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REFLECTION GROUPS OF RANK THREE AND SYSTEMS OF UNIFORMIZATION EQUATIONS

Jiro Sekiguchi

Department of Mathematics,Faculty of Engineering, Tokyo University of Agriculture and Technology, Koganei, Tokyo, 184-8588, Japan, e-mail: sekiguti@cc.tuat.ac.jp

Abstract. In this note, I survey recent progress on systems of uniformization equations along Saito free divisors. In particular, I show concrete forms of such systems along Saito free divisors which are defined as the zero sets of discriminants of complex reflection groups of rank three. A part of the results in this note is a joint work with M. Kato (Univ. Ryukyus)

Keywords: uniformization equations, discriminants of complex reflection groups, Saito free divisors.

Introduction

The notion of Saito free divisors was introduced by K. Saito (cf. [10]). He also formulated and stressed the importance of systems of uniformization equations along such divisors (cf. [9]). In this note, I explain recent progress on systems of uniformization equations along Saito free divisors defined as the zero sets of the discriminants of complex reflection groups of rank three. The hypersurface defined by the discriminant of a real reflection group is a typical example of Saito free divisors. It is known that the discriminant of a complex reflection group of rank three is also a Saito free divisor (cf. [8], [5]). But I don't know whether it is true for the case of arbitrary complex reflection groups. My interests on this subject are to construct (1) Saito free divisors, (2) systems of uniformization equations, and (3) their solutions in a concrete manner. Restricting to the case of three dimensional affine space, I obtained some results on (1), (2). But it is difficult to attack (3) compared with (1), (2). The purpose of this note is to report my results on (1), (2) for the discriminants of irreducible complex reflection groups of rank three. A part of the results of the last three sections are obtained by a joint research with M. Kato (Univ. Ryukyus).

1 Definition of Saito free divisors

Let $F(x) = F(x_1, x_2, \dots, x_n)$ be a reduced polynomial. Then $D = \{x \in \mathbf{C}^n;\ F(x) = 0\}$ is a (weighted homogeneous) Saito free divisor if (C1)+(C2) hold.

(C1) There is a vector field

$$E = \sum_{i=1}^{n} m_i x_i \partial_{x_i}$$

such that $EF = dF$, where $m_1, m_2, \dots, m_n, d$ are positive integers with $0 < m_1 \le m_2 \le \cdots \le m_n$.

(C2) There are vector fields

$$V^i = \sum_{j=1}^{n} a_{ij}(x)\partial_{x_j}\ (i = 1, 2, \dots, n)$$

such that

(i) each $a_{ij}(x)$ is a polynomial of $x_1, x_2, \dots, x_n$,
(ii) $\det(a_{ij}(x)) = cF(x)$ for a non-zero constant c,
(iii) $V^1 = E$, $V^i F(x) = c_i(x)F(x)$ for polynomials $c_i(x)$,
(iv) $[E, V^i] = k_i V^i$ for some constants k_i,
(v) V^i $(j = 1, 2, \dots, n)$ form a Lie algebra over $R = \mathbf{C}[x_1, x_2, \dots, x_n]$

We now give examples of Saito free divisors.

Let

$$f(t) = t^n + x_2 t^{n-2} + x_3 t^{n-3} + \cdots + x_{n-1} t + x_n$$

be a polynomial of nth degree and let $\Delta(x_2, x_3, \dots, x_n)$ be the discriminant of $f(t)$. Then $\Delta = 0$ is a Saito free divisor in $\mathbf{C}^{n-1}$.

More generally, the zero locus of the discriminant of an irreducible real reflection group is a Saito free divisor.

Basic reference of this section is [10].

2 Irreducible complex reflection groups of rank three.

In this section, we collect some results on irreducible complex reflection groups of rank three. A basic reference on complex reflection groups is Shephard-Todd [16] (see also [8]).

Reflection groups treated in this section are real reflection groups of types A_3, B_3, H_3 and complex reflection groups of No.24,No.25,No.26,No.27 in the sense of [16]. The real reflection group of type H_3 is same as the group No. 23 in [16].

Let G be one of the seven groups and let P_1, P_2, P_3 algebraically independent basic G-invariant polynomials and put $k_j = \deg_\xi(P_j)$. We may assume that $k_1 \le k_2 \le k_3$. Let r be the greatest common divisor of k_1, k_2, k_3 and put $k'_j = k_j / r$ $(j = 1, 2, 3)$. For the later convenience, we write x_1, x_2, x_3 for P_1, P_2, P_3. Let $\delta_G(x_1, x_2, x_3)$ be the discriminant of G expressed as a polynomial of x_1, x_2, x_3.

