1(n)|_\gamma$ is (2), then for $|x|>\varepsilon$ $\forall\varepsilon>0$ we have $\triangle_\gamma E(x)=0$, therefore $E(x)$ is B-harmonic in any domain not containing a neighborhood of the origin.

3. GENERALIZED DIVERGENCE THEOREM AND THE SECOND GREEN'S FORMULA FOR THE LAPLACE–BESSEL OPERATOR

The aim of this section is to develop some elements of a field theory for the case when the Laplace-Bessel operator is used instead of the Laplace operator. Here we prove generalized divergence theorem and the second Green's identities relating the bulk with the boundary of a region on which differential Bessel operators act.

Let

$$\nabla'_\gamma=\left(\frac{1}{x_1^{\gamma_1}}\frac{\partial}{\partial x_1},...,\frac{1}{x_n^{\gamma_n}}\frac{\partial}{\partial x_n}\right)$$

is the first weighted operator nabla,

$$\nabla''_\gamma=\left(x_1^{\gamma_1}\frac{\partial}{\partial x_1},...,x_n^{\gamma_n}\frac{\partial}{\partial x_n}\right)$$

is the second weighted operator nabla, then $(\nabla'_\gamma\cdot\nabla''_\gamma)=\Delta_\gamma$, where $\Delta_\gamma=\sum\limits_{j=1}^{n}B_{\gamma_j}$ is Laplace–Bessel operator, $B_{\gamma_j}=\frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}=\frac{\partial^2}{\partial x_j^2}+\frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}$, $j=1,...,n$ is a Bessel operator.

If $\vec{F}=\vec{F}(x)=(F_1(x),...,F_n(x))$ is a vector field, then

$$\mathrm{div}'_\gamma\vec{F}=(\nabla'_\gamma\cdot\vec{F})=\frac{1}{x_1^{\gamma_1}}\frac{\partial F_1}{\partial x_1}+...+\frac{1}{x_i^{\gamma_n}}\frac{\partial F_n}{\partial x_n}$$

is the first weighted divergence,

$$\mathrm{div}''_\gamma\vec{F}=(\nabla''_\gamma\cdot\vec{F})=x_1^{\gamma_1}\frac{\partial F_1}{\partial x_1}+...+x_n^{\gamma_n}\frac{\partial F_n}{\partial x_n}$$

is the second weighted divergence.

In this case the generalized divergence theorem states that the weighted surface integral of a vector field over a closed surface is equal to the weighted volume integral of the first weighted divergence over the region inside the surface.

Theorem 1. *Let G^+ is a domain in $\overline{\mathbb{R}}^n_+$ such that each line perpendicular to the plane $x_i=0$, $i=1,...,n$, either does not intersect G^+ either has one common segment with G^+ (possibly degenerating into a point) of the form*

$$\alpha_i(x')\leq x_i\leq\beta_i(x'),\quad x'{=}(x_1,...,x_{i-1},x_{i+1},...,x_n),\quad i=1,...,n,$$

where α_i, β_i are smooth for $i=1,...,n$. If $\vec{g}{=}(g_1(x),...,g_n(x))$ is a vector field continuously differentiable in G^+ and $\vec{F}{=}(F_1(x),...,F_n(x))$, $F_1(x){=}x_1^{\gamma_1}g_1(x),...,F_n(x){=}x_n^{\gamma_n}g_n(x)$, then

$$\int\limits_{G^+}(\nabla'_\gamma\cdot\vec{F})\,x^\gamma dx=\int\limits_{S^+}(\vec{g}\cdot\vec{\nu})x^\gamma\,dS,\tag{4}$$

where $\vec{\nu}=\vec{e}_1\cos\eta_1+...+\vec{e}_n\cos\eta_n$ is an outer surface normal vector for S^+, η_i is an angle between vector $\vec{\nu}$ and an axe x_j, $\vec{e}_1,...,\vec{e}_n$ is an orthonormal basis in $\mathbb{R}^n$.

