ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2022, Vol. 43, No. 8, pp. 2322–2331. © Pleiades Publishing, Ltd., 2022.

On Certain Elliptic Problems in Sectorial Domains

V. B. Vasilyev[1*] and N. V. Eberlein[2]**

(Submitted by A. M. Elizarov)

[1] *Chair of Applied Mathematics and Computer Modeling, Belgorod State National Research University, Belgorod, 308015 Russia*

[2] *Center of Applied Mathematics, Belgorod State National Research University, Belgorod, 308015 Russia*

Received October 15, 2021; revised March 13, 2022; accepted April 2, 2022

Abstract—A certain conjugation problem for an elliptic pseudo-differential equation in a plane sector is studied in Sobolev–Slobodetskii spaces. Using wave factorization for an elliptic symbol with concrete index we consider Dirichlet and Neumann conditions on sector sides. It permits to reduce the considered boundary value problem to a system of one-dimensional linear integral equations. For a special case it is possible further to reduce the mentioned system to a system of linear algebraic equations with respect to 8 unknown functions.

DOI: 10.1134/S1995080222110300

Keywords and phrases: *elliptic pseudo-differential equation, conjugation problem, sector, system of linear integral equations, system of linear algebraic equations*.

1. INTRODUCTION

The history of pseudo-differential equations is not so long since the term "pseudo-differential operator" has appeared in 60th last century. Now we have a lot of results which are related to such operators and equations [1–4, 7]. These studies are related as a rule to two main problems: boundedness of pseudo-differential operators in different functional spaces and a solvability of corresponding pseudo-differential equations. The second problem is more actual since such equations and related boundary value problems arise in many physical studies.

A theory of boundary value problems for elliptic pseudo-differential equation on manifolds with a smooth boundary was constructed in papers of M. I. Vishik and G. I. Eskin [7]. Unfortunately, it is not applicable for situations of manifolds with a non-smooth boundary. New approaches [5, 6, 12, 15] have appeared for studying equations in non-smooth situations from different points of view and all methods are concentrated around studying model operators near singular points. One of such approaches was developed by the first author, and it is based on a special factorization of an elliptic symbol [15]. This method was used in different situations related to boundary value problems for elliptic pseud-differential equations in canonical non-smooth domains [18–20].

In this paper we use this method for studying one conjugation problem. The problem is a generalization (in some sense) of classical Riemann boundary value problem for analytic functions [9, 10]. Such problems were considered in some papers [12, 13] but the authors have considered partial differential equations only.

*E-mail: vbv57@inbox.ru
**E-mail: 649377@bsu.edu.ru

1.1. Pseudo-Differential Equations and a Conjugation Problem

We consider the following problem in Sobolev–Slobodetskii spaces H^s [7]: find the function

$$U(x) = \begin{cases} u_+(x), & x \in C_+^a, \\ u_-(x), & x \in \mathbb{R}^2 \setminus \overline{C_+^a}, \end{cases}$$

such that $u_+ \in H^s(C_+^a), v_- \in H^s(\mathbb{R}^2 \setminus C_+^a)$ satisfying the equations

$$\begin{cases} (Au_+)(x) = 0, & x \in C_+^a, \\ (Au_-)(x) = 0, & x \in \mathbb{R}^2 \setminus \overline{C_+^a}, \end{cases} \tag{1}$$

where $C_+^a = \{x \in \mathbb{R}^2 : \ x_2 > a|x_1|, \ a > 0\}$, $\Gamma = \partial C_+^a$, A is elliptic pseudo differential operators with symbol $A(\xi)$ satisfying the condition

$$c_1 \le |A(\xi)(1 + |\xi|)^{-\alpha}| \le c_2. \tag{2}$$

The condition (2) means strong ellipticity of the operator A, bit it is not enough for a solvability of equations (1) [7, 15]. We need to describe solvability conditions and possible boundary conditions to guarantee a unique solvability of boundary value problem in a certain Sobolev–Slobodetskii space.

Such problem was first considered in [16] and it was reduced to a system of linear integral equations. The authors have considered special additional conditions and homogeneous symbols to reduce the latter system of linear integral equations to a system of linear algebraic equations [17]. Here we develop and refine the results [16] for homogeneous symbols applying the Mellin transform [11] to obtained system of linear integral equations.

We remind here some definitions related to functional spaces and operators under consideration.

The space $H^s(\mathbb{R}^2)$ is a Hilbert space with the norm

$$||f||_s = \left(\int\limits_{\mathbb{R}^2} |\tilde{f}(\xi)|^2 (1 + |\xi|)^{2s} d\xi \right)^{1/2},$$

where $\tilde{f}$ denotes the Fourier transform $\tilde{f}(\xi) = \int\limits_{\mathbb{R}^2} e^{-ix\,\xi} f(x) dx$. If $D \subset \mathbb{R}^2$ is a domain then $H^s(D)$ is a subspace of $H^s(\mathbb{R}^2)$ consisting of functions with supports in $\overline{D}$.

Let $A(\xi)$ be a measurable function defined in $\mathbb{R}^2$. A pseudo-differential operator A with the symbol $A(\xi)$ defined in a domain D is called the following operator

$$(Au)(x) = \int\limits_{\mathbb{R}^2} e^{ix\,\xi} A(\xi) \tilde{u}(\xi) d\xi, \quad x \in D.$$

1.2. Wave Factorization of an Elliptic Symbol and a General Solution

We need some new objects related to complex analysis.

The symbol $\overset{*}{C}{}_+^a$ denotes a conjugate cone for C_+^a: $\overset{*}{C}{}_+^a = \{x \in \mathbb{R}^2 : x = (x_1, x_2), ax_2 > |x_1|\}$, $C_-^a \equiv -C_+^a$, $T(C_+^a)$ denotes radial tube domain over the cone C_+^a, i.e. domain in a complex space $\mathbb{C}^2$ of the type $\mathbb{R}^2 + iC_+^a$ [8].