In the cases A_3, B_3, H_3, taking G-invariants x_1, x_2, x_3 suitably, $F_{W(A_3)}(x_1, x_2, x_3)$, $F_{W(B_3)}(x_1, x_2, x_3)$, $F_{W(H_3)}(x_1, x_2, x_3)$ are discriminants for G up to a constant factor, respectively, where $F_{A,1}, F_{B,1}, F_{H,1}$ are the polynomials given in Theorem of [13].

	group	order	k_1, k_2, k_3	degree	(k'_1, k'_2, k'_3)
A_3	$W(A_3)$	24	$2, 3, 4$	12	$(2, 3, 4)$
B_3	$W(B_3)$	48	$2, 4, 6$	18	$(1, 2, 3)$
H_3	$W(H_3)$	120	$2, 6, 10$	30	$(1, 3, 5)$
No.24	G_{336}	336	$4, 6, 14$	42	$(2, 3, 7)$
No.25	G_{648}	648	$6, 9, 12$	36	$(2, 3, 4)$
No.26	G_{1296}	1296	$6, 12, 18$	54	$(1, 2, 3)$
No.27	G_{2160}	2160	$6, 12, 30$	90	$(1, 2, 5)$

The concrete forms of discriminants of $W(A_3)$, $W(B_3)$, $W(H_3)$ are as follows:

Type A_3: $16x_1^4x_3 - 4x_1^3x_2^2 - 128x_1^2x_3^2 + 144x_1x_2^2x_3 - 27x_2^4 + 256x_3^3$.

Type B_3: $x_3(x_1^2x_2^2 - 4x_2^3 - 4x_1^3x_3 + 18x_1x_2x_3 - 27x_3^2)$.

Type H_3: $-50x_3^3 + (4x_1^5 - 50x_1^2x_2)x_3^2 + (4x_1^7x_2 + 60x_1^4x_2^2 + 225x_1x_2^3)x_3 - \frac{135}{2}x_2^5$
$-115x_1^3x_2^4 - 10x_1^6x_2^3 - 4x_1^9x_2^2$.

The discriminants for the groups No.25, No.26 are same as those for $W(A_3)$, $W(B_3)$ respectively by taking the basic invariants suitably. On the other hand, those for the groups No.24, No.27 will be given in §5, §.6.

3 Systems of uniformization equations along Saito free divisors.

Let $F(x) = 0$ be a Saito free divisor and let $V^j\,(j = 1, 2, \ldots, n)$ be basic vector fields logarithmic along $F = 0$. Let $u = u(x_1, x_2, \ldots, x_n)$ be an unknown function. Assume that $V^1u = su$ for a constant $s(\neq 0)$ Put $\vec{u} = {}^t(u, V^2u, \ldots, V^nu)$ and consider the system of differential equations

(UE) $V^j\vec{u} = A_j(x)\vec{u} \quad (j = 1, 2, \ldots, n)$

where $A_j(x)\,(j = 1, 2, \ldots, n)$ are $n \times n$ matrices whose entries are polynomials of x.

If (UE) is integrable, it is called a system of uniformization equations with respect to the Saito free divisor $F = 0$.

The system (UE) is written by

$$\text{(UEa)} \quad \begin{cases} V^1u = su \\ V^iV^ju = \displaystyle\sum_{k=1}^{n} h_{ij}^k(x)V^ku \quad (\forall i, j) \end{cases}$$

where $h_{ij}^k(x)$ are polynomials of x. There are n number of fundamental solutions of (UEa). Let $u_j(x)\,(j = 1, 2, \ldots, n)$ be fundamental solutions outside the divisor $F = 0$. Then

$$\varphi(x) = (u_1(x), u_2(x), \ldots, u_n(x))$$

defines a map of $\mathbf{C}^n - \{F = 0\}$ to $\mathbf{C}^n$. The following two problems are fundamental in the study on systems of uniformization equations.

PROBLEM 1: Construct fundamental solutions $u_j(x)\,(j = 1, 2, \ldots, n)$ of (UEa).

PROBLEM 2: Construct the inverse of $\varphi(x)$ in a concrete manner.

These two problems are solved in the case $W(A_3)$ for a special but interesting system of uniformization equations by K. Saito. For the details of the results, see [9].

4 The discriminant of the Coxeter group $W(H_3)$ of type H_3.

A part of the argument in the case of $W(A_3)$ in [9] is applicable to the case of the Saito free divisor defined by the zero locus of the discriminant of the Coxeter group of type H_3. In this section, I will explain the results on this case. For the details of results in this section, see [15].

The discriminant of the polynomial $P(t)$ defined by

$$P(t) = t^6 + y_1t^5 + y_2t^3 + y_3t + \frac{1}{20}y_2^2 - \frac{1}{4}y_1y_3 \tag{4.1}$$

is Δ^2 up to a constant factor, where

$$\Delta = 125y_1^3y_2^4 + 864y_2^5 - 1250y_1^4y_2^2y_3 - 9000y_1y_2^3y_3 + 3125y_1^5y_3^2 + 25000y_1^2y_2y_3^2 + 50000y_3^3. \tag{4.2}$$

Remark 4.1 *The equation $P(t) = 0$ is essentially same as "Die allgemeine Jacobi'sche Gleichung sechsten Grades" (see p.223 in Klein's book [7]).*