Proof. Let i is the fixed natural number between 1 and n inclusively. The part of surface S^+ defined by equation $x_i=\beta_i(x')$ we denote by S^+_u and part of surface S^+ defined by equation $x_i=\alpha_i(x')$ we denote by S^+_d, then

$$(\vec{\nu},e_i)=\begin{cases}-\dfrac{1}{\sqrt{1+\left(\frac{\partial\alpha_i}{\partial x_1}\right)^2+...+(\frac{\partial\alpha_i}{\partial x_{i-1}})^2+\left(\frac{\partial\alpha_i}{\partial x_{i+1}}\right)^2+...+\left(\frac{\partial\alpha_i}{\partial x_n}\right)^2}}, & x\in S^+_d,\\ \dfrac{1}{\sqrt{1+\left(\frac{\partial\beta_i}{\partial x_1}\right)^2+...+\left(\frac{\partial\beta_i}{\partial x_{i-1}}\right)^2+\left(\frac{\partial\beta_i}{\partial x_{i+1}}\right)^2+...+\left(\frac{\partial\beta_i}{\partial x_n}\right)^2}}, & x\in S^+_u.\end{cases}$$

We have

$$\int\limits_{G^+} (\nabla'_\gamma \cdot \vec{F})\, x^\gamma dx = \sum_{i=1}^n \int\limits_{G^+} \frac{1}{x_i^{\gamma_i}} \frac{\partial F_i}{\partial x_i}\, x^\gamma dx.$$

Let consider

$$\int\limits_{G^+} \frac{1}{x_i^{\gamma_i}} \frac{\partial F_i}{\partial x_i}\, x^\gamma dx = \int\limits_Q x_1^{\gamma_1} ... x_{i-1}^{\gamma_{i-1}} x_{i+1}^{\gamma_{i+1}} ... x_n^{\gamma_n} dx_1 ... dx_{i-1} dx_{i+1} ... dx_n \int\limits_{\alpha_i(x')}^{\beta_i(x')} \frac{\partial F_i}{\partial x_i} dx_i,$$

where Q is a projection of G^+ to $x_i = 0$. Integrating by x_i we obtain

$$\int\limits_{G^+} \frac{1}{x_i^{\gamma_i}} \frac{\partial F_i}{\partial x_i}\, x^\gamma dx = \int\limits_Q F_i(x)|_{x_i=\alpha_i(x')}^{x_i=\beta_i(x')} x_1^{\gamma_1} ... x_{i-1}^{\gamma_{i-1}} x_{i+1}^{\gamma_{i+1}} ... x_n^{\gamma_n} dx_1 ... dx_{i-1} dx_{i+1} ... dx_n.$$

Let $(x')^{\gamma'} = x_1^{\gamma_1} ... x_{i-1}^{\gamma_{i-1}} x_{i+1}^{\gamma_{i+1}} ... x_n^{\gamma_n}$, $dx' = dx_1 ... dx_{i-1} dx_{i+1} ... dx_n$, then

$$\int\limits_{G^+} \frac{1}{x_i^{\gamma_i}} \frac{\partial F_i}{\partial x_i}\, x^\gamma dx = \int\limits_Q F_i(x_1, ..., x_{i-1}, \beta_i(x'), x_{i+1}, ..., x_n)(x')^{\gamma'} dx'$$

$$- \int\limits_Q F_i(x_1, ..., x_{i-1}, \alpha_i(x'), x_{i+1}, ..., x_n)(x')^{\gamma'} dx' = \int\limits_Q F_i(x_1, ..., x_{i-1}, \beta_i(x'), x_{i+1}, ..., x_n)(\vec{\nu}, e_i)$$

$$\times \sqrt{1 + \left(\frac{\partial \beta_i}{\partial x_1}\right)^2 + ... + \left(\frac{\partial \beta_i}{\partial x_{i-1}}\right)^2 + \left(\frac{\partial \beta_i}{\partial x_{i+1}}\right)^2 + ... + \left(\frac{\partial \beta_i}{\partial x_n}\right)^2} (x')^{\gamma'} dx'$$

$$+ \int\limits_Q F_i(x_1, ..., x_{i-1}, \alpha_i(x'), x_{i+1}, ..., x_n)(\vec{\nu}, e_i)$$

$$\times \sqrt{1 + \left(\frac{\partial \alpha_i}{\partial x_1}\right)^2 + ... + \left(\frac{\partial \alpha_i}{\partial x_{i-1}}\right)^2 + \left(\frac{\partial \alpha_i}{\partial x_{i+1}}\right)^2 + ... + \left(\frac{\partial \alpha_i}{\partial x_n}\right)^2} (x')^{\gamma'} dx'$$

$$= \int\limits_{S_u^+} F_i(x)(\vec{\nu}, e_i)(x')^{\gamma'} dS_u + \int\limits_{S_d^+} F_i(x)(\vec{\nu}, e_i)(x')^{\gamma'} dS_d$$

$$= \int\limits_{S_u^+} g_i(x)(\vec{\nu}, e_i) x^\gamma dS_u + \int\limits_{S_d^+} g_i(x)(\vec{\nu}, e_i) x^\gamma dS_d = \int\limits_{S^+} g_i(x) \cos\eta_i\, x^\gamma dS.$$

