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Uniqueness Criterion for Solutions of Nonlocal Problems on a Finite Interval for Abstract Singular Equations

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Abstract—For abstract singular equations, nonlocal problems belonging to the class of ill-posed problems are considered. A uniqueness criterion for solutions is established.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Let E be a complex Banach space, and let A, B be linear closed operators in E whose domains $D(A), D(B) \subset E$ are not necessarily dense in E. Consider the following equation:

$$B\left(u''(t) + \frac{k}{t}\,u'(t)\right) = Au(t), \qquad 0 < t < 1, \tag{1.1}$$

which, in the case $B \neq I$, generalizes the abstract Euler–Poisson–Darboux equation. The case $0 < t < T$ can be reduced to the one considered above by changing the variable t to t/T.

In view of the singularity (for $k \neq 0$) of the equation at the point $t = 0$, the setting of boundary and nonlocal conditions depends on the parameter $k \in \mathbb{R}$, and these conditions will be given below. The subsequently imposed nonlocal integral conditions can be interpreted in the spirit of control theory: it is required to find a solution of the differential equation (1.1) with a given initial condition at $t = 0$ and possessing a certain prescribed average value. As was pointed out in [1], conditions of this kind arise, for example, in studying the diffusion of particles in a turbulent plasma, moisture transport processes in capillary-porous media, etc.

An equation of the form (1.1) is called a Sobolev-type equation or a descriptor equation; for a detailed overview of publications on this topic, see, for example, [2]. The Cauchy problem for the singular equation (1.1) with Fredholm operator B was studied above in [3], [4]. Nonlocal problems for equation (1.1) are, in general, ill-posed, but the need to solve ill-posed problems is currently generally accepted (see the introduction in the book [5] and the extensive bibliography therein). For abstract equations of first order, some results on the solvability of nonlocal problems on half-axes were obtained more recently in [6]-[8] and, for second-order singular equations, in [9]-[11]. Nonlocal problems on a finite interval for partial differential equations containing the Bessel differential operator were studied in [12]-[16].

Apparently, there is still no exhaustive uniqueness criterion for the Cauchy problem, but the situation is different, in principle, in the case of boundary and nonlocal problems. We will pose boundary and nonlocal conditions for equation (1.1) for $k \in \mathbb{R}$ and establish appropriate uniqueness criteria for these problems. It will be shown that, just as in the study of boundary-value problems (see [17]), the uniqueness of the solution only depends on the spectral properties of the operators A, B and is related to the distribution of zeros of certain analytic functions. Instead of the uniqueness of the solution, it is more convenient to consider the equivalent problem of the triviality of the solution of the homogeneous problem, which we will do from the very beginning. Since some very general conditions are imposed on the operators A, B, it follows that, in this paper, we will not be concerned with the solvability of nonlocal problems.

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2. THE CASE $k \geq 0$. THE NEUMANN CONDITION FOR $t = 0$

Consider the problem of finding the function $u(t) \in C^1([0,1],E) \cap C^2((0,1],E)$, which, at $t \in (0,1)$, belongs together with its derivatives to the domain $D = D(A) \cap D(B)$, satisfies equation (1.1), the Neumann boundary condition

$$u'(0) = 0, \tag{2.1}$$

as well as a nonlocal condition of the form

$$\int_0^1 t^k (1-t^2)^{\alpha-1} u(t)\,dt = 0, \qquad \alpha > 0. \tag{2.2}$$

A special case of the nonlocal condition (2.2) for $\alpha = 1$ appeared above in [1], [12]. Using the Erdelyi–Kober operator $I^{\alpha}_{0+;2,\eta}$ (see [18, p. 246]), condition (2.2) can also be written in the form

$$\lim_{t\to 1} I^{\alpha}_{0+;2,\eta} u(t) = 0, \qquad I^{\alpha}_{0+;2,\eta} u(t) = \frac{2}{\Gamma(\alpha) t^{2(\alpha+\eta)}} \int_0^t s^{2\eta+1} (t^2 - s^2)^{\alpha-1} u(s)\,ds.$$

We will search for the nontrivial solutions $u(t)$ of the homogeneous problem (1.1), (2.1), (2.2) using the method of separation of variables in the form $u(t) = v(t)h$, where $v(t) \in C^1[0,1] \cap C^2(0,1]$ is a nonzero scalar complex-valued function, and $h \in D$, $h \neq 0$.

Substituting $u(t) = v(t)h$ into problem (1.1), (2.1), (2.2), we obtain the equation

$$\left(v''(t) + \frac{k}{t}\, v'(t)\right) Bh = v(t) Ah, \tag{2.3}$$

and conditions

$$v'(0) = 0, \tag{2.4}$$

$$\int_0^1 t^k (1-t^2)^{\alpha-1} v(t)\,dt = 0. \tag{2.5}$$

Equation (2.3) implies the equality

$$Ah = \frac{v''(t) + k/t v'(t)}{v(t)}\, Bh, \tag{2.6}$$

which must hold on the set $\{t \in (0,1) : v(t) \neq 0\}$.

Obviously, equality (2.6) can only hold if

$$Ah = \lambda Bh \tag{2.7}$$

with some constant $\lambda \in \mathbb{C}$.