To describe the solvability picture for the equations (1) we will introduce the following

Definition 1. Wave factorization for the symbol $A(\xi)$ with respect to the cone C_+^a is called its representation in the form $A(\xi) = A_{\neq}(\xi) A_=(\xi)$, where the factors $A_{\neq}(\xi), A_=(\xi)$ must satisfy the following conditions:

1) $A_{\neq}(\xi), A_=(\xi)$ are defined for all admissible values $\xi \in \mathbb{R}^2$, without may be, the points $\{\xi \in \mathbb{R}^2 : \xi_1^2 = a^2 \xi_2^2\}$;

2) $A_{\neq}(\xi), A_{=}(\xi)$ admit an analytical continuation into radial tube domains $T(\overset{*}{C}{}^a_+)$, $T(\overset{*}{C}{}^a_-)$ respectively with estimates

$$|A^{\pm 1}_{\neq}(\xi + i\tau)| \leq c_1(1 + |\xi| + |\tau|)^{\pm æ},$$

$$|A^{\pm 1}_{=}(\xi - i\tau)| \leq c_2(1 + |\xi| + |\tau|)^{\pm(\alpha - æ)},\ \forall \tau \in \overset{*}{C}{}^a_+ .$$

The number $æ \in \mathbb{R}$ is called an index of wave factorization for the symbol $A(\xi)$.

From this point everywhere below we assume that the wave factorization exists and consider the case $æ - s = 1 + \delta, |\delta| < 1/2$ *and for simplicity we put* $a = 1$.

Then according to the theory [15] general solutions of equations (1) take the following form

$$\tilde{u}_+(\xi) = A^{-1}_{\neq}(\xi)\left(\tilde{c}_0(\xi_1 - \xi_2) + \tilde{d}_0(\xi_1 + \xi_2)\right), \tag{3}$$

$$\tilde{u}_-(\xi) = A^{-1}_{=}(\xi)\left(\tilde{r}_0(\xi_1 - \xi_2) + \tilde{q}_0(\xi_1 + \xi_2)\right), \tag{4}$$

where c_0, d_0, r_0, q_0 are arbitrary functions of one variable, $c_0, d_0 \in H^{s_0}(\mathbb{R}_+)$, $r_0, q_0 \in H^{s_0}(\mathbb{R}_-)$, $s_0 = s - æ + 1/2$.

2. BOUNDARY CONDITIONS AND INTEGRAL EQUATIONS

We have four unknown functions arbitrary functions c_0, d_0, r_0, q_0 from corresponding Sobolev–Slobodetskii spaces. We will choose additional conditions in a special way connecting boundary values of u_+, u_- by linear relations. We change variables

$$\begin{cases} t_1 = \xi_1 - \xi_2, \\ t_2 = \xi_1 + \xi_2 \end{cases}$$

and denote

$$\tilde{U}_+(t) \equiv \tilde{u}_+\left(\frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2}\right), \quad \tilde{U}_-(t) \equiv \tilde{u}_-\left(\frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2}\right),$$

$$a_{\neq}(t) = A_{\neq}\left(\frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2}\right), \quad a_{=}(t) = A_{=}\left(\frac{t_2 + t_1}{2}, \frac{t_2 - t_1}{2}\right),$$

$$\tilde{c}_0(\xi_1 - \xi_2) \equiv C(t_1), \quad \tilde{d}_0(\xi_1 + \xi_2) \equiv D(t_2), \quad \tilde{r}_0(\xi_1 - \xi_2) \equiv R(t_1), \quad \tilde{q}_0(\xi_1 + \xi_2) \equiv Q(t_2).$$

Further, we rewrite equations (3), (4) in the following form

$$\tilde{U}_+(t) = a^{-1}_{\neq}(t)\left(C(t_1) + D(t_2)\right), \quad \tilde{U}_-(t) = a^{-1}_{=}(t)\left(R(t_1) + Q(t_2)\right). \tag{5}$$

Now we integrate the latter equalities first on t_1, then on t_2 and obtain the following relations

$$\int\limits_{-\infty}^{+\infty} \tilde{U}_+(t_1, t_2)dt_1 = \int\limits_{-\infty}^{+\infty} a^{-1}_{\neq}(t_1, t_2)C(t_1)dt_1 + D(t_2)\int\limits_{-\infty}^{+\infty} a^{-1}_{\neq}(t_1, t_2)dt_1,$$

$$\int\limits_{-\infty}^{+\infty} \tilde{U}_+(t_1, t_2)dt_2 = C(t_1)\int\limits_{-\infty}^{+\infty} a^{-1}_{\neq}(t_1, t_2)dt_2 + \int\limits_{-\infty}^{+\infty} a^{-1}_{\neq}(t_1, t_2)D(t_2)dt_2,$$

$$\int\limits_{-\infty}^{+\infty} \tilde{U}_-(t_1, t_2)dt_1 = \int\limits_{-\infty}^{+\infty} a^{-1}_{=}(t_1, t_2)R(t_1)dt_1 + Q(t_2)\int\limits_{-\infty}^{+\infty} a^{-1}_{=}(t_1, t_2)dt_1,$$

$$\int_{-\infty}^{+\infty}\tilde{U}_-(t_1,t_2)dt_2 = R(t_1)\int_{-\infty}^{+\infty}a_=^{-1}(t_1,t_2)dt_2 + \int_{-\infty}^{+\infty}a_=^{-1}(t_1,t_2)Q(t_2)dt_2.$$

