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DESCRIPTION OF A CLASS OF EVOLUTIONARY EQUATIONS IN FERRODYNAMICS

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Abstract. In this paper, we state the problem of constructing evolution equations describing the dynamics of condensed matter with an internal structure. Within the framework of this statement, we describe the class of evolution equations for vector and pseudovector fields on $\mathbb{R}^3$ with an infinitesimal shift defined by a second-order, divergent-type differential operator, which is invariant under translations of $\mathbb{R}^3$ and time translations and is transformed covariantly under rotations of $\mathbb{R}^3$. The case of equations of this class with preserved solenoidality and unimodality of the field is studied separately. A general formula for all operators corresponding to these equations is established.

***Keywords and phrases*:** divergent differential operator, pseudovector field, flux density, unimodality, solenoidality, ferrodynamic equation.

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Dedicated to Academician of the National Academy od Sciences of Ukraine
C. S. Peletminsky

1. Introduction. In theoretical studies of physical problems by means of partial differential equations, the following key question arises: Which physical reasons can serve as the basis for constructing suitable equations? Usually, the reasoning used in the construction of differential equations that adequately describe physical reality is not an object of the theory of differential equations; this situation has developed historically due to the fact that the most famous equations of mathematical physics (for example, the heat equation, the system of hydrodynamical equations for Newtonian fluids, Maxwell's equations, etc.) are reliable from the physical point of view, and mathematical problems appeared earlier are not very important from the mathematical point of view.

The situation changes completely when physicists must solve problems of derivation of adequate differential equations in the study of physical situations that substantially differ from those already well studied. In such cases, well-developed reasonings, such as the Lagrangian or Hamiltonian formalisms (see, e.g., [1, 6, 7, 15]), do not lead to an unambiguous result. This situation arises, for example, in the dynamics of condensed media with a complex internal structure whose instantaneous physical state at each space point $\boldsymbol{x} \in \mathbb{R}^3$ is characterized by a collection of thermodynamical parameters X_a, $a = 1, \dots, N$. Assume that possible values of these parameters belong to a certain domain of an N-dimensional vector space $\mathcal{L}$. Then the state of the medium at each time moment is described by a set of time-depending fields $\boldsymbol{X}(\boldsymbol{x}, t) = \langle X_a(\boldsymbol{x}, t),\ a = 1, \dots, N\rangle$, on $\mathbb{R}^3$. The evolution of the state is described by a system of evolutionary equations of the form

$$\dot{X}_a(\boldsymbol{x}, t) = \Big(\mathsf{L}_a[\boldsymbol{X}]\Big)(\boldsymbol{x}, t), \quad a = 1, \dots, N, \tag{1.1}$$

where the dot means differentiation by t and L_a is a differential operator of order s (in general, nonlinear), $\mathsf{L}_a : \left[C_{s,\mathrm{loc}}(\mathbb{R}^3)\right]^N \to \left[C_{s,\mathrm{loc}}(\mathbb{R}^3)\right]^N$, which can be represented by a set of vector-valued component functions of the fields $\boldsymbol{X}(\boldsymbol{x}, t)$ and their vector derivatives $\left(\bigotimes^n \nabla\right) \otimes \boldsymbol{X}(\boldsymbol{x}, t)$, $n = 1, \dots, s$,

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which belong to $\big[C_{s,\mathrm{loc}}(\mathcal{L}\times(\mathbb{R}^3\times\mathcal{L})\times\cdots\times(\mathbb{R}^{3^s}\times\mathcal{L})\times\mathbb{R}^3)\big]^N$:

$$\Big(\mathsf{L}_a[\boldsymbol{X}]\Big)(\boldsymbol{x},t)=L_a\Big(\boldsymbol{X},\ \nabla\otimes\boldsymbol{X},\ \ldots,\ \underbrace{\nabla\otimes\cdots\otimes\nabla}_{s}\otimes\boldsymbol{X};\ \boldsymbol{x},\ t\Big),\quad a=1,\ldots,N.$$

The functions $X_a(\boldsymbol{x},t)$, $a=1,\ldots,N$, are assumed to be s times continuously differentiable with respect to the components of the vector $\boldsymbol{x}$. Here and below ∇ is the gradient operator in $\mathbb{R}^3$. We denote by $\mathcal{K}_s(\mathcal{L})$ the linear manifold of all such operators.

The problem to be solved by a physicist is the appropriate and adequate construction of the operator L_a. For solving this problem, the physicist must be guided only by the most general physical principles. Moreover, it is necessary to construct a differential operator L_a such that solutions of the system (1.1) satisfy certain conditions that determine in the space $[C_{s,\mathrm{loc}}(\mathbb{R}^3)]^N$ some differentiable manifold $\mathcal{M}\subset[C_{s,\mathrm{loc}}(\mathbb{R}^3)]^N$. In this case, a meaningful mathematical problem occurs. It consists of the description of the whole class of differential operators satisfying the general physical principles mentioned above and additional restrictions. A solution of this problem allows one to choose an appropriate differential operator from the class considered.

In this paper, we are not interested in the well-posedness of initial-boundary-value problems for Eqs. (1.1) defined by operators of the class $\mathcal{K}_s(\mathcal{L})$. We are only interested in the question of whether the differential operators L_a of this class satisfy certain general requirements imposed by the physical nature of the problem.

By the meaning of the physical situation described by Eqs. (1.1), we require that the operator L_a be invariant with respect to the transformation group $\mathbf{T}\otimes\mathbf{R}^3$, where $\mathbf{T}$ is the translation group with respect to the variable t and $\mathbf{R}^3$ is the translation group of the space $\mathbb{R}^3$; moreover, the operator L_a must be covariant with respect to the rotation group $\mathbf{O}_3$ of the space $\mathbb{R}^3$. Naturally, we assume that the linear space $\mathcal{L}$ is transformed by a representation of the group $\mathbf{O}_3$.

Since in what follows we will be interested not in the representations of the group $\mathbf{O}_3$ but in the spaces where they act, we will call the space $\mathcal{L}$ a linear representation, which will not lead to a confusion. In the general case, which is important from the point of view of physical applications, this representation can be decomposed into the product of irreducible spin-tensor representations (see [10]) and a set of scalars (pseudoscalars) invariant under rotations of $\mathbb{R}^3$.

The condition of the invariance under the groups $\mathbf{T}$ and $\mathbf{R}^3$ is satisfied if and only if the vector-valued functions L_a, $a=1,\ldots,N$, are explicitly independent of t and $\boldsymbol{x}$. The covariance condition imposes significant restrictions on the general form of these vector-valued functions and leads to a rather wide linear manifold $\mathcal{K}_s(\mathcal{L})$. A significant narrowing of this manifold occurs when additional conditions are imposed, i.e., when restricting the action of the operator to a suitable manifold $\mathcal{M}$. This is important from the point of view of the significance of the solution in physical applications. Additionally, significant simplification occurs when the physical medium is not anisotropic. Then the representation $\boldsymbol{X}$ does not contain constant tensors of second rank that describe anisotropy. In this case, the medium is said to be *spherically symmetric*.

In this paper, we consider the problem on the description of the manifold $\mathcal{K}_2(\mathbb{R}^3)$, $N=3$, in the case where L_a is a second-order differential operator of divergent type. We discuss the cases where $\boldsymbol{X}$ is a vector or pseudovector field (see, e.g., [12]). Moreover, we examine the case where the field $\boldsymbol{X}$ satisfies additional conditions: unimodality $\boldsymbol{X}^2(\boldsymbol{x},t)=\mathrm{const}$ and solenoidality $(\nabla,\boldsymbol{X})=0$. These conditions determines a manifold $\mathcal{M}_0$ on which the evolutionary equation (1.1) surely possesses such invariants. The solution of this problem is important in the dynamics of solid-state media with electric moment (a vector case) and ferromagnetically ordered media (a pseudovector case).

2. Description of the manifold of divergent operators L_a. According to the general definition of divergent differential operators (see, e.g., [5]), in the case considered where the differential operator L_a is defined by a vector-valued function L_a, $a=1,\ldots,N$, which is invariant under transforms from $\mathbf{T} \otimes \mathbf{R}^3$ and covariant under transforms from the group $\mathbf{O}_3$, we say that Eq. (1.1) is an *equation of divergent type* if the vector-valued function L_a has the form

$$L_a\Big(\boldsymbol{X},\ \nabla \otimes \boldsymbol{X},\ \dots,\ \underbrace{\nabla \otimes \cdots \otimes \nabla}_{s} \otimes \boldsymbol{X}\Big) = \nabla_k S_{a;k}, \quad a = 1, \dots, N, \tag{2.1}$$

where

$$S_{a;k} \equiv S_{a;k}\Big(\boldsymbol{X},\ \nabla \otimes \boldsymbol{X},\ \dots,\ \underbrace{\nabla \otimes \cdots \otimes \nabla}_{s} \otimes \boldsymbol{X}\Big) \in \Big[C_{1,\mathrm{loc}}\Big(\mathcal{L} \times \cdots \times \big(\mathbb{R}^{3s} \times \mathcal{L}\big)\Big)\Big]^{3N}$$

is a vector-valued function with values in $\mathcal{L} \times \mathbb{R}^3$ whose components are labelled by the indexes $a = 1, \dots, N$ and $k = 1, 2, 3$. Here and below, we use the Einstein summation convention: if an index appears twice in a single term, it implies summation of that term over all admissible values of the index.

The importance of the study of evolutionary equations of divergent type is related to the fact that, from the physical point of view, they determine the presence of so-called *local conservation laws* for time-varying fields $X_a(\boldsymbol{x}, t)$, $a = 1, \dots, N$. In this case, the vector-valued functions $\langle S_{a;k},\ a = 1, \dots, N\rangle$, $k = 1, 2, 3$, play the role of flux densities of the corresponding fields.

Thus, the description of all differential operators of divergent type from the manifold $\mathcal{K}_s(\mathcal{L})$ consists of the description of the linear manifold of all differential operators of the form (2.1) such that the vector-valued function $S_{a;k}$ possessed the covariance property. We will define this operator manifold by the same symbol $\mathcal{K}_s(\mathcal{L})$ if this does not lead to a confusion. The following obvious assertion holds.