Thus, the element $h \in D$, $h \neq 0$ must satisfy the operator equation (2.7) with some $\lambda \in \mathbb{C}$, and equation (2.3) becomes

$$v''(t) + \frac{k}{t}\, v'(t) = \lambda v(t). \tag{2.8}$$

The general solution of the ordinary differential equation (2.8) is conveniently written in the form (see, for example, [19])

$$v(t) = c_1 Y_k(t;\lambda) + c_2 t^{1-k} Y_{2-k}(t;\lambda), \qquad c_1, c_2 \in \mathbb{R}, \tag{2.9}$$

where $Y_k(t;A)$ is the solving operator of the Cauchy problem for the Euler–Poisson–Darboux equation ($B = I$ in equation (1.1)), $Y_k(0;A) = I$, $Y_k'(0;A) = 0$, which was constructed in [19], [20] for any values of the parameter $k \in \mathbb{R}$. In the case under consideration, for $k \geq 0$, we have

$$Y_k(t;\lambda) = \Gamma\left(\frac{k}{2} + \frac{1}{2}\right)\left(\frac{t\sqrt{\lambda}}{2}\right)^{1/2-k/2} I_{k/2-1/2}(t\sqrt{\lambda}), \tag{2.10}$$

where $\Gamma(\,\cdot\,)$ is the Euler gamma function and $I_\nu(\,\cdot\,)$ is the modified Bessel function. The scalar function $Y_k(t;\lambda)$ is also called the normalized Bessel function.

A solution of equation (2.3) satisfying the boundary condition (2.4) has the form

$$v(t) = Y_k(t;\lambda). \tag{2.11}$$

To find suitable values of $\lambda \in \mathbb{C}$, it remains to use the nonlocal condition (2.5). Substituting the function (2.10) into (2.5) and using the formula for the shift with respect to the parameter of the solution (see [20]) for the solving operator $Y_k(t;A)$, we obtain the transcendental equation

$$Y_{k+2\alpha}(1;\lambda) = 0. \tag{2.12}$$

Equation (2.12) can also be obtained if, instead of the formula for the shift with respect to the parameter, we use representation (2.10) and the integral 2.15.2.6 from [21].

Denoting $\sqrt{\lambda} = i\mu$ and taking into account representation (2.10), we write the following equation (2.12) in terms of the Bessel function of the first kind $J_\nu(\,\cdot\,)$:

$$\frac{J_{k/2+\alpha-1/2}(\mu)}{\mu^{k/2+\alpha-1/2}} = 0. \tag{2.13}$$

As is known (see [22, Sec. 18.3]), equation (2.13) has an infinite set of positive roots $\mu_m = \mu_m(k,\alpha)$, $m \in \mathbb{N}$. Substituting $\lambda_m = -\mu_m^2$ into (2.11), we determine the functions $v_m(t) = Y_k(t;\lambda_m)$, $m \in \mathbb{N}$, which are nontrivial solutions to problem (2.3)–(2.5), while relation (2.7) turns into the following equations for finding $h_m \neq 0$:

$$Ah_m = \lambda_m B h_m, \qquad m \in \mathbb{N}. \tag{2.14}$$

Let us further assume that, for some $m \in \mathbb{N}$, the pair λ_m, h_m is a solution of equation (2.14). Then we find a nontrivial solution of the homogeneous nonlocal problem (1.1), (2.1), (2.2) of the following form:

$$u_m(t) = Y_k(t;\lambda_m)??h_m. \tag{2.15}$$

Let us now formulate the uniqueness criterion for problem (1.1), (2.1), (2.2).

Theorem 1. *Let $k \geq 0$, $\alpha > 0$, let A, B be linear closed operators in E, and let the nonlocal problem* (1.1)*,* (2.1)*,* (2.2) *have a solution $u(t)$. For this solution to be unique, it is necessary and sufficient that, for no zero λ_m, $m \in \mathbb{N}$, of the function*

$$\Upsilon_{k,\alpha}(\lambda) = Y_{k+2\alpha}(1;\lambda) \tag{2.16}$$

given by equality (2.10)*, does the operator equation* (2.14) *have a solution.*

Proof. To establish necessity, let us assume the opposite. Let, for some λ_m, $m \in \mathbb{N}$ from a countable set of zeros of the function $\Upsilon_{k,\alpha}(\lambda)$ defined by equality (2.16), there exists a solution of equation (2.14) with a vector $h_m \neq 0$. Then the function $u_m(t)$ determined by equality (2.15) serves as a nontrivial solution of the homogeneous nonlocal problem (1.1), (2.1), (2.2), which contradicts the uniqueness of the solution of this problem, and thus necessity is proved.

Let us now prove sufficiency. Suppose that, for any λ_m, $m \in \mathbb{N}$ from a countable set of zeros of the function $\Upsilon_{k,\alpha}(\lambda)$ defined by equality (2.16), equation (2.14) has no solution, and let $u(t)$ be a solution of a homogeneous nonlocal problem (1.1), (2.1), (2.2). Let us show that, in this case, $u(t) \equiv 0$.

Let us introduce the following function $U(\lambda)$ of the variable $\lambda \in \mathbb{C}$ with values in the Banach space E:

$$U(\lambda) = \int_0^1 t^{k+2\alpha} Y_{k+2\alpha}(t;\lambda) w(t)\, dt, \tag{2.17}$$

where the scalar function $Y_{k+2\alpha}(t;\lambda)$ is defined by equality (2.10) (after replacing the parameter k by $k+2\alpha$), is a solution of equation (2.8), and satisfies the conditions $Y_{k+2\alpha}(0;\lambda) = I$, $Y'_{k+2\alpha}(0;\lambda) = 0$, while, in view of the closedness of the operators A, B and the formula for the shift with respect to the parameter (see [20]), the function

$$w(t) = \int_0^1 s^k(1-s^2)^{\alpha-1}u(ts)\,ds,$$

satisfies equation (1.1) after replacing the parameter k by $k+2\alpha$. This fact is not difficult to verify directly.