Then

$$\int\limits_{G^+} (\nabla'_\gamma \cdot \vec{F})\, x^\gamma dx = \sum_{i=1}^n \int\limits_{S^+} g_i(x) \cos\eta_i\, x^\gamma dS = \int\limits_{S^+} (\vec{g} \cdot \vec{\nu}) x^\gamma\, dS,$$

which completes the proof.

Remark 1. Suppose that the domain $G^+ \in \overline{\mathbb{R}}_+^n$ is a union of domains $G_1^+, ..., G_m^+$ without common interior points. Let each domain G_j^+ in $\overline{\mathbb{R}}_+^n$ is such that each line perpendicular to the plane $x_i = 0$, $i = 1, ..., n$, either does not intersect G_j^+ either has only one common with G_j^+ segment (possibly degenerating into a point) of the form

$$\alpha_i^j(x') \leq x_i \leq \beta_i^j(x'), \quad x'{=}(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n), \quad i = 1, ..., n,$$

where α_i, β_i are smooth for i=1, ..., n and $\vec{F}$=$(F_1(x), ..., F_n(x))$, $F_1(x)$=$x_1^{\gamma_1} g_1(x)$, ..., $F_n(x)$=$x_n^{\gamma_n} g_n(x)$, $\vec{g} = (g_1(x), ..., g_n(x))$ is a vector field continuously differentiable in G^+, then the following formula holds

$$\int\limits_{G^+} (\nabla'_\gamma \cdot \vec{F})\, x^\gamma dx = \int\limits_{S^+} (\vec{g} \cdot \vec{\nu}) x^\gamma \, dS, \tag{5}$$

where $S^+ \in \overline{\mathbb{R}}^n_+$ piecewise smooth surface boundary of G^+, $\vec{\nu}$ is a normal vector of the surface S^+.

Theorem 2. *Let G^+ satisfies to the conditions of Remark 1. If $\varphi, \psi \in C^2_{ev}(G^+)$, then the second Green's formula for the Laplace–Bessel operator of the form*

$$\int\limits_{G^+} (\psi \Delta_\gamma \varphi - \varphi \Delta_\gamma \psi)\, x^\gamma dx = \int\limits_{S^+} \left(\psi \frac{\partial \varphi}{\partial \vec{\nu}} - \varphi \frac{\partial \psi}{\partial \vec{\nu}} \right) x^\gamma \, dS \tag{6}$$

is valid.

Proof. Let

$$\vec{F} = \psi \nabla''_\gamma \varphi - \varphi \nabla''_\gamma \psi = \left(\psi \cdot x_1^{\gamma_1} \frac{\partial \varphi}{\partial x_1} - \varphi \cdot x_1^{\gamma_1} \frac{\partial \psi}{\partial x_1}, ..., \psi \cdot x_n^{\gamma_n} \frac{\partial \varphi}{\partial x_n} - \varphi \cdot x_n^{\gamma_n} \frac{\partial \psi}{\partial x_n} \right)$$
$$= \left(x_1^{\gamma_1} \left(\psi \frac{\partial \varphi}{\partial x_1} - \varphi \frac{\partial \psi}{\partial x_1} \right), ..., x_n^{\gamma_n} \left(\psi \frac{\partial \varphi}{\partial x_n} - \varphi \frac{\partial \psi}{\partial x_n} \right) \right),$$

then $\vec{F}$ satisfies conditions of Remark 1. Setting

$$\vec{g} = \left(\psi \frac{\partial \varphi}{\partial x_1} - \varphi \frac{\partial \psi}{\partial x_1}, ..., \psi \frac{\partial \varphi}{\partial x_n} - \varphi \frac{\partial \psi}{\partial x_n} \right)$$