Let us denote

$$\int_{-\infty}^{+\infty}a_{\neq}^{-1}(t_1,t_2)dt_1 \equiv a_1(t_2), \quad \int_{-\infty}^{+\infty}a_{\neq}^{-1}(t_1,t_2)dt_2 \equiv a_2(t_1),$$

$$\int_{-\infty}^{+\infty}a_=^{-1}(t_1,t_2)dt_1 \equiv b_1(t_2), \quad \int_{-\infty}^{+\infty}a_=^{-1}(t_1,t_2)dt_2 \equiv b_2(t_1).$$

Now we choose the following boundary conditions for equations (1):

$$\theta\, u_{+|\partial C_+^a} + \omega\, u_{-|\partial C_+^a} = \mu, \quad \eta\left(\frac{\partial u_+}{\partial n}\right)_{|\partial C_+^a} + \gamma\left(\frac{\partial u_-}{\partial n}\right)_{|\partial C_+^a} = \nu, \tag{6}$$

where θ, ω, η, γ are certain complex numbers taking two different values on sides of ∂C_+^a. Simplest variants of such conditions appear in some applied transmission problems [14].

These given functions μ, ν are defined on ∂C_+^a only. If we use the above change of variables transforming the C_+^a onto the first quatrant, it means that we know the values $\mu_2(y_1)$, $\nu_2(y_1)$ on the straight line $[0,+\infty)\times\{0\}$, and the values $\mu_1(y_2)$, $\nu_1(y_2)$ on the straight line $\{0\}\times[0,+\infty)$. Hence, we know their Fourier transforms $\tilde{\mu}_2(t_1)$, $\tilde{\nu}_2(t_1)$, $\tilde{\mu}_1(t_2)$, $\tilde{\nu}_1(t_2)$. Thus, according to the Fourier transform properties on restriction on a hyper-plane [7]

$$u(x_1,0) = \int_{-\infty}^{+\infty}\tilde{u}(\xi_1,\xi_2)e^{ix_1\xi_1}d\xi_2, \quad u(0,x_2) = \int_{-\infty}^{+\infty}\tilde{u}(\xi_1,\xi_2)e^{ix_2\xi_2}d\xi_1,$$

we have the following relations

$$\theta_1\int_{-\infty}^{+\infty}\tilde{U}_+(t_1,t_2)dt_1 + \omega_1\int_{-\infty}^{+\infty}\tilde{U}_-(t_1,t_2)dt_1 = \tilde{\mu}_1(t_2),$$

$$\theta_2\int_{-\infty}^{+\infty}\tilde{U}_+(t_1,t_2)dt_2 + \omega_2\int_{-\infty}^{+\infty}\tilde{U}_-(t_1,t_2)dt_2 = \tilde{\mu}_2(t_1).$$

Further, $\frac{\partial u_+}{\partial n}$ in variables (x_1,x_2) corresponds to $\frac{\partial U_+}{\partial y_1}$, $\frac{\partial U_+}{\partial y_2}$ in variables (y_1,y_2) in dependence on the corner side

$$\begin{cases} y_1 = x_1 + x_2, \\ y_2 = x_1 - x_2, \end{cases}$$

hence the Fourier transform for $\frac{\partial U_+}{\partial y_1}$, $\frac{\partial U_+}{\partial y_2}$ is equal to $-i\xi_1\tilde{U}_+(\xi_1,\xi_2)$, $-i\xi_2\tilde{U}_+(\xi_1,\xi_2)$ respectively, and according to the Fourier transform properties we have the next two relations for the Fourier images:

$$-i\eta_1\int_{-\infty}^{+\infty}t_1\tilde{U}_+(t_1,t_2)dt_1 - i\gamma_1\int_{-\infty}^{+\infty}t_1\tilde{U}_-(t_1,t_2)dt_1 = \tilde{\nu}_1(t_2),$$

$$-i\eta_2\int_{-\infty}^{+\infty}t_2\tilde{U}_+(t_1,t_2)dt_2 - i\gamma_2\int_{-\infty}^{+\infty}t_2\tilde{U}_-(t_1,t_2)dt_2 = \tilde{\nu}_2(t_1).$$

Additional notations:

$$\int_{-\infty}^{+\infty} t_1 a_{\neq}^{-1}(t_1,t_2)dt_1 \equiv A_1(t_2), \quad \int_{-\infty}^{+\infty} t_2 a_{\neq}^{-1}(t_1,t_2)dt_2 \equiv A_2(t_1),$$

$$\int_{-\infty}^{+\infty} t_1 a_{=}^{-1}(t_1,t_2)dt_1 \equiv B_1(t_2), \quad \int_{-\infty}^{+\infty} t_2 a_{=}^{-1}(t_1,t_2)dt_2 \equiv B_2(t_1),$$

and we have the following relations

$$\int_{-\infty}^{+\infty} t_1 \tilde{U}_+(t_1,t_2)dt_1 = \int_{-\infty}^{+\infty} t_1 a_{\neq}^{-1}(t_1,t_2)C(t_1)dt_1 + D(t_2)A_1(t_2),$$

$$\int_{-\infty}^{+\infty} t_2 \tilde{U}_+(t_1,t_2)dt_2 = C(t_1)A_2(t_1) + \int_{-\infty}^{+\infty} t_2 a_{\neq}^{-1}(t_1,t_2)D(t_2)dt_2,$$

$$\int_{-\infty}^{+\infty} t_1 \tilde{U}_-(t_1,t_2)dt_1 = \int_{-\infty}^{+\infty} t_1 a_{=}^{-1}(t_1,t_2)R(t_1)dt_1 + Q(t_2)B_1(t_2),$$

$$\int_{-\infty}^{+\infty} t_2 \tilde{U}_-(t_1,t_2)dt_2 = R(t_1)B_2(t_1) + \int_{-\infty}^{+\infty} t_2 a_{=}^{-1}(t_1,t_2)Q(t_2)dt_2,$$