Taking into account the closedness of the operators A, B and equality (1.1), we can calculate $AU_\delta(\lambda)$, where

$$U_\delta(\lambda) = \int_\delta^1 t^{k+2\alpha}Y_{k+2\alpha}(t;\lambda)w(t)\,dt, \qquad \delta > 0.$$

After double integrating by parts, we have

$$\begin{aligned} AU_\delta(\lambda) &= \int_\delta^1 t^{k+2\alpha}Y_{k+2\alpha}(t;\lambda)Aw(t)\,dt \\ &= B\int_\delta^1 t^{k+2\alpha}Y_{k+2\alpha}(t;\lambda)\Big(w''(t) + \frac{k+2\alpha}{t}\,w'(t)\Big)\,dt \\ &= t^{k+2\alpha}Y_{k+2\alpha}(t;\lambda)Bw'(t)\big|_\delta^1 - t^{k+2\alpha}Y'_{k+2\alpha}(t;\lambda)Bw(t)\big|_\delta^1 \\ &\quad + B\int_\delta^1 (t^{k+2\alpha}Y''_{k+2\alpha}(t;\lambda) + (k+2\alpha)t^{k+2\alpha-1}Y'_{k+2\alpha}(t;\lambda))w(t)\,dt. \end{aligned}$$

Letting $\delta \to 0$ and taking into account condition (2.2), we obtain

$$AU(\lambda) = Y_{k+2\alpha}(1;\lambda)Bw'(1) + \lambda BU(\lambda). \tag{2.18}$$

Thus, for all numbers λ_m satisfying equation (2.12), equality (2.18) implies the relation

$$AU(\lambda_m) = \lambda_m BU(\lambda_m).$$

By the assumption of the theorem, none of these numbers λ_m can be a solution of the operator equation (2.14). But, in that case, all the $U(\lambda_m)$ must be zero:

$$U(\lambda_m) = 0, \qquad m \in \mathbb{N}. \tag{2.19}$$

Let $\mu_m, m \in \mathbb{N}$ be the positive roots of equation (2.14) arranged in increasing order, and let $(i\mu_m)^2 = \lambda_m$. Then equalities (2.19) take the form

$$U_m = \int_0^1 t^{k/2+\alpha+1/2}J_{k/2+\alpha-1/2}(t\mu_m)w(t)\,dt = 0, \qquad m \in \mathbb{N}. \tag{2.20}$$

Applying a linear continuous functional $f \in E^*$ to the vector coefficients U_m defined by equality (2.20), we obtain the scalar function $\varphi(t) = f(t^{k/2+\alpha}w(t))$, satisfying the conditions

$$f(U_m) = \int_0^1 \sqrt{t}J_{k/2+\alpha-1/2}(t\mu_m)\varphi(t)\,dt = 0, \qquad m \in \mathbb{N}. \tag{2.21}$$

Up to a factor, the scalar coefficients $f(U_m)$ are the coefficients of the Fourier–Bessel series expansion in terms of the functions $J_{k/2+\alpha-1/2}(t\mu_m)$ (see [22, Chap. XVIII], [23]) for the function $\varphi(t) = f(t^{k/2+\alpha}w(t))$.

Therefore, $f(t^{k/2+\alpha}w(t)) \equiv 0$, $0 \le t \le 1$. Since the choice of the functionals $f \in E^*$ was arbitrary, we have $w(t) \equiv 0$, $0 \le t \le 1$; hence, given the representation

$$w(t) = \frac{1}{t^{k+2\alpha-1}}\int_0^t \tau^k(t^2-\tau^2)^{\alpha-1}u(\tau)\,d\tau,$$

after applying the Erdelyi–Kober operator $I^{-\alpha}_{0+;2,\eta}$, we obtain $u(t) \equiv 0$, $0 \le t \le 1$. Hence the solution problems (1.1), (2.1), (2.2) can only be zero. □

Theorem 1 is naturally also valid for the Euler–Poisson–Darboux equation if $B = I$ in equation (1.1), and we will give the corresponding example.

Example. Let $B = I$ in equation (1.1), and let $\alpha = 1$. Consider the following singular operator A given by the Bessel differential expression

$$A = B_{q,x} = \frac{d^2}{dx^2} + \frac{q}{x}\frac{d}{dx}, \qquad q > 0;$$

The operator is defined on the set of functions $D(A) = H^2(0,1) \cap H_0^1(0,1) \subset E = L_2(0,1)$ and acts on the space variable x

The question of the uniqueness of the solution of our nonlocal problem for the hyperbolic equation is reduced to the study of the location of the zeros of the function $I_{q/2-1/2}(\sqrt{z})$, which are the eigenvalues of the operator $B_{q,x}$, and of the zeros of the function $\Upsilon_{k,1}(\lambda)$ defined by equality (2.16).

For $B = I$, $\alpha = 1$, $k \geq 0$, we must study the location of the zeros of the functions $I_{q/2-1/2}(\sqrt{z})$ and $I_{k/2+1/2}(\sqrt{\lambda})$. Depending on the parameters k and q, the specified Bessel functions may or may not have common zeros located on $(-\infty, 0)$; therefore, the solution of boundary-value problems will either be unique or nonunique. For more details on the location of the zeros of Bessel functions, see, for example, [24, Sec. 2]. We also note that an important role in the study of uniqueness is also played by by the intervals of variation of the variables, $0 < t < T$ and $0 < x < l$, because the zeros of each of the indicated Bessel functions change their positions.