we obtain that $\vec{g}$ is continuously differentiable vector field defined in G^+ and

$$(\nabla'_\gamma \cdot \vec{F}) = (\nabla'_\gamma \cdot (\psi \nabla''_\gamma \varphi - \varphi \nabla''_\gamma \psi))$$
$$= \sum_{i=1}^n \left(\frac{1}{x_i^{\gamma_i}} \frac{\partial}{\partial x_i} \left(\psi \cdot x_i^{\gamma_i} \frac{\partial \varphi}{\partial x_i} \right) - \frac{1}{x_i^{\gamma_i}} \frac{\partial}{\partial x_i} \left(\varphi \cdot x_i^{\gamma_i} \frac{\partial \psi}{\partial x_i} \right) \right)$$
$$= \sum_{i=1}^n \left(\frac{1}{x_i^{\gamma_i}} \frac{\partial \psi}{\partial x_i} \cdot x_i^{\gamma_i} \frac{\partial \varphi}{\partial x_i} + \psi \cdot \frac{1}{x_i^{\gamma_i}} \frac{\partial}{\partial x_i} x_i^{\gamma_i} \frac{\partial \varphi}{\partial x_i} - \frac{1}{x_i^{\gamma_i}} \frac{\partial \varphi}{\partial x_i} \cdot x_i^{\gamma_i} \frac{\partial \psi}{\partial x_i} - \varphi \cdot \frac{1}{x_i^{\gamma_i}} \frac{\partial}{\partial x_i} x_i^{\gamma_i} \frac{\partial \psi}{\partial x_i} \right)$$
$$= \sum_{i=1}^n (\psi B_{\gamma_i} \varphi - \varphi B_{\gamma_i} \psi) = \psi \Delta_\gamma \varphi - \varphi \Delta_\gamma \psi,$$

$$(\vec{g} \cdot \vec{\nu}) = \left(\psi \frac{\partial \varphi}{\partial x_1} \cos \eta_1 + ... + \psi \frac{\partial \varphi}{\partial x_n} \cos \eta_n \right) - \left(\varphi \frac{\partial \psi}{\partial x_1} \cos \eta_1 + ... + \varphi \frac{\partial \psi}{\partial x_n} \cos \eta_n \right) = \psi \frac{\partial \varphi}{\partial \vec{\nu}} - \varphi \frac{\partial \psi}{\partial \vec{\nu}}.$$

Now we can easily get (6) by applying (5).

4. MEAN-VALUE THEOREM FOR B-HARMONIC FUNCTIONS

In this section we obtain mean-value theorem for B-harmonic functions. This theorem states that the value of a B-harmonic function at a point is equal to its weighted spherical mean over part of a sphere centered at that point. Weighted spherical mean in this case constructed with the help of multidimensional generalized translation.

Weighted spherical mean (see [13–17]) of function $u(x)$, $x \in \overline{\mathbb{R}}^n_+$ for $n \geq 2$ is

$$(M_t^\gamma u)(x) = (M_t^\gamma)_x[u(x)] = \frac{1}{|S_1^+(n)|_\gamma} \int\limits_{S_1^+(n)} {}^\gamma \mathbf{T}_x^{t\theta} u(x) \theta^\gamma dS, \tag{7}$$

where $\theta^\gamma=\prod\limits_{i=1}^n \theta_i^{\gamma_i}$, $S_1^+(n)=\{\theta{:}|\theta|=1,\theta\in\mathbb{R}^n_+\}$ is a part of a sphere in $\mathbb{R}^n_+$, $|S_1^+(n)|_\gamma$ is given by (2) and ${}^\gamma\mathbf{T}_x^{t\theta}$ is the multidimensional generalized translation (3). For $n=1$ let $M_t^\gamma[f(x)]={}^\gamma T_x^t f(x)$.