Theorem 2.1 (see, e.g., [4]). *For $\mathcal{L}^3$-valued functions*

$$S_{a;k} \in \Big[C_{1,\mathrm{loc}}\Big(\mathcal{L} \times \big(\mathbb{R}^3 \times \mathcal{L}\big) \times \cdots \times \big(\mathbb{R}^{3(s-1)} \times \mathcal{L}\big)\Big)\Big]^{3N}$$

satisfying the condition

$$\nabla_k S_{a;k} = 0,$$

the following representation holds:

$$S_{a;k} = \varepsilon_{klm} \nabla_l Z_{a;m},$$

where

$$Z_{a;m}\Big(\boldsymbol{X},\ \nabla \otimes \boldsymbol{X},\ \dots,\ \underbrace{\nabla \otimes \cdots \otimes \nabla}_{s} \otimes \boldsymbol{X}\Big) \in \Big[C_{1,\mathrm{loc}}\big(\mathcal{L} \times \cdots \times (\mathbb{R}^{3(s-2)} \times \mathcal{L})\big)\Big]^{3N}$$

is a vector-valued function and ε_{klm} is the Levi-Civita pseudotensor in $\mathbb{R}^3$.

Therefore, to describe the manifold $\mathcal{K}_s(\mathcal{L})$, we must describe all vector-valued functions $S_{a;k}$ satisfying the covariance condition under rotations of the space, which represent actions of differential operator of $(s-1)$th order defined up to an arbitrary function $\varepsilon_{klm} \nabla_l Z_{a;m}$ indicated in the lemma.

Below, we are interesting in the manifold $\mathcal{K}_2(\mathcal{L})$ of Eqs. (1.1) with the second-order differential operator $\mathsf{L}_a : \big[C_{2,\mathrm{loc}}(\mathbb{R}^3)\big]^N \to \big[C_{2,\mathrm{loc}}(\mathbb{R}^3)\big]^N$. In this case, the functions $S_{a;k}$ are represented by the actions of quasilinear first-order differential operators on the field $\boldsymbol{X}$,

$$S_{a;k}\big(\boldsymbol{X}, \nabla \otimes \boldsymbol{X}\big) = T_{a,b;k,m}(\boldsymbol{X}) \nabla_m X_b + U_{a;\,k}(\boldsymbol{X}). \tag{2.2}$$

Here summation over $b = 1, \dots, N$ is assumed. Due to the covariance of the functions $S_{a;k}\big(\boldsymbol{X}, \nabla \otimes \boldsymbol{X}\big)$, the coefficients $T_{a,b;k,m}(\boldsymbol{X})$ and the functions $U_{a;k}(\boldsymbol{X})$ are, respectively, tensor- and vector-valued functions only of the values of the fields $X_a(\boldsymbol{x}, t)$. (In what follows, we do not distinguish between covariant and contravariant indexes since the space $\mathbb{R}^3$ is Euclidean, see [12]).)

The formula (2.2) implies that to find all operators from $\mathcal{K}_2(\mathcal{L})$, we must describe the linear manifold of all vector-valued functions of the form (2.2) up to the term $\varepsilon_{klm} \nabla_l Z_{a;m}(\boldsymbol{X})$, where $Z_{a;m}(\boldsymbol{X})$ is an arbitrary, twice continuously differentiable vector-valued function.

Since the functions $T_{a,b;k,m}(\boldsymbol{X})$ and $U_{a;k}(\boldsymbol{X})$ do not depend explicitly on $\boldsymbol{x} \in \mathbb{R}^3$, it suffices to describe linear manifolds of all tensor-valued functions $T_{a,b;k,m}(\boldsymbol{X})$ and all vector-valued functions $U_{a;k}(\boldsymbol{X})$ as function of the vector $\boldsymbol{X} \in \mathcal{L}$ in the spaces $\left[C_{1,\mathrm{loc}}(\mathcal{L})\right]^{(3N)^2}$ and $\left[C_{1,\mathrm{loc}}(\mathcal{L})\right]^{3N}$, respectively.

Note that the functions $T_{a,b;k,m}(\boldsymbol{X})$ and $U_{a;k}(\boldsymbol{X})$ form representations of the group $\mathbf{O}_3$ in the representation spaces $\left(\mathcal{L} \times \mathbb{R}^3\right)^2$ and $\mathcal{L} \times \mathbb{R}^3$, respectively. Let

$$\left\{T^{(\alpha)}_{a,b;k,m};\ \alpha \in \mathcal{T}\right\}, \quad \left\{U_{a;k}(\boldsymbol{X});\ \beta \in \mathcal{U}\right\}$$

be finite sets of functions that form bases of these representations. Functions from these bases are linearly independent monomials with respect to the tensor product in an algebra with a fixed set of generators. This set consists of irreducible representations, which form the representation $\boldsymbol{X}$, and the second-rank tensor $\boldsymbol{\delta}$ (the Kronecker delta) and the third-rank pseudotensor $\boldsymbol{\varepsilon}$ (the Levi-Civita symbol), which are universal for $\mathbb{R}^3$.

Using the basis decompositions of an arbitrary representation of the group, we can describe the manifold considered as follows.

Theorem 2.2. *The continuously differentiable tensor-valued function $T_{a,b;k,m}(\boldsymbol{X}) : \mathcal{L} \to (\mathcal{L} \times \mathbb{R}^3)^2$ and the continuously differentiable vector-valued function $U_{a;k}(\boldsymbol{X}) : \mathcal{L} \to \mathcal{L} \times \mathbb{R}^3$ are defined by the formulas*

$$T_{a,b;k,m}(\boldsymbol{X}) = \sum_{\alpha \in \mathcal{T}} f^{(\alpha)}(\boldsymbol{X}) T^{(\alpha)}_{a,b;k,m}(\boldsymbol{X}), \quad U_{a;k}(\boldsymbol{X}) = \sum_{\beta \in \mathcal{U}} g^{(\beta)}(\boldsymbol{X}) U^{(\beta)}_{a;k}(\boldsymbol{X}),$$

where the coefficients of the decompositions of $f^{(\alpha)}$, $\alpha \in \mathcal{T}$, and $g^{(\beta)}$, $\beta \in \mathcal{U}$, are continuously differentiable (scalar-valued) functions of the set of variables, which consists of invariants of the representation $\mathcal{L}$ with respect to the action of the group $\mathbf{O}_3$.

Based on the linear functional manifolds $T_{a,b;k,m}(\boldsymbol{X})$, $\alpha \in \mathcal{T}$, and $U_{a;k}(\boldsymbol{X})$, $\beta \in \mathcal{U}$, we obtain the following description of the manifold $\mathcal{K}_s(\mathcal{L})$.

Theorem 2.3. *The set of vector-valued functions $T^{(\alpha)}_{a,b;k,m}(\boldsymbol{X}) \nabla_m X_b$, $\alpha \in \mathcal{T}$, is linearly independent if $\boldsymbol{X} \in \left[C_{2,\mathrm{loc}}(\mathbb{R}^3)\right]^N$.*

Proof. Assume that the set of vector-valued functions $T^{(\alpha)}_{a,b;k,m}(\boldsymbol{X}) \nabla_m X_b$, $\alpha \in \mathcal{T}$, is linearly dependent, i.e., there exists coefficients $c^{(\alpha)}$, $\alpha \in \mathcal{T}$, such that

$$\sum_{\alpha \in \mathcal{T}} c^{(\alpha)} T^{(\alpha)}_{a,b;k,m}(\boldsymbol{X}) \nabla_m X_b = 0.$$

Due to the arbitrariness of the field $\boldsymbol{X}$, we linearize this equation near the function $\boldsymbol{X} = \text{const}$:

$$\sum_{\alpha \in \mathcal{T}} c^{(\alpha)} T^{(\alpha)}_{a,b;k,m}(\boldsymbol{X}) \nabla_m \delta X_b = 0.$$

We set in this equality $\delta X_b = A_b \exp(\boldsymbol{k}, \boldsymbol{x})$ with an arbitrary constant set A_b, $b = 1, \ldots, N$, and a vector $\boldsymbol{k} \in \mathbb{R}^3$; here and below, $(\cdot, \cdot)$ is the scalar product in $\mathbb{R}^3$. We obtain the equality

$$\sum_{\alpha \in \mathcal{T}} c^{(\alpha)} T^{(\alpha)}_{a,b;k,m}(\boldsymbol{X}) A_b k_m = 0.$$

Differentiating this equality by the components of these vectors and taking into account the arbitrariness of the vectors $\boldsymbol{k} \in \mathbb{R}^3$ and $\boldsymbol{A} = \langle A_b,\ b = 1, \ldots, N\rangle \in \mathcal{L}$, we arrive at the identity

$$\sum_{\alpha \in \mathcal{T}} c^{(\alpha)} T^{(\alpha)}_{a,b;k,m}(\boldsymbol{X}) = 0.$$

This means the linear dependence of the basis functions of the linear representation formed by the tensor-valued functions $T^{(\alpha)}_{a,b;k,m}(\boldsymbol{X})$. $\square$

Corollary 2.1. *All vector-valued functions $S_{a;k}(\boldsymbol{X},\nabla\otimes\boldsymbol{X})$ that define elements of the linear manifold $\mathcal{K}_s(\mathcal{L})$ can be represented by the formula*

$$S_{a;k}(\boldsymbol{X},\nabla\otimes\boldsymbol{X})=\sum_{\alpha\in\mathcal{T}}f^{(\alpha)}(\boldsymbol{X})T^{(\alpha)}_{a,b;k,m}(\boldsymbol{X})\nabla_m X_b+\sum_{\beta\in\mathcal{U}}g^{(\beta)}(\boldsymbol{X})U^{(\beta)}_{a;k}(\boldsymbol{X}), \tag{2.3}$$

up to a function $\varepsilon_{klm}\nabla_l Z_{a;m}$, $k=1,2,3$, where $Z_{a;m}(\boldsymbol{X})$, $a=1,\dots,N$, $m=1,2,3$, is an arbitrary, twice continuously differentiable vector-valued function.