If $A = -B_{q,x}$ or $A = iB_{q,x}$, where i is the imaginary unit, then the eigenvalues of the operator A lie either on $(0, +\infty)$, or on the imaginary axis and do not lie on $(-\infty, 0)$; therefore, the corresponding nonlocal problems have a unique solution.

3. THE CASE $k < 1$. THE DIRICHLET CONDITION FOR $t = 0$

Consider the problem of finding the function $u(t) \in C([0,1], E) \cap C^2((0,1], E)$ which, for $t \in (0,1)$, belongs together with its derivatives to the domain $D = D(A) \cap D(B)$, satisfies equation (1.1), the Dirichlet boundary condition

$$u(0) = 0, \tag{3.1}$$

as well as a nonlocal condition of the form

$$\int_0^1 t(1-t^2)^{\beta-1} u(t)\, dt = 0, \qquad \beta > 0 \tag{3.2}$$

or, using the Erdelyi–Kober operator, $I^{\beta}_{0+;2,0} u(t)|_{t=1} = 0$.

As in Sec. 2, to find the nontrivial solutions of the homogeneous problem (1.1), (3.1), (3.2), it is necessary, from the general solution (2.9), to choose a solution satisfying the Dirichlet condition $v(0) = 0$. Such a solution has the form

$$v(t) = t^{1-k} Y_{2-k}(t; \lambda), \tag{3.3}$$

where the function $Y_{2-k}(t; \lambda)$ is given by equality (2.10).

To determine the values of $\lambda \in \mathbb{C}$, we will use the nonlocal condition

$$\int_0^1 t(1-t^2)^{\beta-1} v(t)\, dt = 0,$$

substituting into which the function (3.3) and using the formula for the shift with respect to the parameter for $k < 0$ (see [25]), we obtain the transcendental equation

$$Y_{2\beta+2-k}(1; \lambda) = 0. \tag{3.4}$$

Denoting $\sqrt{\lambda} = i\mu$ and in view of the representation (2.10), in terms of the Bessel function of the first kind $J_\nu(\,\cdot\,)$, we write equation (3.4) in the form

$$\frac{J_{\beta+1/2-k/2}(\mu)}{\mu^{\beta+1/2-k/2}} = 0. \tag{3.5}$$

Equation (3.5) has an infinite set of positive roots $\mu_m, m \in \mathbb{N}$ arranged in increasing order. Substituting $\lambda_m = -\mu_m^2$ into (3.3), we obtain the functions $v_m(t) = t^{1-k}Y_{2-k}(t;\lambda_m)$, $m \in \mathbb{N}$, which are nontrivial solutions of problem (1.1), (3.1), (3.2), while relation (2.7) turns into the following equations for finding $h_m \neq 0$:

$$Ah_m = \lambda_m B h_m, \qquad m \in \mathbb{N}. \tag{3.6}$$

Let us further assume that, for some $m \in \mathbb{N}$, the pair λ_m, h_m is a solution of equation (3.6). Then we define a nontrivial solution of the homogeneous nonlocal problem (1.1), (3.1), (3.2) of the following form:

$$u_m(t) = t^{1-k}Y_{2-k}(t;\lambda_m)??h_m. \tag{3.7}$$

For the problem considered in this section, the following uniqueness criterion for the solution holds.

Theorem 2. *Let $k < 1$, $\beta > 0$, and let A, B be linear closed operators in E. Suppose that the nonlocal problem* (1.1), (3.1), (3.2) *has a solution $u(t)$. For this solution to be unique, it is necessary and sufficient that, for no $\lambda_m = \lambda_m(k,\beta)$, $m \in \mathbb{N}$, which is the zero of the function*

$$\Psi_{k,\beta}(\lambda) = Y_{2\beta+2-k}(1;\lambda), \tag{3.8}$$

where the function $Y_{2\beta+2-k}(t;\lambda)$ is given by equality (2.10)*, does the operator equation* (3.6) *have a solution.*

Proof. To prove necessity, suppose the opposite. Let, for some λ_m, $m \in \mathbb{N}$, from a countable set of zeros of the function $\Psi_{k,\beta}(\lambda)$ defined by equality (3.8), there exist a solution of equation (3.6) with a vector $h_m \neq 0$. Then the function $u_m(t)$ defined by equality (3.7) serves as a nontrivial solution of the homogeneous nonlocal problem (1.1), (3.1), (3.2), which contradicts the uniqueness of the solution of this problem, and thus necessity is proved.

Let us now prove sufficiency. Suppose that, for any λ_m from countable set of zeros of the function $\Psi_{k,\beta}(\lambda)$ defined by equality (3.8), equation (3.6) has no solution, and let $u(t)$ be a solution of a homogeneous nonlocal problem (1.1), (3.1), (3.2). Let us show that, in this case, $u(t) \equiv 0$.

Let us introduce the function $V(\lambda)$ of the variable $\lambda \in \mathbb{C}$ with values in the Banach space E,

$$V(\lambda) = \int_0^1 tY_{2\beta+2-k}(t;\lambda)w(t)\,dt, \tag{3.9}$$

where the function

$$w(t) = t^{2\beta}\int_0^1 s(1-s^2)^{\beta-1}u(ts)\,ds,$$

by virtue of the formula for the shift with respect to the parameter for $k < 1$ (see [25]) (which, however, is easy to verify directly) and the closedness of the operators A, B satisfies equation (1.1) after replacing the parameter k by $k - 2\beta$.