and we obtain the the following 4×4-system of linear integral equations of second kind with respect to unknown functions C, D, R, Q of one variable

$$\begin{cases} \theta_1 \int\limits_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1,t_2)C(t_1)dt_1 + \theta_1 D(t_2)a_1(t_2) \\ +\omega_1 \int\limits_{-\infty}^{+\infty} a_{=}^{-1}(t_1,t_2)R(t_1)dt_1 + \omega_1 Q(t_2)b_1(t_2) = \tilde{\mu}_1(t_2), \\ \theta_2 C(t_1)a_2(t_1) + \theta_2 \int\limits_{-\infty}^{+\infty} a_{\neq}^{-1}(t_1,t_2)D(t_2)dt_2 \\ +\omega_2 R(t_1)b_2(t_1) + \omega_2 \int\limits_{-\infty}^{+\infty} a_{=}^{-1}(t_1,t_2)Q(t_2)dt_2 = \tilde{\mu}_2(t_1), \\ \eta_1 \int\limits_{-\infty}^{+\infty} t_1 a_{\neq}^{-1}(t_1,t_2)C(t_1)dt_1 + \eta_1 D(t_2)A_1(t_2) \\ +\gamma_1 \int\limits_{-\infty}^{+\infty} t_1 a_{=}^{-1}(t_1,t_2)R(t_1)dt_1 + \gamma_1 Q(t_2)B_1(t_2) = i\tilde{\nu}_1(t_2), \\ \eta_2 C(t_1)A_2(t_1) + \eta_2 \int\limits_{-\infty}^{+\infty} t_2 a_{\neq}^{-1}(t_1,t_2)D(t_2)dt_2 \\ +\gamma_2 R(t_1)B_2(t_1) + \gamma_2 \int\limits_{-\infty}^{+\infty} t_2 a_{=}^{-1}(t_1,t_2)Q(t_2)dt_2 = i\tilde{\nu}_2(t_1). \end{cases} \tag{7}$$

Indeed, we have proved the following

Theorem 1. *Let μ,ν be given functions from spaces $H^{s-1/2}(\mathbb{R})$, $H^{s-3/2}(\mathbb{R})$ respectively, and μ_j, ν_j, $j=1,2$, be their restrictions on $H^{s-1/2}(\mathbb{R}_\pm)$, $H^{s-3/2}(\mathbb{R}_\pm)$ respectively. Then the conjugation problem (1), (7) has unique solution u_+, u_- if and only if the system of linear integral equations (8) has unique solution $C,D,R,Q \in \tilde{H}^{s-æ+1/2}(\mathbb{R})$.*

3. HOMOGENEOUS SYMBOLS AND THE MELLIN TRANSFORM

Lemma 1. *Let the factors $A_{\neq}(\xi)$, $A_{=}(\xi)$ ore homogeneous functions of order æ and $\alpha-$ æ respectively. Then the functions $a_k(t_{3-k})$, $b_k(t_{3-k})$, $A_k(t_{3-k})$, $B_k(t_{3-k})$, $k=1,2$, are homogeneous functions of orders $1-$ æ, $1-\alpha+$ æ, $2-$ æ, $2-\alpha+$ æ, respectively.*

A simple proof for the Lemma 1 is obtained by direct calculations and be found in [17].

Theorem 2. *Let æ $=\alpha/2$ and the factors $A_{\neq}(\xi), A_{=}(\xi)$ ore homogeneous functions of order $\alpha/2$ and differentiable out of an origin, $b_j(t_{3-j})\neq 0$, $B_j(t_{3-j})\neq 0$, $j=1,2$, $\forall t_1,t_2\neq 0$. Then the system (7) is equivalent to a certain system of linear algebraic equations.*

Proof. Indeed, we divide two sides of the first equation from (7) by $b_2(t_1)$, and the second one by $b_1(t_2)$. Further, we divide two sides of the third equation from (7) by $B_2(t_1)$, and the fourth one by $B_1(t_2)$. Thus, we a new system in which we have the following factors and kernels of integral equations

$$a_k(t_{3-k})b_k^{-1}(t_{3-k}),\quad A_k(t_{3-k})B_k^{-1}(t_{3-k}),$$

$$a_{\neq}^{-1}(t_1,t_2)b_k^{-1}(t_{3-k}),\quad t_k a_{\neq}^{-1}(t_1,t_2)B_k^{-1}(t_{3-k}),$$

$$a_{=}^{-1}(t_1,t_2)b_k^{-1}(t_{3-k}),\quad t_k a_{=}^{-1}(t_1,t_2)B_k^{-1}(t_{3-k}), k=1,2.$$

According to Lemma 1 the factors a_k, b_k and A_k, B_k have the same order of homogeneity, so that the functions $a_k(t_{3-k})b_k^{-1}(t_{3-k})$, $A_k(t_{3-k})B_k^{-1}(t_{3-k})$, are homogeneous of order 0. It means that these functions take only two values depending on sign of the variable. We will denote these values by l_{k1}, l_{k2} and L_{k1}, L_{k1} for positive and negative values of a variable.

Now let us consider the kernels of integral operators. It is essential that the kernels are homogeneous of order -1. We will verify one of them, for example $t_k a_{\neq}^{-1}(t_1,t_2)B_k^{-1}(t_{3-k})$. Let us denote $t_1 a_{\neq}^{-1}(t_1,t_2)B_1^{-1}(t_2)\equiv K(t_1,t_2)$. Then we have

$$K(\lambda t_1,\lambda t_2)=\lambda t_1 a_{\neq}^{-1}(\lambda t_1,\lambda t_2)B_1^{-1}(\lambda t_2)=\lambda^{1-æ}t_1 a_{\neq}^{-1}(t_1,t_2)\lambda^{æ-2}B_1^{-1}(t_2)=\lambda^{-1}K(t_1,t_2),$$

since $a_{\neq}(t_1,t_2)$ is homogeneous of order æ, and $B_1(t_2)$ is homogeneous of order $2-$ æ according to Lemma 1.