Since for all sets of arbitrary coefficient functions $f^{(\alpha)}(\boldsymbol{X})$, $\alpha\in\mathcal{T}$, and $g^{(\beta)}(\boldsymbol{X})$, $\beta\in\mathcal{U}$, the expression (2.3) for the function $S_{a;k}(\boldsymbol{X},\nabla\otimes\boldsymbol{X})$, cannot be equal $\varepsilon_{klm}\nabla_l Z_{a;m}$, where $Z_{a;m}(\boldsymbol{X})$, $m=1,2,3$, $a=1,\dots,N$, is some vector-valued function, we conclude that the following assertion holds.

Corollary 2.2. *The formula* (2.1) *provides a complete description of the linear manifold $\mathcal{K}_s(\mathcal{L})$ of differential operators $\mathsf{L}_a[\boldsymbol{X}]$, where the vector-valued functions $S_{a;k}(\boldsymbol{X},\nabla\otimes\boldsymbol{X})$ are defined by the formula* (2.3).

Thus, the description of the manifold $\mathcal{K}_s(\mathcal{L})$ is reduced to the description of the set of basis functions $T^{(\alpha)}_{a,b;k,m}$, $\alpha\in\mathcal{T}$, and $U^{(\beta)}_{a;k}$, $\beta\in\mathcal{U}$, of the corresponding representations of the group $\mathbf{O}_3$, whose argument set consists of the set $\boldsymbol{X}$ of fields from the representation space $\mathcal{L}$ and which are covariant under transformations from of the group $\mathbf{O}_3$.

3. Description of the manifold $\mathcal{K}_2(\mathbb{R}^3)$ for vector fields. Based on the general formulation of the problem given in the previous section, we now state the specific problem on the description of the manifold $\mathcal{K}_2(\mathcal{L})$ of all evolutionary equations for vector fields. In this case, the set $\boldsymbol{X}$ consists of the components of the vector $\boldsymbol{P}=\langle P_j;\ j=1,2,3\rangle$, i.e., $\mathcal{L}=\mathbb{R}^3$.

Introducing the corresponding notation for the components of the flux density $S_{j;k}(\boldsymbol{P},\nabla\otimes\boldsymbol{P})$ and the coefficients $U^{(\beta)}_{j;k}(\boldsymbol{P})$ and $T_{j,l;k,m}(\boldsymbol{P})$ of the decomposition (2.2), we rewrite it in the form

$$S_{j;k}(\boldsymbol{P},\nabla\otimes\boldsymbol{P})=T_{j,l;k,m}(\boldsymbol{P})\nabla_m P_l+U_{j;k}(\boldsymbol{P}).$$

Applying elements of the group $\mathbf{O}_3$ to the left-hand side of this equality transforms it as a second-rank tensor. For this reason, the coefficients $T_{j,l;k,m}(\boldsymbol{P})$ and $U_{j;k}(\boldsymbol{P})$ are fourth- and second-rank tensors, respectively. Thus, according to Corollary 2.2, the description of all differential operators from $\mathcal{K}_2(\mathbb{R}^3)$ is reduced to the definition of the sets $\mathcal{T}$ and $\mathcal{U}$, respectively, of all basis functions for representations of the group $\mathbf{O}_3$ in the form of tensors of fourth and second rank. We search for required basis functions as monomials with respect to the tensor product. The set of generators of the tensor algebra in the spherically symmetric case consists of the fundamental tensors of the second rank $\boldsymbol{\delta}$, the third-rank pseudotensor $\boldsymbol{\varepsilon}$, and the vector $\boldsymbol{P}$.

Technically, the solution of the corresponding algebraic problem is not difficult. We formulate the final result as follows.

Theorem 3.1 (see [13])**.** *The set $\mathcal{U}$ consists of the two tensors $\boldsymbol{\delta}$ and $\boldsymbol{P}\otimes\boldsymbol{P}$. The set $\mathcal{T}$ consists of the ten elements of the following types*:

(1) *the three tensors of the form $\boldsymbol{\delta}\otimes\boldsymbol{\delta}$*;
(2) *the six tensors $\boldsymbol{P}\otimes\boldsymbol{P}\otimes\boldsymbol{\delta}$*;
(3) *the tensor $\boldsymbol{P}\otimes\boldsymbol{P}\otimes\boldsymbol{P}\otimes\boldsymbol{P}$.*

Proof. For the proof, one must enumerate all available possibilities of constructing monomials from generators such that monomials obtained lead to tensor representations of the required type.

The test of linear independence of all monomials obtained $T^{(\alpha)}_{j,l;k,m}(\boldsymbol{P})$ is reduced to the test of the linear independence of monomials separately for each of the groups indicated in the statement of the theorem since elements of different groups differ by the powers of the vector $\boldsymbol{P}$: it is not involved in the elements of the first group, it is involved in the elements of the second group quadratically, and its power in the unique element of the third group is equal to four. Therefore, the element $\boldsymbol{P}\otimes\boldsymbol{P}\otimes\boldsymbol{P}\otimes\boldsymbol{P}$ is linearly independent of the other elements. The test of the linear independence of the elements in the first and second groups consists of the analysis of the equations for the coefficients $c^{(\alpha)}$, $\alpha = 1,\dots,9$:

$$c^{(1)}T^{(1)}_{j,l;k,m}(\boldsymbol{P}) + c^{(2)}T^{(2)}_{j,l;k,m}(\boldsymbol{P}) + c^{(3)}T^{(3)}_{j,l;k,m}(\boldsymbol{P}) = 0$$

for the first group and

$$\sum_{\alpha=4}^{9} c^{(\alpha)}T^{(\alpha)}_{j,l;k,m}(\boldsymbol{P}) = 0$$

for the second group. In turn, the proof of the equality $c^{(\alpha)} = 0$, $\alpha = 1,\dots,9$, is reduced to the nondegeneracy test of two simple systems of homogeneous linear equations for the coefficients $c^{(\alpha)}$, $\alpha = 1,2,3$, and $c^{(\alpha)}$, $\alpha = 4,\dots,9$, respectively. □

Since in the spherically symmetric case, there exists a unique invariant $\boldsymbol{P}^2$ of the rotation group, the general formula (2.2) for the case of vector fields leads one to the description of all flux densities of fields that define all manifolds $\mathcal{K}_2(\mathbb{R}^3)$ of divergent-type differential operators acting in the space of twice continuously differentiable vector fields $\boldsymbol{P}$ on $\mathbb{R}^3$,

$$S_{j;k}(\boldsymbol{P},\nabla\otimes\boldsymbol{P}) = g^{(1)}(\boldsymbol{P}^2)\delta_{jk} + g^{(2)}(\boldsymbol{P}^2)P_jP_k + \sum_{\alpha=1}^{10} f^{(\alpha)}(\boldsymbol{P}^2)T^{(\alpha)}_{j,l;k,m}(\boldsymbol{P})\nabla_m P_l, \tag{3.1}$$

where we change the arbitrary coefficient functions

$$f^{(\alpha)}(\boldsymbol{P}) \Rightarrow f^{(\alpha)}(\boldsymbol{P}^2), \quad \alpha = 1,\dots,10, \qquad g^{(\beta)}(\boldsymbol{P}) \Rightarrow g^{(\beta)}(\boldsymbol{P}^2), \quad \beta = 1,2,$$

by continuously differentiable functions on $(0,\infty)$ depending on the unique invariant $\boldsymbol{P}^2$. The convolutions of $T^{(\alpha)}_{j,l;k,m}(\boldsymbol{P})\nabla_m P_l$ with arbitrary coefficients $f^{(\alpha)}(\boldsymbol{P}^2)$ in the formula (3.1) are represented in the notation of the vector algebra in $\mathbb{R}^3$ by the following list corresponding to the order used in the statement of Theorem 3.1:

$$\delta_{jk}(\nabla,\boldsymbol{P}), \quad \nabla_j P_k \pm \nabla_k P_j; \tag{3.2}$$

$$\delta_{jk}(\boldsymbol{P},\nabla)\boldsymbol{P}^2, \quad P_k\nabla_j\boldsymbol{P}^2 \pm P_j\nabla_k\boldsymbol{P}^2, \quad P_j(\boldsymbol{P},\nabla)P_k \pm P_k(\boldsymbol{P},\nabla)P_j, \quad P_jP_k(\nabla,\boldsymbol{P}); \tag{3.3}$$

$$P_jP_k(\boldsymbol{P},\nabla)\boldsymbol{P}^2. \tag{3.4}$$

Here we present the flux densities in the form of symmetric and skew-symmetric combinations, taking into account their irreducibility.

Thus, we obtain the following theorem.

Theorem 3.2 (see [13]). *All divergent-type differential operators $\mathsf{L}_j[\boldsymbol{P}]$ from $\mathcal{K}_2(\mathbb{R}^3)$*

$$\mathsf{L}_j[\boldsymbol{P}] = \nabla_k S_{j;k}\big(\boldsymbol{P},\nabla\otimes\boldsymbol{P}\big)$$

on the space of twice continuously differentiable vector fields $\boldsymbol{P}$ on $\mathbb{R}^3$ are defined by the formula (3.1), *where $g^{(1)}$, $g^{(2)}$, $f^{(\alpha)}$, $\alpha = 1,\dots,10$, are arbitrary continuously differentiable functions of $\boldsymbol{P}^2$ and the tensors $T^{(\alpha)}_{j,l;k,m}(\boldsymbol{P})$ are defined by the formulas* (3.2)–(3.4).

4. Description of the manifold $\mathcal{K}_2(\mathbb{R}^3)$ for pseudovector fields. Now we consider the manifold $\mathcal{K}_2(\mathbb{R}^3)$ of all second-order differential operators of divergent type for pseudovector fields. In this case, the set $\boldsymbol{X}$ consists of the components of the pseudovector $\boldsymbol{M} = \langle M_j;\ j = 1,2,3\rangle$ in the space $\mathcal{L} = \mathbb{R}^3$. Introducing the notation for the components of the flux densities $S_{j;k}(\boldsymbol{M},\nabla\otimes\boldsymbol{M})$ and for the functions $U^{(\beta)}_{j;k}(\boldsymbol{M})$ and $T_{j,l;k,m}(\boldsymbol{M})$ similarly to Sec. 3, we rewrite the formula (2.2) in the form

$$S_{j;k}\big(\boldsymbol{M},\nabla\otimes\boldsymbol{M}\big) = T_{j,l;k,m}(\boldsymbol{M})\nabla_m X_l + U_{j;k}(\boldsymbol{M}).$$

Since the action of elements of the group $\mathbf{O}_3$ on the left-hand side of this equality transforms it as a second-rank tensor in the case of continuous rotations of the space and changes the sign in the case of reflections of $\mathbb{R}^3$, we conclude that $S_{j;k}(\boldsymbol{M}, \nabla \otimes \boldsymbol{M})$ is a second-rank pseudotensor. For this reason, the coefficients $T_{j,l;k,m}(\boldsymbol{M})$ and $U_{j;\,k}(\boldsymbol{M})$ are a fourth-rank tensor and a second-rank pseudotensor, respectively.