As in the proof of Theorem 1, we first calculate $AU_\delta(\lambda)$, where

$$V_\delta(\lambda) = \int_\delta^1 tY_{2\beta+2-k}(t;\lambda)w(t)\,dt, \qquad \delta > 0.$$

Twice integrating by parts, we obtain

$$\begin{aligned} AV_\delta(\lambda) &= \int_\delta^1 tY_{2\beta+2-k}(t;\lambda)Aw(t)\,dt = B\int_\delta^1 tY_{2\beta+2-k}(t;\lambda)\biggl(w''(t) + \frac{k-2\beta}{t}\,w'(t)\biggr)\,dt \\ &= tY_{2\beta+2-k}(t;\lambda)Bw'(t)\big|_\delta^1 + ((k-2\beta-1)Y_{2\beta+2-k}(t;\lambda) - tY'_{2\beta+2-k}(t;\lambda))Bw(t)\big|_\delta^1 \end{aligned}$$

$$+ B\int_\delta^1 t\biggl(Y''_{2\beta+2-k}(t;\lambda)+\frac{2\beta+2-k}{t}\,Y'_{2\beta+2-k}(t;\lambda)\biggr)w(t)\,dt.$$

Letting $\delta\to 0$ and taking into account condition (3.2), we see that

$$AV(\lambda)=Y_{2\beta+2-k}(1;\lambda)Bw'(1)+\lambda BV(\lambda). \tag{3.10}$$

Thus, for all numbers λ_m, satisfying equation (3.4), equality (3.10) implies the relation

$$AV(\lambda_m)=\lambda_m BV(\lambda_m).$$

By the assumption of the theorem, none of such numbers λ_m can be a solution of the operator equation (3.6). But then all the values of $V(\lambda_m)$ must be zero,

$$V(\lambda_m)=0,\qquad m\in\mathbb{N}. \tag{3.11}$$

As in the proof of Theorem 1, it follows from equalities (3.11) that the function $w(t)$ vanishes and so does $u(t)$. Thus, the solution of problem (1.1), (3.1), (3.2) can only vanish. The theorem is proved. □

Remark 1. For the values of the parameter $0\le k<1$, Theorem 1 with the Neumann condition for $t=0$, and Theorem 2 with the Dirichlet condition for $t=0$ hold simultaneously.

Because of the presence of a, generally speaking, irreversible operator B, Sobolev-type equations (1.1) are also said to be degenerate. In the next section, we will consider nonlocal problems for the class of so-called degenerate equations. Note that the Cauchy problem for such a class of degenerate equations was investigated above in [26].

4. NONLOCAL PROBLEMS FOR DEGENERATE DIFFERENTIAL EQUATIONS WITH POWER-LAW DEGENERATION

As applications of Theorems 1 and 2 in the Banach space E, we will consider the following equation degenerate in the variable t:

$$t^\gamma v''(t)+bt^{\gamma-1}v'(t)=Av(t),\qquad 0<t<T. \tag{4.1}$$

Let $0<\gamma<2$, $b\in\mathbb{R}$. The value of the parameter γ, $0<\gamma<2$, indicates a weak degeneration of equation (4.1), in contrast to the case of a strong degeneration, $\gamma>2$, which will also be considered in what follows. For $\gamma=2$, the Euler equation is obtained; this equation, as is known, reduces to a nondegenerate equation.

The setting of boundary conditions at the point of degeneration $t=0$ depends on the coefficients b and $\gamma>0$ of the equations, and these boundary conditions will be given below.

For $b<1$, we consider the problem of finding function $v(t)\in C([0,T],E)\cap C^2((0,T],E)$, $t\in(0,T)$, belonging to $D(A)$ and satisfying equation (4.1), the Dirichlet condition

$$v(0)=0, \tag{4.2}$$

as well as a nonlocal condition of the form

$$\int_0^T t^{1-\gamma}(1-\nu^2t^{2-\gamma})^{\beta-1}v(t)\,dt=0,\qquad \beta>0,\quad \nu=\frac{2}{2-\gamma}\,. \tag{4.3}$$

The replacement of the independent variable and the unknown function

$$t=\left(\frac{\tau}{\nu}\right)^{\nu},\qquad \nu=\frac{2}{2-\gamma}\,,\qquad v(t)=v\biggl(\left(\frac{\tau}{\nu}\right)^{\nu}\biggr)=w(\tau),$$

$$v'(t)=\left(\frac{\tau}{\nu}\right)^{1-\nu}w'(\tau),\qquad v''(t)=\left(\frac{\tau}{\nu}\right)^{2(1-\nu)}\biggl(w''(\tau)+\frac{1-\nu}{\tau}\,w'(\tau)\biggr),$$

reduces the degenerating equation (4.1) to the Euler–Poisson–Darboux equation

$$w''(\tau)+\frac{k}{\tau}\,w'(\tau)=Aw(\tau),\qquad \tau\in[0,l], \tag{4.4}$$

where $k = b\nu - \nu + 1$, $\nu = 2/(2-\gamma)$, $l = \nu T^{1/\nu}$. Further, conditions (4.2), (4.3) become, respectively, the following conditions:

$$w(0) = 0, \qquad \int_0^l \tau(1-\tau^2)^{\beta-1} w(\tau)\, d\tau = 0. \tag{4.5}$$

We have already studied the resulting problem (4.4), (4.5) in Sec. 3 and, returning to the original nonlocal problem (4.1)–(4.3) and using Theorem 2, we formulate the following uniqueness criterion for a weakly degenerating equation:

Theorem 3. *Let $0 < \gamma < 2$, $b < 1$, $\beta > 0$, let A be a closed linear operator in E, and let the boundary-value problem* (4.1)–(4.3) *have a solution $v(t)$. For this solution to be unique, it is necessary and sufficient that no zero λ_m, $m \in \mathbb{N}$, given by equality* (2.10) *of the function $Y_{2\beta+2-k}(l;\lambda)$, where $k = b\nu - \nu + 1$, $\nu = 2/(2-\gamma)$, $l = \nu T^{1/\nu}$, be an eigenvalue of the operator A.*

As mentioned above, the formulation of the boundary condition at the point of degeneration $t = 0$ depends on the coefficient b. Now suppose that, in equation (4.4), the coefficient $b > \gamma/2$. In this case, instead of the Dirichlet condition (4.2), we must impose the Neumann weighting condition

$$\lim_{t\to 0+} t^{\gamma/2} v'(t) = 0, \tag{4.6}$$

and, instead of the nonlocal condition (4.3), we must impose a nonlocal condition of the form

$$\int_0^T t^{b-\gamma}(1-\nu^2 t^{2-\gamma})^{\alpha-1} v(t)\, dt = 0, \qquad \alpha > 0. \tag{4.7}$$

In this case, just as in Theorem 3, but using Theorem 1 instead of Theorem 2, we can formulate a uniqueness criterion.

Theorem 4. *Let $0 < \gamma < 2$, $b > \gamma/2$, $\alpha > 0$, and let A be a linear closed operator in E. Suppose that the boundary-value problem* (4.1), (4.6), (4.7) *has a solution $v(t)$. For this solution to be unique, it is necessary and sufficient that no zero λ_m, $m \in \mathbb{N}$, of the function $Y_{k+2\alpha}(l;\lambda)$, where $k = b\nu - \nu + 1$, $\nu = 2/(2-\gamma)$, $l = \nu T^{1/\nu}$, given by by equality* (2.10)*, be an eigenvalue of the operator A.*

For the values of the parameter b, satisfying the inequality $\gamma/2 < b < 1$, Theorem 3 with the Dirichlet condition for $t = 0$, and Theorem 4 with the Neumann weight condition for $t = 0$ hold simultaneously.

Further, let us consider equation (4.1) in the case of strong degeneracy where $\gamma > 2$. The replacement of the independent variable and the unknown function

$$t = \left(-\frac{\tau}{\nu}\right)^{-\nu}, \qquad \nu = \frac{2}{2-\gamma}, \qquad v(t) = qv\left(\left(-\frac{\tau}{\nu}\right)^{-\nu}\right) = w(\tau)$$

reduces equation (4.1) to an Euler–Poisson–Darboux equation of the form

$$w''(\tau) + \frac{p}{\tau} w'(\tau) = Aw(\tau), \qquad 0 < \tau < l, \tag{4.8}$$

where $p = 1 + 2(b-1)/(\gamma-2)$, $l = -\nu T^{-1/\nu}$.

In the case of strong degeneracy, the setting of the boundary conditions at the point of degeneracy $t = 0$ also depends on the coefficient b. Uniqueness criteria for nonlocal problems for the Euler–Poisson–Darboux equation (4.8), to which the nonlocal problems considered above are reduced, are also contained in Theorems 1 and 2; therefore, just as for Theorems 3 and 4, we establish the following criteria.

Theorem 5. *Let $\gamma > 2$, $b < 1$, $\beta > 0$, let A be a closed linear operator in E, and let there exist a solution of equation* (4.1) *satisfying the conditions*

$$v(0) = 0, \qquad \int_0^T t^{\gamma-3}(1-\nu^2 t^{\gamma-2})^{\beta-1} v(t)\,dt = 0.$$

For the solution to be unique, it is necessary and sufficient that no zero λ_m, $m \in \mathbb{N}$, of the function $Y_{2\beta+2-p}(l;\lambda)$, where $p = 2(b-1)/(\gamma-2)+1$, $l = 2/(2-\gamma)T^{\gamma/2-1}$, given by equality (2.10) *be an eigenvalue of the operator A.*

Theorem 6. *Let $\gamma > 2$, $b > 2-\gamma/2$, $\alpha > 0$, let A be a linear closed operator in E, and let there exist a solution of equation* (4.1) *satisfying the conditions*

$$\lim_{t\to 0+} t^{2-\gamma/2} v'(t) = 0, \qquad \int_0^T t^{b+\gamma-4}(1-\nu^2 t^{\gamma-2})^{\alpha-1} v(t)\,dt = 0.$$

For the solution to be unique, it is necessary and sufficient that no zero λ_m, $m \in \mathbb{N}$, of the function $Y_{p+2\alpha}(l;\lambda)$, where $p = 2(b-1)/(\gamma-2)+1$, $l = 2/(2-\gamma)T^{\gamma/2-1}$, given by equality (2.10) *be an eigenvalue of the operator A.*

Finally, we establish a uniqueness criterion for an abstract analogue of a differential equation degenerating with respect to the space variable with power-law degeneration. For $\omega > 0$, we consider the equation

$$v''(t) = t^{\omega} A v(t), \qquad 0 < t < T \tag{4.9}$$

and, along with the Neumann condition at the point $t = 0$

$$v'(0) = 0, \tag{4.10}$$

we define the nonlocal condition

$$\int_0^T t^{\omega+2}(1-\mu^2 t^{2\omega})^{\alpha-1} v(t)\,dt = 0, \qquad \alpha > 0, \quad \mu = \frac{2}{\omega+2}\,. \tag{4.11}$$

If A is the operator of differentiation with respect to the space variable x, for example, the operator $Av(t,x) = v''_{xx}(t,x)$, then equation (4.9) is a degenerate hyperbolic equation, generalizing the Tricomi equation, but has a different type of degeneration as compared to the previous degenerating equations. Therefore,it is natural to call the abstract equation (3.9) a degenerating equation.