To rewrite the system (7) we introduce new notations in the following way. Let us denote

$$k(t_1,t_2)=a_{\neq}^{-1}(t_1,t_2)b_1^{-1}(t_2),\quad m(t_1,t_2)=a_{=}^{-1}(t_1,t_2)b_1^{-1}(t_2),$$

$$n(t_1,t_2)=a_{\neq}^{-1}(t_1,t_2)b_2^{-1}(t_1),\quad p(t_1,t_2)=a_{=}^{-1}(t_1,t_2)b_2^{-1}(t_1),$$

$$K(t_1,t_2)=t_1 a_{\neq}^{-1}(t_1,t_2)B_1^{-1}(t_2),\quad M(t_1,t_2)=t_1 a_{=}^{-1}(t_1,t_2)B_1^{-1}(t_2),$$

$$N(t_1,t_2)=t_2 a_{\neq}^{-1}(t_1,t_2)B_2^{-1}(t_1),\quad P(t_1,t_2)=t_2 a_{=}^{-1}(t_1,t_2)B_2^{-1}(t_1),$$

and then we construct the following kernels defined in the first quadrant. We put for all $t_1,t_2>0$

$$k_{11}(t_1,t_2)=k(t_1,t_2);\quad k_{12}(t_1,t_2)=k(t_1,-t_2),$$
$$k_{21}(t_1,t_2)=k(-t_1,t_2),\quad k_{22}(t_1,t_2)=k(-t_1,-t_2),$$

and analogously we introduce $m_{ij},n_{ij},p_{ij},K_{ij},M_{ij},N_{ij},P_{ij},i,j=1.2$.

Further, we introduce new unknown functions for $t_1,t_2>0$ as follows

$$C_1(t_1)=C(t_1),\quad C_2(t_1)=C(-t_1),\quad R_1(t_1)=R(t_1),\quad R_2(t_1)=R(-t_1),$$

and similarly

$$D_1(t_2)=D(t_2),\quad D_2(t_2)=D(-t_2),\quad Q_1(t_2)=Q(t_2),\quad Q_2(t_2)=Q(-t_2).$$

Thus, using these notations we can rewrite the system (7) in the following way

$$
\begin{cases}
\theta_1 \int\limits_0^{+\infty} k_{11}(t_1,t_2)C_1(t_1)dt_1 + \theta_1 \int\limits_0^{+\infty} k_{21}(t_1,t_2)C_2(t_1)dt_1 + \theta_1 l_{11} D_1(t_2) \\
+\omega_1 \int\limits_0^{+\infty} m_{11}(t_1,t_2)R_1(t_1)dt_1 + \omega_1 \int\limits_0^{+\infty} m_{21}(t_1,t_2)R_2(t_1)dt_1 + \omega_1 Q_1(t_2) = \tilde{\mu}_{11}(t_2), \\
\theta_1 \int\limits_0^{+\infty} k_{12}(t_1,t_2)C_1(t_1)dt_1 + \theta_1 \int\limits_0^{+\infty} k_{22}(t_1,t_2)C_2(t_1)dt_1 + \theta_1 l_{12} D_2(t_2) \\
+\omega_1 \int\limits_0^{+\infty} m_{12}(t_1,t_2)R_1(t_1)dt_1 + \omega_1 \int\limits_0^{+\infty} m_{22}(t_1,t_2)R_2(t_1)dt_1 + \omega_1 Q_2(t_2) = \tilde{\mu}_{12}(t_2), \\
\theta_2 C_1(t_1) l_{21} + \theta_2 \int\limits_0^{+\infty} n_{11}(t_1,t_2)D_1(t_2)dt_2 + \theta_2 \int\limits_0^{+\infty} n_{21}(t_1,t_2)D_2(t_2)dt_2 \\
+\omega_2 R_1(t_1) + \omega_2 \int\limits_0^{+\infty} p_{11}(t_1,t_2)Q_1(t_2)dt_2 + \omega_2 \int\limits_0^{+\infty} p_{21}(t_1,t_2)Q_2(t_2)dt_2 = \tilde{\mu}_{21}(t_1), \\
\theta_2 C_2(t_1) l_{22} + \theta_2 \int\limits_0^{+\infty} n_{12}(t_1,t_2)D_1(t_2)dt_2 + \theta_2 \int\limits_0^{+\infty} n_{22}(t_1,t_2)D_2(t_2)dt_2 \\
+\omega_2 R_2(t_1) + \omega_2 \int\limits_0^{+\infty} p_{12}(t_1,t_2)Q_1(t_2)dt_2 + \omega_2 \int\limits_0^{+\infty} p_{22}(t_1,t_2)Q_2(t_2)dt_2 = \tilde{\mu}_{22}(t_1), \\
\eta_1 \int\limits_0^{+\infty} K_{11}(t_1,t_2)C_1(t_1)dt_1 + \eta_1 \int\limits_0^{+\infty} K_{21}(t_1,t_2)C_2(t_1)dt_1 + \eta_1 D_1(t_2) L_{11} \\
+\gamma_1 \int\limits_0^{+\infty} M_{11}(t_1,t_2)R_1(t_1)dt_1 + \gamma_1 \int\limits_0^{+\infty} M_{21}(t_1,t_2)R_2(t_1)dt_1 + \gamma_1 Q_1(t_2) = \tilde{\nu}_{11}(t_2), \\
\eta_1 \int\limits_0^{+\infty} K_{12}(t_1,t_2)C_1(t_1)dt_1 + \eta_1 \int\limits_0^{+\infty} K_{22}(t_1,t_2)C_2(t_1)dt_1 + \eta_1 D_2(t_2) L_{12} \\
+\gamma_1 \int\limits_0^{+\infty} M_{12}(t_1,t_2)R_1(t_1)dt_1 + \gamma_1 \int\limits_0^{+\infty} M_{22}(t_1,t_2)R_2(t_1)dt_1 + \gamma_1 Q_2(t_2) = \tilde{\nu}_{12}(t_2), \\
\eta_2 C_1(t_1) L_{21} + \eta_2 \int\limits_0^{+\infty} N_{11}(t_1,t_2)D_1(t_2)dt_2 + \eta_2 \int\limits_0^{+\infty} N_{21}(t_1,t_2)D_2(t_2)dt_2 \\
+\gamma_2 R_1(t_1) + \gamma_2 \int\limits_0^{+\infty} P_{11}(t_1,t_2)Q_1(t_2)dt_2 + \gamma_2 \int\limits_0^{+\infty} P_{21}(t_1,t_2)Q_2(t_2)dt_2 = \tilde{\nu}_{21}(t_1), \\
\eta_2 C_2(t_1) L_{22} + \eta_2 \int\limits_0^{+\infty} N_{12}(t_1,t_2)D_1(t_2)dt_2 + \eta_2 \int\limits_0^{+\infty} N_{22}(t_1,t_2)D_2(t_2)dt_2 \\
+\gamma_2 R_2(t_1) + \gamma_2 \int\limits_0^{+\infty} P_{12}(t_1,t_2)Q_1(t_2)dt_2 + \gamma_2 \int\limits_0^{+\infty} P_{22}(t_1,t_2)Q_2(t_2)dt_2 = \tilde{\nu}_{22}(t_1),
\end{cases} \tag{8}
$$