The weighted spherical mean $M_t^\gamma[f(x)]$ is the transmutation operator intertwining $(\Delta_\gamma)_x$ and $(B_{n+|\gamma|-1})_t$ for the $f\in C^2_{ev}$ (see [11]):

$$(B_{n+|\gamma|-1})_t M_t^\gamma[f(x)] = M_t^\gamma[(\Delta_\gamma)_x f(x)].$$

Theorem 3. *Let $n+|\gamma|>2$. If u is B-harmonic in a domain Ω and if the part of a sphere $S^+_{r_0,x}(n)$ is contained in Ω, then $u(x)=(M_r^\gamma u)(x)$ for $0<r\le r_0$.*

Proof. Since operator ${}^{\gamma_i}T_{x_i}^{y_i}$ of function $u\in C^2_{ev}$ is a transmutation operator with the following intertwining property

$${}^{\gamma_i}T_{x_i}^{y_i}(B_{\gamma_i})_{x_i}u(x)=(B_{\gamma_i})_{y_i}{}^{\gamma_i}T_{x_i}^{y_i}u(x),$$

then if u is B-harmonic in a domain Ω then ${}^\gamma\mathbf{T}_x^y u$ is harmonic in Ω_1. That is, B-harmonicity is preserved under generalized translations. Therefore, we can consider only the case when $x=0$. Let E is a subdomain of Ω satisfies to the conditions of Remark 1 such that ∂E consists of smooth pieces and $\partial E\subset\Omega$. Applying formula (6) we obtain

$$\int\limits_{\partial E}\frac{\partial u}{\partial\vec{\nu}}\,x^\gamma dS=\int\limits_E \Delta_\gamma u(x)x^\gamma dx=0, \tag{8}$$

where $\frac{\partial}{\partial\vec{\nu}}$ is differentiation in the direction of the outward directed normal to ∂E and dS is the element of surface area on ∂E.

Let $x\in\mathbb{R}_n^+$ and $v(x)=|x|^{2-n-|\gamma|}$, then for $|x|>\varepsilon\ \forall\varepsilon>0$ we have $\triangle_\gamma v(x)=0$, so v is B-harmonic in any domain not containing a neighborhood of the origin.

Suppose $S^+_{\varepsilon,0}(n)$ and $S^+_{r,0}(n)$ be the surfaces of the parts of spheres centered in origin of radii ε and r correspondingly and Ω^* is the shell domain between $S^+_{\varepsilon,0}(n)$ and $S^+_{r,0}(n)$. Applying formula (6) to the functions u and v we obtain

$$0=\int\limits_{\Omega^*}(u\Delta_\gamma v-v\Delta_\gamma u)\,x^\gamma dx=\int\limits_{\partial\Omega^*}\left(u\frac{\partial v}{\partial\vec{\nu}}-v\frac{\partial u}{\partial\vec{\nu}}\right)\,x^\gamma\,dS. \tag{9}$$

On the coordinate planes $x_i=0$, $i=1,...,n$ the the surface integrals in the right side of (9) are equal to zero. In the parts of a spheres $S^+_{\varepsilon,0}(n)$ and $S^+_{r,0}(n)$ the function $v(x)$ is constant so by (8) we get

$$\int\limits_{\partial\Omega^*}v\frac{\partial u}{\partial\vec{\nu}}\,x^\gamma\,dS=0.$$

Therefore we obtain from (9)

$$\int\limits_{\partial\Omega^*}u\frac{\partial v}{\partial\vec{\nu}}\,x^\gamma\,dS=(2-n-|\gamma|)\left(\int\limits_{S^+_{r,0}(n)}u(x)|x|^{1-n-|\gamma|}\,x^\gamma\,dS-\int\limits_{S^+_{\varepsilon,0}(n)}u(x)|x|^{1-n-|\gamma|}\,x^\gamma\,dS\right)=0.$$

Consequently

$$r^{1-n-|\gamma|}\int\limits_{S^+_{r,0}(n)}u(x)\,x^\gamma\,dS=\varepsilon^{1-n-|\gamma|}\int\limits_{S^+_{\varepsilon,0}(n)}u(x)\,x^\gamma\,dS$$

and

$$(M_r^\gamma u)(0)=\frac{1}{|S_1^+(n)|_\gamma}\int\limits_{S_1^+(n)}{}^\gamma u(r\theta)\theta^\gamma dS=\{r\theta=x\}=\frac{1}{|S_1^+(n)|_\gamma r^{n+|\gamma|-1}}\int\limits_{S^+_{r,0}(n)}u(x)\,x^\gamma\,dS$$

$$= \frac{1}{|S_1^+(n)|_\gamma \varepsilon^{n+|\gamma|-1}} \int\limits_{S^+_{\varepsilon,0}(n)} u(x)\, x^\gamma \, dS \to u(0), \quad \varepsilon \to 0.$$

This proves Theorem 3.

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