The replacement of the independent variable and the unknown function

$$t = \left(\frac{\tau}{\mu}\right)^{\mu}, \qquad \mu = \frac{2}{\omega+2}\,, \qquad v(t) = \left(\frac{\tau}{\mu}\right)^{\mu} w(\tau)$$

reduces equation (4.9) to an Euler–Poisson–Darboux equation of the form

$$w''(\tau) + \frac{\mu+1}{\tau}\, w'(\tau) = Aw(\tau), \qquad \tau > 0.$$

Since the parameter of the Euler–Poisson–Darboux equation satisfies inequality $k = \mu + 1 > 1$, using Theorem 1, we obtain the following criterion.

Theorem 7. *Let $\omega > 0$, $\alpha > 0$, let A be a closed linear operator in E, and let the boundary-value problem* (4.9)–(4.11) *have a solution $v(t)$. For this solution to be unique, it is necessary and sufficient, that no zero λ_m, $m \in \mathbb{N}$, of the function $Y_{q+2\alpha}(l;\lambda)$, where $q = (\omega+4)/(\omega+2)$, $l = 2T^{\omega/2+1}/(\omega+2)$, given by equality* (2.10) *be an eigenvalue of the operator A.*

5. NONLOCAL CONDITION OF THE SECOND KIND. THE CASE $k \geq 0$. THE NEUMANN CONDITION FOR $t = 0$

Instead of the nonlocal condition (2.2) for equation (1.1), we consider a condition of the form

$$a \int_0^1 t^k u(t)\, dt + bu'(1) = 0, \qquad a \neq 0, \quad b \neq 0. \tag{5.1}$$

A nonlocal condition of this kind for partial differential equations appeared earlier in [13], [14]. In this section, we will establish the corresponding uniqueness theorems for a solution satisfying condition (5.1).

Let $k \geq 0$. As in Sec. 2, in order to find nontrivial solutions of the homogeneous problem (1.1), (2.1), (5.1), for the function (2.11), we must choose the appropriate values of $\lambda \in \mathbb{C}$ so that the function $v(t) = Y_k(t; \lambda)$ satisfies the condition

$$a \int_0^1 t^k v(t)\, dt + bv'(1) = 0, \qquad a \neq 0, \quad b \neq 0.$$

Using the formula for the shift with respect to the parameter and the equality

$$Y_k'(t; \lambda) = \frac{\lambda t}{k+1}\, Y_{k+2}(t; \lambda), \tag{5.2}$$

we obtain the following transcendental equation for finding λ:

$$(a + b\lambda) Y_{k+2}(1; \lambda) = 0, \tag{5.3}$$

Note that, in order to obtain equation (5.3), we put $\alpha = 1$ in the Erdelyi–Kober operator $I^{\alpha}_{0+;2,\eta}$ appearing in condition (5.1) and, besides, as compared to equation (2.12), in addition to the zeros of the Bessel function, one more zero $\lambda_0 = -a/b$ appears. Therefore, the uniqueness criterion takes the following form.

Theorem 8. *Let $k \geq 0$, and let A, B be linear closed operators in E. Suppose that the nonlocal problem* (1.1)*,* (2.1)*,* (5.1) *has a solution $u(t)$. For this solution to be unique, it is necessary and, in the case $u(t) \in C^3((0,1], D)$, also sufficient that, for no λ_m, $m \in \mathbb{N}_0$, which is a zero of the function*

$$\Phi_{k,a,b}(\lambda) = (a + b\lambda) Y_{k+2}(1; \lambda), \tag{5.4}$$

where the function $Y_{k+2}(t; \lambda)$ is given by equality (2.10)*, does the operator equation* (2.14) *have a solution.*

Proof. *Necessity* is established just as for Theorem 1 and, for the proof of *sufficiency*, instead of the function $w(t)$ introduced in Theorem 1, we must use the function

$$w_{a,b}(t) = a \int_0^t s^k u(s)\, ds + \frac{b}{t}\, u'(t).$$

It is easy to verify that, for $u(t) \in C^3((0,1], D)$, the function $w_{a,b}(t)$ satisfies equation (1.1) after replacing the parameter k by $k + 2$. Then, as in Theorem 1, we obtain $w_{a,b}(t) \equiv 0$, $0 < t \leq 1$. Thus, after differentiating the function $w_{a,b}(t)$, in order to find $u(t)$, we obtain the following problem:

$$u''(t) - \frac{1}{t}\, u'(t) + \frac{a}{b}\, t^{k+1} u(t) = 0, \tag{5.5}$$

$$u'(0) = 0, \tag{5.6}$$

$$a \int_0^1 s^k u(s)\, ds + bu'(1) = 0. \tag{5.7}$$

The solution of the ordinary differential equation (5.5) in the Banach space E satisfying condition (5.6) has the form (see 2.162 (1a) [27])

$$u(t) = t J_\nu(z t^{k/2+3/2}) \eta, \qquad \nu = \frac{2}{k+3}\,, \qquad z = \frac{2}{k+3} \sqrt{\frac{a}{b}}\,, \qquad \eta \in E. \tag{5.8}$$