Thus, the description of operators from $\mathcal{K}_2(\mathcal{L})$ consists of the definition of the sets $\mathcal{T}$ and $\mathcal{U}$ of all linearly independent monomials with respect to the tensor product that represent fourth-rank tensors and second-rank pseudotensors, respectively. As in Sec. 3, we examine the spherically symmetric case, where the generators of the tensor algebra are the tensor $\boldsymbol{\delta}$, the pseudotensor $\boldsymbol{\varepsilon}$, and the pseudovector $\boldsymbol{M}$. A simple algebraic analysis leads us to the following assertion.

Theorem 4.1 (see [14])**.** *The set $\mathcal{U}$ is empty. The set $\mathcal{T}$ consists of the* 26 *elements listed below*:

(1) *the three tensors of the form* $\boldsymbol{\delta} \otimes \boldsymbol{\delta}$;
(2) *the six tensors* $\boldsymbol{M} \otimes \boldsymbol{M} \otimes \boldsymbol{\delta}$;
(3) *the six tensors* $\boldsymbol{\delta} \otimes (\boldsymbol{M}\hat{}\,\boldsymbol{\varepsilon})$;
(4) *the four tensors* $\boldsymbol{M} \otimes \boldsymbol{\varepsilon}$;
(5) *the six tensors* $\boldsymbol{M} \otimes \boldsymbol{M} \otimes (\boldsymbol{M}\hat{}\,\boldsymbol{\varepsilon})$;
(6) *the tensor* $\boldsymbol{M} \otimes \boldsymbol{M} \otimes \boldsymbol{M} \otimes \boldsymbol{M}$.

Here $\hat{}$ *means convolution.*

Proof. The proof is based on the fact that in the constructions of monomials as four-rank tensors, the total number of factors that are equal to the pseudovector $\boldsymbol{M}$ or the pseudotensor $\boldsymbol{\varepsilon}$ must be even. Moreover, the product may contain no more than one Levi-Civita symbol due to the tensor identity

$$\varepsilon_{ijk}\varepsilon_{lmn} = \det \begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{pmatrix}$$

(see, e.g., [11]), which expresses the tensor product of the Levi-Civita symbols through the linear combination of monomials, which does not contain the Levi-Civita symbol.

The test of the linear independence of all monomials obtained $T^{(\alpha)}_{j,l;k,m}(\boldsymbol{M})$ is reduced to the test of their linear independence within each of the groups listed in the statement of the theorem. These groups contain all monomials that have the same powers of the pseudovector $\boldsymbol{M}$; namely, the elements of the first group do not contain this pseudovector; the elements of the third and fourth groups depend linearly on it; the elements of the second group depend on it quadratically, and its powers in the elements of the fifth and sixth groups are equal to 3 and 4, respectively. The element $\boldsymbol{M}\otimes\boldsymbol{M}\otimes\boldsymbol{M}\otimes\boldsymbol{M}$ is linearly independent of the other elements. The test of the linear independence of elements in all other groups consists of the analysis of the equations for the coefficients $c^{(\alpha)}$, $\alpha = 1, \ldots, 25$, which are listed in the order specified above:

$$c^{(1)}T^{(1)}_{j,l;k,m}(\boldsymbol{M}) + c^{(2)}T^{(2)}_{j,l;k,m}(\boldsymbol{M}) + c^{(3)}T^{(3)}_{j,l;k,m}(\boldsymbol{M}) = 0,$$
$$\sum_{\alpha=10}^{19} c^{(\alpha)}T^{(\alpha)}_{j,l;k,m}(\boldsymbol{M}) = 0, \quad \sum_{\alpha=4}^{9} c^{(\alpha)}T^{(\alpha)}_{j,l;k,m}(\boldsymbol{M}) = 0, \quad \sum_{\alpha=20}^{25} c^{(\alpha)}T^{(\alpha)}_{j,l;k,m}(\boldsymbol{M}) = 0.$$

The analysis of these equations is reduced to the nondegeneracy test of the systems of homogeneous linear equations for the coefficients $c^{(\alpha)}$, $\alpha = 1, \ldots, 25$, of these equations; the first two systems consist of 3 and 10 equations, respectively, whereas each of the other two systems contain 6 equations. □

Since in the spherically symmetric case considered now, there exists a unique invariant $\boldsymbol{M}^2$ of the rotation group, the general representation (2.2) for the case of a pseudovector field allows one to describe all flux densities for the manifold $\mathcal{K}_2(\mathbb{R}^3)$ of differential operators of divergent type in the space of twice continuously differentiable pseudovector fields $\boldsymbol{M}$ on $\mathbb{R}^3$,

$$S_{j;k}(\boldsymbol{M}, \nabla \otimes \boldsymbol{M}) = \sum_{\alpha=1}^{26} f^{(\alpha)}(\boldsymbol{M}^2) T^{(\alpha)}_{j,l;k,m}(\boldsymbol{M}) \nabla_m M_l, \tag{4.1}$$

where we change the arbitrary coefficient functions $f^{(\alpha)}(\boldsymbol{M}) \Rightarrow f^{(\alpha)}(\boldsymbol{M}^2)$, $\alpha = 1, \ldots, 26$, by continuously differentiable functions on $(0, \infty)$ depending on the unique invariant $\boldsymbol{M}^2$.

The convolutions $T^{(\alpha)}_{j,l;k,m}(\boldsymbol{M}) \nabla_m M_l$ in the formula (4.1) are first-order quasilinear differential operators; we represent them in the notation of the vector algebra in $\mathbb{R}^3$ by the following list corresponding to the order used in the statement of Theorem 4.1:

$$\delta_{jk}(\nabla, \boldsymbol{M}), \quad \nabla_j M_k \pm \nabla_k M_j; \tag{4.2}$$

$$\delta_{jk}(\boldsymbol{M}, \nabla)\boldsymbol{M}^2, \quad M_k \nabla_j \boldsymbol{M}^2 \pm M_j \nabla_k \boldsymbol{M}^2, \quad M_j(\boldsymbol{M}, \nabla) M_k \pm M_k(\boldsymbol{M}, \nabla) M_j, \quad M_j M_k(\nabla, \boldsymbol{M}); \tag{4.3}$$

$$\delta_{jk}(\boldsymbol{M}, [\nabla, \boldsymbol{M}]),\ [\boldsymbol{M}, \nabla]_j M_k \pm [\boldsymbol{M}, \nabla]_k M_j,\ \varepsilon_{jln} M_n \nabla_k M_l \pm \varepsilon_{kln} M_n \nabla_j M_l,\ \varepsilon_{jkm} M_m (\nabla, \boldsymbol{M}); \tag{4.4}$$

$$M_j [\nabla, \boldsymbol{M}]_k \pm M_k [\nabla, \boldsymbol{M}]_j, \quad \varepsilon_{jkl}(\boldsymbol{M}, \nabla) M_l, \quad \varepsilon_{jkl} \nabla_l \boldsymbol{M}^2; \tag{4.5}$$

$$M_j M_k (\boldsymbol{M}, [\nabla, \boldsymbol{M}]), \quad \varepsilon_{jkm} M_m (\boldsymbol{M}, \nabla) \boldsymbol{M}^2, \tag{4.6}$$

$$M_j [\boldsymbol{M}, (\boldsymbol{M}, \nabla)\boldsymbol{M}]_k \pm M_k [\boldsymbol{M}, (\boldsymbol{M}, \nabla)\boldsymbol{M}]_j, \quad M_j [\boldsymbol{M}, \nabla]_k \boldsymbol{M}^2 \pm M_k [\boldsymbol{M}, \nabla]_j \boldsymbol{M}^2; \tag{4.7}$$

$$M_j M_k (\boldsymbol{M}, \nabla) \mathbf{M}^2. \tag{4.8}$$

As in the case of vector fields, we present for monomials their symmetric and skew-symmetric combinations, taking into account their linear independence.

Using the sum (4.1) of all linear differential operators with arbitrary coefficients $f^{(\alpha)}(\boldsymbol{M}^2)$, we arrive at the following assertion.

Theorem 4.2. *All operators $\mathsf{L}_j[\boldsymbol{M}] \in \mathcal{K}_2(\mathcal{L})$ in the case of a pseudovector field $\boldsymbol{M}$ can be represented by the following formula*:

$$\mathsf{L}_j[\boldsymbol{M}] = \nabla_k S_{j;\,k}\big(\boldsymbol{M}, \nabla \otimes \boldsymbol{M}\big) = \sum_{\alpha=1}^{26} \nabla_k f^{(\alpha)}(\boldsymbol{M}^2) T^{(\alpha)}_{j,l;k,m}(\boldsymbol{M}) \nabla_m M_l,$$

where the first-order operators $T^{(\alpha)}_{j,l;k,m}(\boldsymbol{M}) \nabla_m M_l$, $\alpha = 1, \ldots, 26$, are listed in (4.2)–(4.8).

5. Class $\mathcal{K}_2(\mathbb{R}^3) \cap \mathcal{M}_0(\mathbb{R}^3)$ for vector fields. In Secs. 5 and 6, we present the main results of this paper. We describe manifolds that consist of all operators L_j from $\mathcal{K}_2(\mathbb{R}^3)$, which belong to the operator class $\mathcal{M}_0(\mathbb{R}^3)$ and preserve the unimodality and solenoidality properties during the evolution of the field $\boldsymbol{M}$. The requirement of preserving these properties of the field is just the special constraint in constructing suitable operators adequate to the physical conditions of the problem. In this paper, we show that the class $\mathcal{K}_2(\mathbb{R}^3) \cap \mathcal{M}_0(\mathbb{R}^3)$ is trivial, i.e., consists of the zero operator.