where right hand sides are defined as follows for all $t_1 > 0$, $t_2 > 0$;

$$
\tilde{\mu}_{jk}(t_{3-j}) = \begin{cases} \tilde{\mu}_j(t_{3-j}) b_j^{-1}(t_{3-j}), & k = 1, \\ \tilde{\mu}_j(-t_{3-j}) b_j^{-1}(-t_{3-j}), & k = 2, \end{cases}
$$

$$
\tilde{\nu}_{jk}(t_{3-j}) = \begin{cases} i\tilde{\nu}_j(t_{3-j}) B_j^{-1}(t_{3-j}), & k = 1, \\ i\tilde{\nu}_j(-t_{3-j}) B_j^{-1}(-t_{3-j}), & k = 2, \end{cases} \qquad j = 1, 2.
$$

Since all kernels of integral operators are homogeneous of order -1 it is convenient to use the Mellin transform [11]. Since we suppose the factors $A_{\neq}$, $A_{=}$ are differentiable, then the Mellin transform is applicable. The functions under the integral can be assumed to be smooth enough, taking into account further approximation in H^s-spaces. Using well known properties of the Mellin transform we obtain the following (8×8)-system of linear algebraic equations with respect to unknown functions C_k, D_k, R_k,

$Q_k, k=1,2,$

$$\begin{cases}\theta_1\hat{k}_{11}(\lambda)\hat{C}_1(\lambda)+\theta_1\hat{k}_{21}(\lambda)\hat{C}_2(\lambda)+\theta_1 l_{11}\hat{D}_1(\lambda)\\ +\omega_1\hat{m}_{11}(\lambda)\hat{R}_1(\lambda)+\omega_1\hat{m}_{21}(\lambda)\hat{R}_2(\lambda)+\omega_1\hat{Q}_1(\lambda)=\hat{\mu}_{11}(\lambda),\\ \theta_1\hat{k}_{12}(\lambda)\hat{C}_1(\lambda)+\theta_1\hat{k}_{22}(\lambda)\hat{C}_2(\lambda)+\theta_1 l_{12}\hat{D}_2(\lambda)\\ +\omega_1\hat{m}_{12}(\lambda)\hat{R}_1(\lambda)+\omega_1\hat{m}_{22}(\lambda)\hat{R}_2(\lambda)+\omega_1\hat{Q}_2(\lambda)=\hat{\mu}_{12}(\lambda),\\ \theta_2 l_{21}\hat{C}_1(\lambda)+\theta_2\hat{n}_{11}(\lambda)\hat{D}_1(\lambda)+\theta_2\hat{n}_{21}(\lambda)\hat{D}_2(\lambda)\\ +\omega_2\hat{R}_1(\lambda)+\omega_2\hat{p}_{11}(\lambda)\hat{Q}_1(\lambda)+\omega_2\hat{p}_{21}(\lambda)\hat{Q}_2(\lambda)=\hat{\mu}_{21}(\lambda),\\ \theta_2 l_{22}\hat{C}_2(\lambda)+\theta_2\hat{n}_{12}(\lambda)\hat{D}_1(\lambda)+\theta_2\hat{n}_{22}(\lambda)\hat{D}_2(\lambda)\\ +\omega_2\hat{R}_2(\lambda)+\omega_2\hat{p}_{12}(\lambda)\hat{Q}_1(\lambda)+\omega_2\hat{p}_{22}(\lambda)\hat{Q}_2(\lambda)=\hat{\mu}_{22}(\lambda),\\ \eta_1\hat{K}_{11}(\lambda)\hat{C}_1(\lambda)+\eta_1\hat{K}_{21}(\lambda)\hat{C}_2(\lambda)+\eta_1 L_{11}\hat{D}_1(\lambda)\\ +\gamma_1\hat{M}_{11}(\lambda)\hat{R}_1(\lambda)+\gamma_1\hat{M}_{21}(\lambda)\hat{R}_2(\lambda)+\gamma_1\hat{Q}_1(\lambda)=\hat{\nu}_{11}(\lambda),\\ \eta_1\hat{K}_{12}(\lambda)\hat{C}_1(\lambda)+\eta_1\hat{K}_{22}(\lambda)\hat{C}_2(\lambda)+\eta_1 L_{12}\hat{D}_2(\lambda)\\ +\gamma_1\hat{M}_{12}(\lambda)\hat{R}_1(\lambda)+\gamma_1\hat{M}_{22}(\lambda)\hat{R}_2(\lambda)+\gamma_1\hat{Q}_2(\lambda)=\hat{\nu}_{12}(\lambda),\\ \eta_2 L_{21}\hat{C}_1(\lambda)+\eta_2\hat{N}_{11}(\lambda)\hat{D}_1(\lambda)+\eta_2\hat{N}_{21}(\lambda)\hat{D}_2(\lambda)\\ +\gamma_2\hat{R}_1(\lambda)+\gamma_2\hat{P}_{11}(\lambda)\hat{Q}_1(\lambda)+\gamma_2\hat{P}_{21}(\lambda)\hat{Q}_2(\lambda)=\hat{\nu}_{21}(\lambda),\\ \eta_2 L_{22}\hat{C}_2(\lambda)+\eta_2\hat{N}_{12}(\lambda)\hat{D}_1(\lambda)+\eta_2\hat{N}_{22}(\lambda)\hat{D}_2(\lambda)\\ +\gamma_2\hat{R}_2(\lambda)+\gamma_2\hat{P}_{12}(\lambda)\hat{Q}_1(\lambda)+\gamma_2\hat{P}_{22}(\lambda)\hat{Q}_2(\lambda)=\hat{\nu}_{22}(\lambda).\end{cases} \tag{9}$$