Substituting the function (5.8) into condition (5.7) and taking into account integral 1.8.1.21 from [21], we obtain

$$a\int_0^1 s^{k+1}J_\nu(zs^{k/2+3/2})\,ds\eta + b\biggl(J_\nu(z)+\frac{k+3}{2}\,zJ'_\nu(z)\biggr)\eta$$
$$=\biggl(-\sqrt{ab}J_{\nu-1}(z)+\frac{2^{2-\nu}a}{\Gamma(\nu)(k+3)z^{(2k+4)/(k+3)}}+bJ_\nu(z)+\frac{b(k+3)}{2}\,zJ'_\nu(z)\biggr)\eta$$
$$=\frac{2^{2-\nu}a}{\Gamma(\nu)(k+3)z^{(2k+4)/(k+3)}}\,\eta=0; \qquad (5.9)$$

here we have used the following well-known equality for the derivative of the Bessel function:

$$zJ'_\nu(z)=-\nu J_\nu(z)+zJ_{\nu-1}(z).$$

Equality (5.9) implies $\eta=0$ and, therefore, $u(t)\equiv 0$. □

6. NONLOCAL CONDITION OF THE SECOND KIND. THE CASE $k<1$. THE DIRICHLET CONDITION FOR $t=0$

Let $k<1$. In this case, for equation (1.1), instead of the nonlocal condition of the second kind (5.1), we must use the following condition of the form

$$a\int_0^1 tu(t)\,dt+b\lim_{t\to 1}(t^{k-1}u(t))'=0, \qquad a\neq 0, \quad b\neq 0. \qquad (6.1)$$

As in Sec. 2, in order to find the nontrivial solutions of the homogeneous problem (1.1), (3.1), (6.1), we must choose the appropriate values of $\lambda\in\mathbb{C}$ for the function (2.11) so that the function $v(t)=Y_k(t;\lambda)$ satisfies the condition

$$a\int_0^1 tv(t)\,dt+b\lim_{t\to 1}(t^{k-1}v(t))'=0, \qquad a\neq 0, \quad b\neq 0.$$

Using the formula for the shift with respect to the parameter and equality (5.7), we obtain the following transcendental equation for finding λ:

$$(a+b\lambda)Y_{4-k}(1;\lambda)=0, \qquad (6.2)$$

and the uniqueness statement takes the following form.

Theorem 9. *Let $k<1$, and let A, B be linear closed operators in E. Suppose that the nonlocal problem* (1.1), (3.1), (6.1) *has a solution $u(t)$. For this solution to be unique, it is necessary, and, in the case $u(t)\in C^3((0,1],D)$ also sufficient, that, for no λ_m, $m\in\mathbb{N}_0$, which is a zero of the function*

$$\Theta_{k,a,b}(\lambda)=(a+b\lambda)Y_{4-k}(1;\lambda), \qquad (6.3)$$

where the function $Y_{4-k}(t;\lambda)$ is given by equality (2.10)*, does the operator equation* (2.14) *have a solution.*

Proof. *Necessity* is established just as for Theorem 2, but, to prove *sufficiency*, we must replace the function $w(t)$ introduced in Theorem 2 by the function

$$w_{a,b}(t)=a\int_0^t su(s)\,ds+\frac{b}{t}(t^{k-1}u(t))'.$$

It is easy to verify that, for $u(t)\in C^3((0,1],D)$, the function $w_{a,b}(t)$ satisfies equation (1.1) after replacing the parameter k by $4-k$. For details about operators of motion with respect to the parameter of solutions of singular equations, see in [28]. Then, as in Theorem 2, we obtain $w_{a,b}(t)\equiv 0$, $0<t\leq 1$.

Thus, after differentiating the function $w_{a,b}(t)$, we obtain the following problem for finding $u(t)$:

$$t^2u''(t)+(2k-3)tu'(t)+\biggl(\frac{a}{b}\,t^{5-k}+(k-1)(k-3)\biggr)u(t)=0, \qquad (6.4)$$

$$u(0)=0, \tag{6.5}$$

$$a\int_0^1 tu(t)\,dt+b\lim_{t\to 1}(t^{k-1}u(t))'=0. \tag{6.6}$$

The solution of the ordinary differential equation (6.4) in the Banach space E satisfying condition (6.5) has the form (see 2.162 (1a) [27])

$$u(t)=t^{2-k}J_\nu(zt^{5/2-k/2})\eta,\qquad \nu=\frac{2}{5-k}\,,\qquad z=\frac{2}{5-k}\sqrt{\frac{a}{b}}\,,\qquad \eta\in E. \tag{6.7}$$

Substituting the function (6.7) into condition (6.6) and taking into account integral 1.8.1.21 from [21], we obtain

$$\begin{aligned} &a\int_0^1 t^{3-k}J_\nu(zt^{5/2-k/2})\,dt\eta+b\biggl(J_\nu(z)+\frac{5-k}{2}\,zJ'_\nu(z)\biggr)\eta\\ &\quad=\biggl(-\sqrt{ab}J_{\nu-1}(z)+bJ_\nu(z)+\frac{b(5-k)}{2}\,zJ'_\nu(z)\biggr)\eta+\frac{2^{2-\nu}a}{\Gamma(\nu)(5-k)z^{(8-2k)/(5-k)}}\,\eta\\ &\quad=\frac{2^{2-\nu}a}{\Gamma(\nu)(5-k)z^{(8-2k)/(5-k)}}\,\eta=0. \end{aligned}\tag{6.8}$$

Equality (6.8) implies $\eta=0$ and, therefore, $u(t)\equiv 0$. The theorem is proved. □

As applications of Theorems 8 and 9, we can obtain, just as in Sec. 4, the corresponding theorems for degenerating differential equations with power-law degeneration.

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