Let a field $\boldsymbol{P}(\boldsymbol{x}, t)$ possess the unimodality property, $\boldsymbol{P}^2(\boldsymbol{x}, t) = P^2 = \text{const}$. Then all functions $f^{(\alpha)}(\boldsymbol{P}^2)$, $\alpha = 1, \ldots, 10$, and $g^{(\beta)}(\boldsymbol{P}^2)$, $\beta = 1, 2$, are constant. Moreover, under this condition, the flux densities $S_{j;k}$ defined by the expressions (3.3) and (3.4), which have the derivatives $\nabla_j \boldsymbol{P}^2$, are equal to zero. Moreover, if the field $\boldsymbol{P}(\boldsymbol{x}, t)$ is solenoidal, $(\nabla, \boldsymbol{P}) = 0$, then the flux densities vanish since the expressions (3.2) and (3.3) contain the divergence of the field considered. As a result, if the field $\boldsymbol{P}$ is unimodal and solenoidal, then the general equation $\dot{P}_j = \mathsf{L}_j[\boldsymbol{P}]$, which governs its dynamics and contains the differential operator described in Theorem 3.2 and the formula (3.1), takes the form

$$\begin{aligned}\dot{P}_j = g^{(2)}(\boldsymbol{P}, \nabla) P_j + \big(f^{(2)} - f^{(3)}\big) \Delta P_j \\ + f^{(7)} \nabla_k (P_j (\boldsymbol{P}, \nabla) P_k + P_k (\boldsymbol{P}, \nabla) P_j) + f^{(8)} \nabla_k (P_j (\boldsymbol{P}, \nabla) P_k - P_k (\boldsymbol{P}, \nabla) P_j).\end{aligned} \tag{5.1}$$

Here we used the numeration of the constants $f^{(\alpha)}$ according to the order of the corresponding flux densities described by the formulas (3.2)–(3.4).

An operator $\mathsf{L}_j[\boldsymbol{P}]$ preserves the unimodality and solenoidality properties of a filed $\boldsymbol{P}(\boldsymbol{x}, t)$ if and only if for any twice continuously differentiable field on $\mathbb{R}^3$, the following identities hold:

$$\nabla_j \mathsf{L}_j[\boldsymbol{P}] = 0, \quad P_j \mathsf{L}_j[\boldsymbol{P}] = 0. \tag{5.2}$$

Theorem 5.1. $\mathcal{K}_2(\mathbb{R}^3) \cap \mathcal{M}_0(\mathbb{R}^3) = \{0\}$.

Proof. We show that if the field $\boldsymbol{P}$ possesses the properties $\boldsymbol{P}^2 = P^2$ and $(\nabla, \boldsymbol{P}) = 0$, then Eqs. (5.2) are valid only in the case where $f^{(3)} = f^{(2)}$, $f^{(7)} = f^{(8)} = 0$.

We scalarly multiply the right-hand side of Eq. (5.2) by $\boldsymbol{P}$. Since $(\boldsymbol{P}, (\boldsymbol{P}, \nabla)\boldsymbol{P}) = (\nabla, \boldsymbol{P})\boldsymbol{P}^2 = 0$, taking into account the relation $(\boldsymbol{P}, \mathsf{L}[(\boldsymbol{P})]) = 0$, we obtain the equality

$$\begin{aligned}&\big(f^{(2)} - f^{(3)}\big) P_j \Delta P_j \\ &\quad + f^{(7)} P_j \nabla_k (P_j (\boldsymbol{P}, \nabla) P_k + P_k (\boldsymbol{P}, \nabla) P_j) + f^{(8)} P_j \nabla_k (P_j (\boldsymbol{P}, \nabla) P_k - P_k (\boldsymbol{P}, \nabla) P_j) = 0.\end{aligned} \tag{5.3}$$

Since, by the assumption, $\boldsymbol{P}^2 = \text{const}$, we have $P_j \nabla_k P_j = 0$ and $\nabla_k P_k = 0$; therefore,

$$P_j \nabla_k P_j (\boldsymbol{P}, \nabla) P_k = P^2 \nabla_k P_l \nabla_l P_k + (P_j \nabla_k P_j) \cdot (\boldsymbol{P}, \nabla) P_k = P^2 \operatorname{Sp} \mathsf{A}^2, \tag{5.4}$$

where we have introduced the matrix $A_{kl} = (\mathsf{A})_{kl} \equiv \nabla_k P_l$. Similarly we obtain

$$P_j \nabla_k P_k (\boldsymbol{P}, \nabla) P_j = \nabla_k P_k P_j (\boldsymbol{P}, \nabla) P_j - \big((\boldsymbol{P}, \nabla) P_j\big) \cdot \big((\boldsymbol{P}, \nabla) P_j\big) = -(P_l A_{lj})^2. \tag{5.5}$$

Finally,

$$P_j \Delta P_j = P_j \nabla_k \nabla_k P_j = \nabla_k (P_j \nabla_k P_j) - (\nabla_k P_j) \cdot (\nabla_k P_j) = -\operatorname{Sp} \mathsf{A}\mathsf{A}^T. \tag{5.6}$$

Thus, substituting the expressions (5.4)–(5.6) into (5.3), we have

$$c^{(1)} \operatorname{Sp}(\mathsf{A}\mathsf{A}^T) + c^{(2)} \operatorname{Sp} \mathsf{A}^2 + c^{(3)} (P_l A_{lj})^2 = 0, \tag{5.7}$$

where

$$c^{(1)} = f^{(3)} - f^{(2)}, \quad c^{(2)} = \big(f^{(7)} + f^{(8)}\big) P^2, \quad c^{(3)} = f^{(8)} - f^{(7)}.$$

Substitute the decomposition $P_j = P_j^{(0)} + A_{jk}^{(0)} x_k + \dots$ into (5.7), where $\boldsymbol{P}^{(0)} = \boldsymbol{P}(0)$ is a constant vector' and $A_{jk}^{(0)}$ are the matrix elements at the spatial point $\boldsymbol{x} = 0$. In Eq. (5.7), consider the term of zeroth power with respect to $\boldsymbol{x}$, which occurs after the decomposition of the left-hand side. The matrix A is arbitrary, except for the elements $A_{jk}^{(0)}$, which must satisfy the following two conditions: $A_{jj}^{(0)} = 0$, due to the relation $(\nabla, \boldsymbol{P})_0 = 0$, and the vector $P_j^{(0)}$ is its left eigenvector with zero eigenvalue. Substituting into Eq. (5.7) consecutively the matrices A

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

with the vector $P_l^{(0)} = \langle 0, 0, 1 \rangle$, we obtain, according to the order of the substitution, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, i.e., $f^{(2)} = f^{(3)}$, $f^{(7)} = f^{(8)} = 0$.

After substituting these equalities into (5.1), one must verify whether the equation

$$\dot{P}_j = g^{(2)} (\boldsymbol{P}, \nabla) P_j, \quad j = 1, 2, 3,$$

may possess an invariant $(\nabla, \boldsymbol{P}) = 0$ with a continuously differentiable field $\boldsymbol{P}$ on $\mathbb{R}^3$ under the condition $\boldsymbol{P}^2 = \text{const}$. Assuming this and applying the divergence operation to both sides of the equation, we have

$$0 = \nabla_j (\boldsymbol{P}, \nabla) P_j = (\boldsymbol{P}, \nabla) \boldsymbol{P} + (\nabla_j P_l) \cdot (\nabla_l P_j) = \operatorname{Sp} \mathsf{A}^2;$$

in the general case, this is impossible for the matrix A with properties stated above. It suffices to take a symmetric matrix. $\square$

6. Description of the class $\mathcal{K}_2(\mathbb{R}^3) \cap \mathcal{M}_0(\mathbb{R}^3)$ for pseudovector fields. Let a pseudovector field $\boldsymbol{M}(\boldsymbol{x}, t)$ possess the unimodality property, $\boldsymbol{M}^2(\boldsymbol{x}, t) = M^2 = \text{const}$. All functions $f^{(\alpha)}(\boldsymbol{M}^2)$, $\alpha = 1, \dots, 26$, becomes constant and the flux densities $S_{j;k}$ defined by the expressions (4.3) and (4.5)–(4.8), which have the derivatives $\nabla_j \boldsymbol{M}^2$, are equal to zero. Such densities are densities with numbers 4–6, 19, 21, and 24–26 from the list presented above. If, in addition, the field $\boldsymbol{M}(\boldsymbol{x}, t)$ is solenoidal, i.e., $(\nabla, \boldsymbol{M}) = 0$, then the flux densities with numbers 1, 9, and 15 also vanish due to the divergence $(\nabla, \boldsymbol{M})$.

As a result, under the unimodality and solenoidality conditions, the general equation $\dot{M}_j = \mathsf{L}_j[\boldsymbol{M}]$, which governs the dynamics of the field $\boldsymbol{M}$ and contains the differential operator described in Theorem 4.2 and the formula (4.1), takes the form

$$\begin{aligned} \dot{M}_j = {} & \big(f^{(2)} - f^{(3)}\big)\Delta M_j + \big(f^{(7)} + f^{(8)}\big)\nabla_k\big(M_j(\boldsymbol{M}, \nabla)M_k\big) + \big(f^{(7)} - f^{(8)}\big)\nabla_k\big(M_k(\boldsymbol{M}, \nabla)M_j\big) \\ & + f^{(10)}\nabla_j\big(\boldsymbol{M}, [\nabla, \boldsymbol{M}]\big) + \big(f^{(11)} + f^{(12)}\big)\nabla_k\big([\boldsymbol{M}, \nabla]_j M_k\big) + \big(f^{(11)} - f^{(12)}\big)\nabla_k\big([\boldsymbol{M}, \nabla]_k M_j\big) \\ & + \varepsilon_{jln}\big(f^{(13)} + f^{(14)}\big)\nabla_k(M_n \nabla_k M_l) + \varepsilon_{kln}\big(f^{(13)} - f^{(14)}\big)\nabla_k(M_n \nabla_j M_l) \\ & + \big(f^{(16)} + f^{(17)}\big)\nabla_k\big(M_j[\nabla, \boldsymbol{M}]_k\big) + \big(f^{(16)} - f^{(17)}\big)\nabla_k\big(M_k[\nabla, \boldsymbol{M}]_j\big) \\ & + \varepsilon_{jkl} f^{(18)}\nabla_k(\boldsymbol{M}, \nabla)M_l + f^{(20)}\nabla_k M_j M_k\big(\boldsymbol{M}, [\nabla, \boldsymbol{M}]\big) \\ & + \big(f^{(22)} + f^{(23)}\big)\nabla_k\Big(M_j\big[\boldsymbol{M}, (\boldsymbol{M}, \nabla)\boldsymbol{M}\big]_k\Big) + \big(f^{(22)} - f^{(23)}\big)\nabla_k\Big(M_k\big[\boldsymbol{M}, (\boldsymbol{M}, \nabla)\boldsymbol{M}\big]_j\Big). \end{aligned} \tag{6.1}$$

As in the previous section, we numerate the constants $f^{(\alpha)}$ according the corresponding densities in the formulas (4.2)–(4.7).