Let us introduce a (8×8)-matrix of the system (9)

$$\mathcal{A}(\lambda)=\begin{pmatrix}\theta_1\hat{k}_{11}(\lambda) & \theta_1\hat{k}_{21}(\lambda) & \theta_1 l_{11} & 0 & \omega_1\hat{m}_{11}(\lambda) & \omega_1\hat{m}_{21}(\lambda) & \omega_1 & 0\\ \theta_1\hat{k}_{12}(\lambda) & \theta_1\hat{k}_{22}(\lambda) & 0 & \theta_1 l_{12} & \omega_1\hat{m}_{12}(\lambda) & \omega_1\hat{m}_{22}(\lambda) & 0 & \omega_1\\ \theta_2 l_{21} & 0 & \theta_2\hat{n}_{11}(\lambda) & \theta_2\hat{n}_{21}(\lambda) & \omega_2 & 0 & \omega_2\hat{p}_{11}(\lambda) & \omega_2\hat{p}_{21}(\lambda)\\ 0 & \theta_2 l_{22} & \theta_2\hat{n}_{12}(\lambda) & \theta_2\hat{n}_{22}(\lambda) & 0 & \omega_2 & \omega_2\hat{p}_{12}(\lambda) & \omega_2\hat{p}_{22}(\lambda)\\ \eta_1\hat{K}_{11}(\lambda) & \eta_1\hat{K}_{21}(\lambda) & \eta_1 L_{11} & 0 & \gamma_1\hat{M}_{11}(\lambda) & \gamma_1\hat{M}_{21}(\lambda) & \gamma_1 & 0\\ \eta_1\hat{K}_{12}(\lambda) & \theta_1\hat{K}_{22}(\lambda) & 0 & \eta_1 L_{12} & \gamma_1\hat{M}_{12}(\lambda) & \gamma_1\hat{M}_{22}(\lambda) & 0 & \gamma_1\\ \eta_2 L_{21} & 0 & \eta_2\hat{N}_{11}(\lambda) & \eta_2\hat{N}_{21}(\lambda) & \gamma_2 & 0 & \gamma_2\hat{P}_{11}(\lambda) & \gamma_2\hat{P}_{21}(\lambda)\\ 0 & \eta_2 L_{22} & \eta_2\hat{N}_{12}(\lambda) & \eta_2\hat{N}_{22}(\lambda) & 0 & \gamma_2 & \gamma_2\hat{P}_{12}(\lambda) & \gamma_2\hat{P}_{22}(\lambda)\end{pmatrix}.$$

The system (9) with the matrix $\mathcal{A}(\lambda)$ is the required system of linear algebraic equations. □

Remark. Let us note that if we deal with integrals

$$\int_0^{+\infty} K(x,y)u(y)dy, \quad \int_0^{+\infty} K(x,y)u(x)dx,$$

then we use different kernel for the Mellin transform, namely $K(x,1)$, $K(1,y)$, respectively.

4. THE SOLVABILITY CONDITION

Theorem 3. *Under assumptions of Theorem 2 the condition*

$$inf|\det\mathcal{A}(\lambda)|>0, \quad \Re\lambda=1/2 \tag{10}$$

is necessary and sufficient for a unique solvability of the problem (1), (6).

Proof. Indeed, the Theorem 2 permits to reduce the system (8) to the system (9). The condition (10) is a necessary and sufficient condition for the unique solvability of such systems and the applicability of the inverse Mellin transform. □

A priori estimates for a solution of the problem (1), (6) can be obtained by the methods described in [15]. We will give these estimates in next papers.