An operator $\mathsf{L}_j[\boldsymbol{M}]$ preserves the unimodality and solenoidality properties of the field $\boldsymbol{M}(\boldsymbol{x}, t)$ if and only if for any twice continuously differentiable pseudovector field $\boldsymbol{M}$ on $\mathbb{R}^3$, the following identities hold:

$$\nabla_j \mathsf{L}_j[\boldsymbol{M}] = 0, \quad M_j \mathsf{L}_j[\boldsymbol{M}] = 0. \tag{6.2}$$

We examine these equalities for arbitrary fields $\boldsymbol{M}$ from the class $\mathcal{K}_2(\mathbb{R}^3) \cap \mathcal{M}_0(\mathbb{R}^3)$. The analysis of Eqs. (6.2) consists of the following seven steps.

I. The first equality (6.2) yields the following equation for the coefficients $f^{(\alpha)}$:

$$\begin{aligned} & f^{(7)}\nabla_j\nabla_k\Big(M_j(\boldsymbol{M}, \nabla)M_k + M_k(\boldsymbol{M}, \nabla)M_j\Big) + f^{(10)}\Delta\Big(\boldsymbol{M}, [\nabla, \boldsymbol{M}]\Big) \\ & \quad + f^{(11)}\nabla_j\nabla_k\Big([\boldsymbol{M}, \nabla]_j M_k + [\boldsymbol{M}, \nabla]_k M_j\Big) + f^{(13)}\nabla_j\nabla_k\Big(\varepsilon_{jln} M_n \nabla_k M_l + \varepsilon_{kln} M_n \nabla_j M_l\Big) \\ & \quad + f^{(16)}\nabla_j\nabla_k\Big(M_j[\nabla, \boldsymbol{M}]_k + M_k[\nabla, \boldsymbol{M}]_j\Big) + f^{(20)}\nabla_j\nabla_k M_j M_k\big(\boldsymbol{M}, [\nabla, \boldsymbol{M}]\big) \\ & \quad + f^{(22)}\nabla_j\nabla_k\Big(M_j\Big[\boldsymbol{M}, (\boldsymbol{M}, \nabla)\boldsymbol{M}\Big]_k + M_k\big[\boldsymbol{M}, (\boldsymbol{M}, \nabla)\boldsymbol{M}\big]_j\Big) = 0, \end{aligned} \tag{6.3}$$

whereas the second to the equation

$$\begin{aligned} & \big(f^{(2)} - f^{(3)}\big)M_j \Delta M_j + f^{(10)} M_j \nabla_j\big(\boldsymbol{M}, [\nabla, \boldsymbol{M}]\big) + \varepsilon_{kln}\big(f^{(13)} - f^{(14)}\big)M_j \nabla_k(M_n \nabla_j M_l) \\ & \quad + \big(f^{(7)} + f^{(8)}\big)M_j \nabla_k\big(M_j(\boldsymbol{M}, \nabla)M_k\big) + \big(f^{(7)} - f^{(8)}\big)M_j \nabla_k\big(M_k(\boldsymbol{M}, \nabla)M_j\big) \\ & \quad + \big(f^{(11)} + f^{(12)}\big)M_j \nabla_k\big([\boldsymbol{M}, \nabla]_j M_k\big) + \big(f^{(16)} - f^{(17)}\big)M_j \nabla_k\big(M_k[\nabla, \boldsymbol{M}]_j\big) \\ & \quad + \varepsilon_{jkl} f^{(18)} M_j \nabla_k(\boldsymbol{M}, \nabla)M_l + f^{(20)} M_j \nabla_k M_j M_k\big(\boldsymbol{M}, [\nabla, \boldsymbol{M}]\big) \\ & + \big(f^{(22)} + f^{(23)}\big)M_j \nabla_k\Big(M_j\big[\boldsymbol{M}, (\boldsymbol{M}, \nabla)\boldsymbol{M}\big]_k\Big) + \big(f^{(22)} - f^{(23)}\big)M_j \nabla_k\Big(M_k\big[\boldsymbol{M}, (\boldsymbol{M}, \nabla)\boldsymbol{M}\big]_j\Big); \end{aligned} \tag{6.4}$$

in (6.4), we take into account the identities

$$M_j \nabla_k [\boldsymbol{M}, \nabla]_k M_j = M_j \nabla_k M_j [\nabla, \boldsymbol{M}]_k = \varepsilon_{jln} M_j \nabla_k (M_n \nabla_k M_l) = 0.$$

We assume that the coefficient $f^{(\alpha)}$ of these equations are universal, i.e., independent of the length of the vector $\boldsymbol{M}$. Then these two equations are split into series of equations according to the power of the field $\boldsymbol{M}$ in the corresponding terms. Namely, from Eq. (6.3) we obtain the equations

$$f^{(10)} \Delta\big(\boldsymbol{M}, [\nabla, \boldsymbol{M}]\big) + 2 f^{(11)} \nabla_j \nabla_k [\boldsymbol{M}, \nabla]_j M_k + 2\varepsilon_{jln} f^{(13)} \nabla_j \nabla_k M_n \nabla_k M_l + 2 f^{(16)} \nabla_j \nabla_k M_j [\nabla, \boldsymbol{M}]_k = 0 \quad (6.5)$$

with the terms of power 2, to the equation

$$f^{(7)} \nabla_j \nabla_k M_j (\boldsymbol{M}, \nabla) M_k = 0 \quad (6.6)$$

with the term of power 3, and the equation

$$f^{(20)} \nabla_j \nabla_k M_j M_k \big(\boldsymbol{M}, [\nabla, \boldsymbol{M}]\big) + 2 f^{(22)} \nabla_j \nabla_k M_j \Big[\boldsymbol{M}, (\boldsymbol{M}, \nabla)\boldsymbol{M}\Big]_k = 0 \quad (6.7)$$

with the term of power 4. Similarly, from Eq. (6.4) we obtain the equations

$$\big(f^{(2)} - f^{(3)}\big) M_j \Delta M_j = 0; \quad (6.8)$$

$$f^{(10)} M_j \nabla_j \big(\boldsymbol{M}, [\nabla, \boldsymbol{M}]\big) + \varepsilon_{jkl} f^{(18)} M_j \nabla_k \big((\boldsymbol{M}, \nabla) M_l\big) + \varepsilon_{kln} \big(f^{(13)} - f^{(14)}\big) M_j \nabla_k (M_n \nabla_j M_l) + \big(f^{(11)} + f^{(12)}\big) M_j \nabla_k \big([\boldsymbol{M}, \nabla]_j M_k\big) + \big(f^{(16)} - f^{(17)}\big) M_j \nabla_k \big(M_k [\nabla, \boldsymbol{M}]_j\big) = 0; \quad (6.9)$$

$$\big(f^{(7)} + f^{(8)}\big) M_j \nabla_k \big(M_j (\boldsymbol{M}, \nabla) M_k\big) + \big(f^{(7)} - f^{(8)}\big) M_j \nabla_k \big(M_k (\boldsymbol{M}, \nabla) M_j\big) = 0; \quad (6.10)$$

$$f^{(20)} M_j \nabla_k M_j M_k \big(\boldsymbol{M}, [\nabla, \boldsymbol{M}]\big) + \big(f^{(22)} + f^{(23)}\big) M_j \nabla_k \Big(M_j \big[\boldsymbol{M}, (\boldsymbol{M}, \nabla)\boldsymbol{M}\big]_k\Big) + \big(f^{(22)} - f^{(23)}\big) M_j \nabla_k \Big(M_k \big[\boldsymbol{M}, (\boldsymbol{M}, \nabla)\boldsymbol{M}\big]_j\Big) = 0. \quad (6.11)$$

II. Equality (6.8) for unimodal fields $\boldsymbol{M}$ holds only under the condition $(\nabla_k M_l)(\nabla_k M_l) = 0$; this implies that $\boldsymbol{M} = \text{const}$. Then $f^{(2)} = f^{(3)}$. In the general case of a unimodal solenoidal field, we have

$$\nabla_j \nabla_k M_j (\boldsymbol{M}, \nabla) M_k \neq 0.$$

Then Eq. (6.6) implies that $f^{(7)} = 0$. In this case, due to the relation

$$M_j (\nabla_k M_j) \cdot (\boldsymbol{M}, \nabla) M_k = 0,$$

we obtain from (6.10)

$$f^{(8)} M_j M_k \nabla_k (\boldsymbol{M}, \nabla) M_j) = 0.$$

Since in the general case, the expression of the coefficient $f^{(8)}$ in nonzero, we have $f^{(8)} = 0$.

III. We find all other zero coefficients $f^{(\alpha)}$ under the condition that Eqs. (6.5), (6.7), (6.9), and (6.11) are fulfilled. We find relationships between the coefficients by linearizing these equalities near the constant field $\boldsymbol{M}^{(0)}$. In these equations, we set $\boldsymbol{M} = \boldsymbol{M}^{(0)} + \boldsymbol{m}$, where, due to the solenoidality and unimodality of the field $\boldsymbol{M}$, the field $\boldsymbol{m}$ is also solenoidal and lies in the plane orthogonal to $\boldsymbol{M}^{(0)}$. The linearization of these equations with respect to the field $\boldsymbol{m}$ leads to the following equalities, which must identically hold for an arbitrary field satisfying the conditions $(\boldsymbol{M}^{(0)}, \boldsymbol{m}) = 0$ and $(\nabla, \boldsymbol{m}) = 0$:

$$\big(f^{(10)} + 2 f^{(13)}\big)\big(\boldsymbol{M}^{(0)}, [\nabla, \Delta \boldsymbol{m}]\big) = 0, \quad (6.12)$$

$$\big(f^{(20)} - 2 f^{(22)}\big)\big(\boldsymbol{M}^{(0)}, \nabla\big)^2 \big(\boldsymbol{M}^{(0)}, [\nabla, \boldsymbol{m}]\big) = 0, \quad (6.13)$$

$$\Big(f^{(10)} + f^{(13)} - f^{(14)} + f^{(16)} - f^{(17)} + f^{(18)}\Big)\big(\boldsymbol{M}^{(0)}, \nabla\big)\big(\boldsymbol{M}^{(0)}, [\nabla, \boldsymbol{m}]\big) = 0, \quad (6.14)$$

$$\big(f^{(20)}-f^{(22)}-f^{(23)}\big)M^2\big(\boldsymbol{M}^{(0)},\nabla\big)\big(\boldsymbol{M}^{(0)},[\nabla,\boldsymbol{m}]\big)=0. \tag{6.15}$$

Substituting $\boldsymbol{m}=\boldsymbol{A}\exp(\boldsymbol{x},\boldsymbol{k})$ into Eqs. (6.12)–(6.15), where the vectors $\boldsymbol{k}$ and $\boldsymbol{A}$ are such that $(\boldsymbol{A},\boldsymbol{M}^{(0)})=0$ and $(\boldsymbol{A},\boldsymbol{k})=0$, respectively, we obtain in each equality nonzero factors depending on the field $\boldsymbol{m}$. Then

$$f^{(10)}+2f^{(13)}=0,\quad f^{(20)}=2f^{(22)},\quad f^{(22)}=f^{(23)}, \tag{6.16}$$

$$f^{(18)}=f^{(13)}+f^{(14)}+f^{(17)}-f^{(16)}. \tag{6.17}$$

Taking into account Eq. (6.16), we transform Eq. (6.7) to the form

$$f^{(20)}\Big(\nabla_j\nabla_kM_jM_k\big(\boldsymbol{M},[\nabla,\boldsymbol{M}]\big)+\nabla_j\nabla_kM_j\big[\boldsymbol{M},(\boldsymbol{M},\nabla)\boldsymbol{M}\big]_k\Big)=0, \tag{6.18}$$

IV. Now we simplify Eqs. (6.5), (6.7), (6.9), and (6.11). Substitute the field

$$M_j=A_j\cos(\boldsymbol{x},\boldsymbol{k})+B_j\sin(\boldsymbol{x},\boldsymbol{k}),\quad \text{where } k_jA_j=k_jB_j=A_jB_j=0,$$

into Eq. (6.18); note that it belongs to the manifold $\mathcal{M}_0(\mathbb{R}^3)$. Then

$$(\boldsymbol{M},\nabla)\boldsymbol{M}=0,\quad \big(\boldsymbol{M},[\nabla,\boldsymbol{M}]\big)=\varepsilon_{jkl}k_kA_jB_l.$$

Hence (6.18) is reduced to the equality

$$f^{(20)}(\nabla_kM_j)(\nabla_jM_k)=0=f^{(20)}(\operatorname{Sp}\mathsf{A}^2).$$

We can choose a traceless matrix $(\mathsf{A})_{jk}\equiv A_{jk}=\nabla_jM_k$, which is diagonal at the point $\boldsymbol{x}=0$; therefore, $\operatorname{Sp}\mathsf{A}^2\neq0$. Hence $f^{(20)}=0$ and $f^{(22)}=f^{(23)}=0$.

Taking into account (6.16), we reduce Eq. (6.5) to the form

$$f^{(13)}\Big(\varepsilon_{jln}\nabla_jM_n\Delta M_l-\Delta\big(\boldsymbol{M},[\nabla,\boldsymbol{M}]\big)\Big)+\nabla_j\nabla_k\Big(f^{(11)}[\boldsymbol{M},\nabla]_jM_k+f^{(16)}M_j[\nabla,\boldsymbol{M}]_k\Big)=0. \tag{6.19}$$

Using the unimodality and solenoidality conditions of the field $\boldsymbol{M}$, we transform terms (6.19) as follows:

$$\nabla_j\nabla_k[\boldsymbol{M},\nabla]_jM_k=\varepsilon_{jln}B_{ljk}A_{nk},\quad \nabla_j\nabla_kM_j[\nabla,\boldsymbol{M}]_k=\varepsilon_{jln}A_{nk}B_{ljk},$$

$$\varepsilon_{jln}\nabla_j(M_n\Delta M_l)-\Delta\big(\boldsymbol{M},[\nabla,\boldsymbol{M}]\big)=-2\varepsilon_{jln}\big[A_{jl}B_{nkk}+A_{kn}B_{ljk}\big],$$

where $B_{ljk}\equiv\nabla_j\nabla_kM_l$ is a third-rank tensor, symmetric with respect to the pair of the last indexes. Substituting these expressions into Eq. (6.19), we obtain

$$\varepsilon_{jln}\Big[\big(f^{(11)}+f^{(16)}\big)A_{nk}B_{ljk}-2f^{(13)}\big(A_{kn}B_{ljk}+A_{jl}B_{nkk}\big)\Big]=0. \tag{6.20}$$

V. Now we examine the possibility of the local fulfillment of Eq. (6.20) in a neighborhood of an arbitrary fixed point $\boldsymbol{x}_0$ with nonzero coefficients. Without loss of generality, we assume that $\boldsymbol{x}_0=0$. We use the power decomposition of the field $\boldsymbol{M}$ at this point:

$$M_j=M_j^{(0)}+A_{jk}x_k+\frac{1}{2}B_{jkl}x_kx_l+o\big(|\boldsymbol{x}|^2\big). \tag{6.21}$$

The coefficients of the decomposition must satisfy the conditions

$$\boldsymbol{M}^2=M_j^{(0)}M_j^{(0)}+2M_j^{(0)}A_{jk}x_k+\Big(A_{jk}A_{jl}+M_j^{(0)}B_{jkl}\Big)x_kx_l+o\big(|\boldsymbol{x}|^2\big)=M^2$$

and, therefore,

$$2M_j^{(0)}A_{jk}x_k+A_{jk}A_{jl}x_kx_l+M_j^{(0)}B_{jkl}x_kx_l+o\big(|\boldsymbol{x}|^2\big)=0.$$

Moreover, the following relation must hold:

$$\nabla_jM_j=A_{jj}+B_{jjk}x_k+o\big(|\boldsymbol{x}|\big)=0.$$

The decomposition (6.21) holds if and only if the coefficients A_{jk} and B_{jkl} satisfy the following conditions:

$$B_{jkl} = B_{jlk}, \quad A_{jj} = 0, \quad M_j^{(0)} A_{jk} = 0, \quad B_{jjk} = 0, \quad A_{jk}A_{jl} + M_j^{(0)} B_{jkl} = 0. \tag{6.22}$$

VI. We prove that $f^{(13)} = 0$. Without loss of generality, we assume that $M_3^{(0)} = M$. Then from (6.22) we have

$$A_{j3} = 0, \quad A_{mk}A_{ml} + MB_{3kl} = 0, \quad k, l = 1, 2, 3.$$

We set $A_{kl} = \delta_{k3}(1 - \delta_{l3})$. Then

$$(\mathsf{A}^T \mathsf{A})_{kl} = (1 - \delta_{k3})(1 - \delta_{l3});$$

due to (6.22), we see that $B_{3kl} = 0$ if at least one of the indexes is equal to 3 and $B_{3kl} = -M^{-1}$ for $k, l \neq 3$.

The elements B_{jkl}, where $j = 1, 2$, can be chosen arbitrarily, but they must satisfy the conditions

$$B_{jjk} = 0, \quad B_{jkl} = B_{jlk}.$$

We set

$$B_{jkl} = b^{(j)} B_{kl}, \quad j = 1, \quad k, l = 1, 2, 3,$$

where the matrix B_{kl} is symmetric. Consider the condition $B_{jjk} = 0$. Since $B_{33k} = 0$, for $k = 3$ we have

$$B_{113} + B_{223} = b^{(1)} B_{13} + b^{(2)} B_{23} \equiv b^{(1)} b_1 + b^{(2)} b_2 = 0.$$

This equality can be satisfied for any given numbers $b^{(1)}$ and $b^{(2)}$ by an appropriate choice of the numbers b_1 and b_2.

For $k = 1, 2$, the condition $B_{jjk} = 0$ yields

$$b^{(1)} B_{1k} + b^{(2)} B_{2k} = 0,$$

i.e., $B_{2k} = \lambda B_{1k}$, where $\lambda = -b^{(1)}/b^{(2)} \in \mathbb{R}$ is an arbitrary number. Finally, since the matrix B_{kl} is symmetric, we obtain

$$B_{kl} = b \begin{pmatrix} 1 & \lambda \\ \lambda & \lambda^2 \end{pmatrix}, \quad b = B_{11}.$$

Using the tensors A_{jk} and B_{jkl} introduced above, we conclude that for $b^{(1)} = b^{(2)}$, the following equalities hold:

$$\varepsilon_{jln} A_{nk} B_{ljk} = bb^{(2)}(1 - \lambda^2), \quad \varepsilon_{jln} A_{jl} B_{nkk} = 0, \quad \varepsilon_{jln} A_{kn} B_{ljk} = 0,$$

where we have set $B_{133} = B_{233}$ for the unknown coefficients B_{k33}, $k = 1, 2$. Substituting these values into (6.20), we have $2bb^{(2)}(1 - \lambda^2) f^{(13)} = 0$; this implies $f^{(13)} = 0$.