Appendix

5. PROPERTIES OF THE MELLIN TRANSFORM

For convenience of a reader we will give here certain facts on the Mellin transform and will show how it can be applied to special integral equations. The Mellin transform is defined by formula

$$\hat{f}(s) = \int\limits_0^\infty f(x)x^{s-1}dx, \quad s = \sigma + i\tau,$$

at least for functions $f(x) \in C_0^\infty(\mathbb{R}_+)$. The integral converges for all complex s and it is an entire analytic function. If we change variable $x = e^t$, then the Mellin transform passes into the Fourier transform of function $f(e^t)$:

$$\hat{f}(s) = \int\limits_{-\infty}^\infty e^{(\sigma+i\tau)} f(e^t)dt, \quad s = \sigma + i\tau.$$

Thus, all properties of the Mellin transform can be obtained from corresponding properties of the Fourier transform. Particularly, the inversion formula of the Mellin transform for $f(x) \in C_0^\infty(\mathbb{R})$ has the following form

$$f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^\infty \hat{f}(s)t^{-s}d\tau, \quad s = \sigma + i\tau.$$

Parceval equality for Mellin transform

$$\int\limits_0^{+\infty} t^{2\sigma-1}|f(t)|^2dt = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \left|\hat{f}(s)\right|^2 d\tau, \quad s = \sigma + i\tau,$$

particularly, for $\sigma = 1/2$ we have

$$\int\limits_0^{+\infty} |f(t)|^2dt = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \left|\hat{f}(s)\right|^2 d\tau, \quad s = 1/2 + i\tau,$$

or, in other words,

$$\int\limits_0^{+\infty} |f(t)|^2dt = \frac{1}{2\pi i}\int\limits_{1/2-i\infty}^{1/2+i\infty} \left|\hat{f}(s)\right|^2 ds,$$

meaning the right integral as

$$\lim_{y\to\infty}\int\limits_{1/2-iy}^{1/2+iy} \left|\hat{f}(s)\right|^2 ds.$$

If we have the integral

$$\int\limits_0^{+\infty} K(t_1, t_2)u(t_1)dt_2,$$

in which the kernel $K(t_1, t_2)$ is a homogeneous function of order -1, then after applying the Mellin transform we obtain the following expression

$$\int\limits_0^{+\infty} t_1^{\lambda-1} \left(\int\limits_0^{+\infty} K(t_1, t_2) u(t_1) dt_2 \right) dt_1.$$

The change of variable in the inner integral $t_1 = xt_2$ leads to the following integral

$$\int\limits_0^{+\infty} t_2^{\lambda-1} x^{\lambda-1} \left(\int\limits_0^{+\infty} t_2 K(xt_2, t_2) u(t_2) dt_1 \right) dx,$$

and after rearrangements of integrals we obtain the following product

$$\int\limits_0^{+\infty} t_2^{\lambda-1} u(t_2) dt_2 \int\limits_0^{+\infty} x^{\lambda-1} K(x, 1) dx = \hat{u}(\lambda) \hat{K}(\lambda),$$

where $\hat{u}$ denotes the Mellin transform of u.

6. CONCLUSION

We consider here one of simple case of conjugation problems since we have assumed the restriction $n+1$. more complicated case for arbitrary $n \in \mathbb{N}$ can be considered in the same way, and we will try to demonstrate it in forthcoming papers.

REFERENCES

1. L. Hörmander, *Analysis of Partial Differential Operators* (Springer, Berlin, 1983), Vols. I–IV.
2. M. Taylor, *Pseudodifferential Operators* (Princeton Univ. Press, Princeton, 1981).
3. F. Treves, *Introduction to Pseudodifferential Operators and Fourier Integral Operators* (Springer, New York, 1980).
4. S. Rempel and B.-W. Schulze, *Index Theory of Elliptic Boundary Problems* (Akademie, Berlin, 1982).
5. B.-W. Schulze, *Boundary Value Problems and Singular Pseudo-Differential Operators* (Wiley, Chichester, 1998).
6. S. A. Nazarov and B. A. Plamenevsky, *Elliptic Problems in Domains with Piecewise Smooth Boundaries* (Walter de Gruyter, Berlin, 1994).
7. G. Eskin, *Boundar Value Problems for Elliptic Pseudodifferential Equations* (AMS, Providence, RI, 1981).
8. V. S. Vladimirov, *Methods of the Theory of Generalized Functions* (Taylor and Francis, London, 2002).
9. F. D. Gakhov, *Boundary Value Problems* (Dover, Mineola, NY, 1981).
10. N. I. Muskhelishvili, *Singular Integral Equations* (North-Holland, Amsterdam, 1976).
11. E. C. Titchmarsch, *Introduction to the Theory of Fourier Integrals* (Oxford Univ. Press, London, 1948).
12. M. Borsuk, *Transmission Problems for Elliptic Second-Order Equations in Non-Smooth Domains, Frontiers in Mathematics* (Birkhäuser, Basel, 2010).
13. G. Eskin, "Transmission problems for equations of principal type with two variables," Tr. Mosk. Mat. Ob-va **21**, 245–292 (1970).
14. M. Costabel and E. Stephan, "A direct boundary integral equations method for trasmission problems," J. Math. Anal. Appl. **106**, 367–413 (1985).
15. V. B. Vasil'ev, *Wave Factorization of Elliptic Symbols: Theory and Applications. Introduction to the Theory of Boundary Value Problems in Non-Smooth Domains* (Kluwer Academic, Dordrecht, 2000).
16. V. B. Vasilyev, "On some transmission problems in a plane corner," Tatra Mt. Math. Publ. **63**, 291–301 (2015).
17. V. Vasilyev and N. Eberlein, "On solvability conditions for a certain conjugation problem," Axioms **10**, 234 (2021).
18. V. B. Vasilyev, "Pseudo-differential equations, wave factorization, and related problems," Math. Meth. Appl. Sci. **41**, 9252–9263 (2018).
19. V. B. Vasil'ev, "Boundary value problems for elliptic pseudodifferential equations in a multidimensional cone," Differ. Equat. **56**, 1324–1334 (2020).
20. V. B. Vasilyev, "On certain 3D limit boundary value problem," Lobachevskii J. Math. **41**, 913–921 (2020).