VII. Since $f^{(13)} = 0$ and $\varepsilon_{jln} A_{nk} B_{ljk}$ is nonzero in the general case, the relation (6.20) implies $f^{(11)} = -f^{(16)}$. Taking into account this equality together with $f^{(13)} = 0$ and Eqs. (6.16) and (6.17), we rewrite Eq. (6.9) in the following form:

$$\begin{aligned} f^{(12)} M_j \nabla_k [\boldsymbol{M}, \nabla]_j M_k - f^{(14)} M_j \nabla_k \Big(\varepsilon_{kln} M_n \nabla_j M_l - \varepsilon_{jkl} (\boldsymbol{M}, \nabla) M_l \Big) \\ + f^{(16)} M_j \nabla_k \Big(M_k [\nabla, \boldsymbol{M}]_j - [\boldsymbol{M}, \nabla]_j M_k - \varepsilon_{jkl} (\boldsymbol{M}, \nabla) M_l \Big) \\ - f^{(17)} M_j \nabla_k \Big(M_k [\nabla, \boldsymbol{M}]_j - \varepsilon_{jkl} (\boldsymbol{M}, \nabla) M_l \Big) = 0. \end{aligned}$$

Using the solenoidality and unimodality of the field $\boldsymbol{M}$ and introducing the notation $A_{jk} = \nabla_j M_k$ and $B_{jkl} = \nabla_k \nabla_l M_j$, after simple transformations we reduce this equation to the form

$$\big(f^{(12)} - f^{(14)} - f^{(17)}\big) \varepsilon_{klm} M_l A_{kj} A_{jm} - f^{(14)} \varepsilon_{jkm} M_l A_{kj} A_{lm} - 2 f^{(16)} \varepsilon_{jlm} M_j M_k B_{mkl} = 0;$$

here the left-hand side contains the decomposition over linearly independent invariants consisting of the tensors A_{jk} and B_{jkl} and the pseudovector M_j. The equation obtained, which holds for any solenoidal, unimodal, twice continuously differentiable field $\boldsymbol{M}$ on $\mathbb{R}^3$, implies

$$f^{(16)} = 0, \quad f^{(14)} = 0, \quad f^{(12)} = f^{(17)}.$$

Introducing the notation $f^{(12)} = \gamma$, taking into account all restrictions for the coefficients $f^{(\alpha)}$ obtained above, and using (6.1), we obtain the following general form of differential operators of the class $\mathcal{K}_2(\mathbb{R}^3) \cap \mathcal{M}_0(\mathbb{R}^3)$:

$$\mathsf{L}_j[\boldsymbol{M}] = \gamma \nabla_k \Big([\boldsymbol{M}, \nabla]_j M_k - [\boldsymbol{M}, \nabla]_k M_j + M_j [\nabla, \boldsymbol{M}]_k - M_k [\nabla, \boldsymbol{M}]_j + \varepsilon_{jkl} (\boldsymbol{M}, \nabla) M_l \Big). \tag{6.23}$$

Thus, we have proved the following main theorem, which provides a description of the class $\mathcal{K}_2(\mathbb{R}^3) \cap \mathcal{M}_0(\mathbb{R}^3)$ for pseudovector fields.

Theorem 6.1. *Consider the linear manifold $\mathcal{K}_2(\mathbb{R}^3) \cap \mathcal{M}_0(\mathbb{R}^3)$ of all divergent-type differential equations*

$$\dot{M}_j = \mathsf{L}_j[\boldsymbol{M}]$$

for pseudovector solenoidal unimodal fields $\boldsymbol{M}$ on $\mathbb{R}^3$ from the space $[C_{2,\mathrm{loc}}(\mathbb{R}^3)]^3$ with second-order differential operators $\mathsf{L}_j[\boldsymbol{M}]$ acting in this space and covariant under the action of elements the group $\boldsymbol{O}_3$ that leave invariant any manifold

$$\Big\{ \boldsymbol{M} : \ (\nabla, \boldsymbol{M}) = 0; \ \boldsymbol{M}^2 = M^2, \ \boldsymbol{x} \in \mathbb{R}^3 \Big\} \subset \big[C_{2,\mathrm{loc}}(\mathbb{R}^3) \big]^3$$

of this space with an arbitrary value of $M^2 \in (0, \infty)$. This linear manifold has dimension one; *it is described by the formula* (6.23) *with an arbitrary constant $\gamma \in \mathbb{R}$.*

7. Conclusion. The idea on which this study is based is related to the problem of describing the dynamics of complexly arranged condensed media at the macroscopic level. The specific result obtained in this work is related to the well-known problem in the physics of magnetism of constructing an adequate evolutionary equation for the field $\boldsymbol{M}$ of the magnetization density with irreversible dynamics (see [1–3, 15]). The well-known evolutionary Landau–Lifshitz equation (see, e.g., [8, 9]) in the spherically symmetric case has the form

$$\dot{\boldsymbol{M}} = \gamma [\boldsymbol{M}, \Delta \boldsymbol{M}].$$

It is a divergent-type equation since

$$[\boldsymbol{M}, \Delta \boldsymbol{M}]_j = \nabla_k \varepsilon_{jln} M_l \nabla_k M_n;$$

obviously, it preserves the unimodality of the field but does not preserve the solenoidality. Moreover, it does not possess the dissipation property since the real part of the eigenvalues of the (3×3)-matrix, which is the symbol of the linearization of the operator $\mathsf{L}_j[\boldsymbol{M}]$, are zero. In our opinion, attempts of overcoming these disadvantages of the evolutionary ferrodynamics equation based on generalizations of the Landau–Lifshitz equation are not successful (see, e.g., [2, 3]). The operator $\mathsf{L}_j[\boldsymbol{M}] = \gamma[\boldsymbol{M}, \Delta \boldsymbol{M}]$ is not involved in the right-hand side of the evolutionary equation

$$\dot{M}_j = \gamma \nabla_k \Big([\boldsymbol{M}, \nabla]_j M_k - [\boldsymbol{M}, \nabla]_k M_j + M_j [\nabla, \boldsymbol{M}]_k - M_k [\nabla, \boldsymbol{M}]_j + \varepsilon_{jkl} (\boldsymbol{M}, \nabla) M_l \Big)$$

with the operator $\mathsf{L}_j[\cdot]$ defined by the formula (6.23); Theorem 6.1 explains this fact.

Note that, due to the existence of two invariants, the symbol of the linearization of the operator defined by the formula (6.23) has two zero eigenvalues. Therefore, the third eigenvalue is necessarily real. By an appropriate choice of the sign of the constant γ, one can achieve that this operator will be parabolic.

In this work, we are considered only the simplest formulations of problems related to irreducible vector representations, which could be of interest for physical applications. An obvious generalization of the problems studied consists of the rejection of spherical symmetry, i.e., in using the tensor algebra with the set of generators extended by a second-rank tensor characterizing the asymmetry of the medium in the construction of all possible tensor coefficients, both in the vector and pseudovector cases. With this formulation of the problem, a much richer set of tensor coefficients arises that determine the densities $S_{j;k}$, $j = 1, 2, 3$; this leads to a more laborious analysis of possibilities emerged.

Another way of generalization consists of refusing of the unimodality and solenoidality conditions. In this case, classes of admissible equation become much wider. In particular, in the first case—for pseudovector fields—such a class contains the Landau–Lifshitz equation mentioned above; in the second case, we obtain the well-known Navier–Stokes equation for incompressible fluids.

Finally, the broadest generalizations of the problem solved in this work follow from its general formulation presented in Sec. 2. Moreover, from the point of view of applications, it is expedient to study evolution equations with differential operators that are not divergent but can be represented in the form

$$\dot{X}_a = (\nabla_k S_{a;\,k})(X) + H_a(X), \quad a = 1, \dots, N,$$

where $H_a(X)$ is a function on the space of values of the complete set of local thermodynamical parameters, which is transformed covariantly under the action of the group $\mathbf{O}_3$ and serves as a "self-consistent source" of the field from the physical point of view.

REFERENCES

1. A. F. Andreev and V. I. Marchenko, "Macroscopic theory of spin waves," *Zh. Eksp. Teor. Fiz.*, **70**, No. 4, 1522–1532 (1976).
2. V. G. Bar'yakhtar, *Life in Science* [in Russian], Naukova Dumka, Kiev (2010).
3. V. G. Bar'yakhtar, V. G. Belykh, and T. K. Soboleva, "Macroscopic theory of relaxation of collective excitations in disordered and noncollinear magnets," *Teor. Mat. Fiz.*, **77**, No. 2, 311–318 (1988).
4. B. B. Delgado and R. M. Porter, "General solution of the inhomogeneous div-curl system and consequences," *Adv. Appl. Clifford Alg.*, **24**, No. 7, 3015–3037 (2017).
5. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin (1983).
6. A. A. Isaev, M. Yu. Kovalevsky, and S. V. Peletminsky, "Hamiltonian approach to the theory of condensed matter with spontaneously broken symmetry," *Fiz. Elem. Chast. Atom. Yad.*, **27**, No. 2, 431–492 (1996).
7. E. I. Kats and V. V. Lebedev, *Dynamics of Liquid Crystals* [in Russian], Nauka, Moscow (1988).
8. A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, *Nonlinear Magnetization Waves* [in Russian], Naukova Dumka, Kiev (1988).
9. L. D. Landau and E. M. Lifshitz, "On the theory of dispersion of the magnetic permeability of ferromagnetics," in: *L. D. Landau, Collected Works* [in Russian], Nauka, Moscow (1969), pp. 128.
10. G. Ya. Lyubarsky, *Group Theory and Its Applications in Physics* [in Russian], GIFML, Moscow (1958).
11. A. J. Mc Connell, *Application of Tensor Analysis*, Dover, New York (1957).
12. P. K. Rashevsky, *Riemannian Geometry and Tensor Analysis* [in Russian], Nauka, Moscow (1967).
13. A. V. Subbotin, "Description of the class of divergent-type evolutionary equations for vector fields," *Nauch. Ved. Belgorod Univ. Ser. Mat. Fiz.*, **50**, No. 4, 492–497 (2018).

14. Yu. P. Virchenko and A. E. Ponamareva, “Construction of a general evolutionary equation for a pseudo-vector solenoidal field with a local conservation law,” *Nauch. Ved. Belgorod Univ. Ser. Mat. Fiz.*, **50**, No. 2, 224–232 (2018).
15. D. V. Volkov, “Phenomenological Lagrangians,” *Fiz. Elem. Chast. Atom. Yad.*, **4**, No. 1, 3–41 (1973).

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