The polynomial Δ is regarded as the discriminant of the group $W(H_3)$. In fact, the substitution of the variables (y_1, y_2, y_3) with (x_1, x_2, x_3) defined by the relations

$$\left\{ \begin{array}{lcl} y_1 & = & -4x_1 \\ y_2 & = & 10x_1^3 - 25x_2 \\ y_3 & = & -4x_1^5 + 50x_1^2x_2 - 50x_3 \end{array} \right. \tag{4.3}$$

implies that Δ coincides with the determinant of the matrix M up to a constant factor, where M is defined by

$$M = \begin{pmatrix} x_1 & 3x_2 & 5x_3 \\ 3x_2 & 2x_3 + 2x_1^2x_2 & 7x_1x_2^2 + 2x_1^4x_2 \\ 5x_3 & 7x_1x_2^2 + 2x_1^4x_2 & \frac{1}{2}(15x_2^3 + 4x_1^4x_3 + 18x_1^3x_2^2) \end{pmatrix} \tag{4.4}$$

and $\det M$ is the discriminant of $W(H_3)$ (cf. [17]). In the sequel, we always regard $P(t)$ as a polynomial of t and x. The hypersurface defined as the zero set of the polynomial $f_0 = \det M$ is an example of Saito free divisors. To show this, we define vector fields V_0, V_1, V_2 by

$$\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = M \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}$$

Then we have

$$[V_0, V_1] = 2V_1, \ [V_0, V_2] = 4V_2,$$

$$[V_1, V_2] = (4x_1^3x_2 + 2x_2^2)V_0 + 4x_1x_2V_1$$

and

$$\begin{array}{l} V_0f_0 = 15f_0, \\ V_1f_0 = 2x_1^2f_0, \\ V_2f_0 = 2x_1(2x_1^3 + 5x_2)f_0 \end{array}$$

Remark 4.2 *We note that*

$$P(-x_1) = \frac{125}{4}x_2^2. \tag{4.5}$$

This implies that $(-x_1, 5\sqrt{5}/2 \cdot x_2)$ *is a point on the hyperelliptic curve* $s^2 = P(t)$ *on* (s, t) *plane.*

Consider the system of differential equations

$$V_i \begin{pmatrix} u \\ V_1u \\ V_2u \end{pmatrix} = B_{i+1} \begin{pmatrix} u \\ V_1u \\ V_2u \end{pmatrix} \quad (i = 0, 1, 2) \tag{4.6}$$

The system (4.6) is a system of uniformization equations along the Saito free divisor $f_0(x) = 0$. Here B_j are defined as follows:

$$B_1 = \begin{pmatrix} s_0 & 0 & 0 \\ 0 & 2+s_0 & 0 \\ 0 & 0 & 4+s_0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{2}{225}x_1 \left\{ \begin{matrix} (8+70s_1-100s_1^2+8s_0 \\ +35s_1s_0+2s_0^2)x_1^3 \\ +(-180+825s_1-750s_1^2 \\ -90s_0+75s_1s_0)x_2 \end{matrix} \right\} & \frac{1}{15}(8+5s_1+4s_0)x_1^2 & s_1 \\ \frac{1}{900} \left\{ \begin{matrix} (-128+80s_1+100s_1^2-128s_0 \\ +40s_1s_0-32s_0^2)x_1^6 \\ +10(-32-400s_1+550s_1^2+328s_0 \\ -20s_1s_0-8s_0^2)x_1^3x_2 \\ +(-4500s_1+5625s_1^2+1800s_0)x_2^2 \\ +(2400-3000s_1+1200s_0)x_1x_3 \end{matrix} \right\} & \frac{1}{15}x_1 \left\{ \begin{matrix} (8+5s_1+4s_0)x_1^3 \\ +10(8+5s_1+s_0)x_2 \end{matrix} \right\} & \frac{1}{15}(4-5s_1+2s_0)x_1^2 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{900} \left\{ \begin{matrix} (-128+80s_1+100s_1^2-128s_0 \\ +40s_1s_0-32s_0^2)x_1^6 \\ +10(-32-400s_1+550s_1^2-32s_0 \\ -20s_1s_0-8s_0^2)x_1^3x_2 \\ +s_1(-4500+5625s_1)x_2^2 \\ +100(24-30s_1+12s_0)x_1x_3 \end{matrix} \right\} & \frac{1}{15}x_1 \left\{ \begin{matrix} (8+5s_1+4s_0)x_1^3 \\ +10(2+5s_1+s_0)x_2 \end{matrix} \right\} & \frac{1}{15}(4-5s_1+2s_0)x_1^2 \\ \frac{1}{450} \left\{ \begin{matrix} (-128+80s_1+100s_1^2-128s_0 \\ +40s_1s_0-32s_0^2)x_1^8 \\ +(80-500s_1+500s_1^2-280s_0 \\ +200s_1s_0-160s_0^2)x_1^5x_2 \\ +25(-104-130s_1+325s_1^2+40s_0 \\ +25s_1s_0-8s_0^2)x_1^2x_2^2 \\ +100(12-15s_1+24s_0)x_1^3x_3 \\ +50(60-75s_1+30s_0)x_2x_3 \end{matrix} \right\} & \frac{1}{4}(4+5s_1)x_2(4x_1^3+5x_2) & \frac{x_1}{15} \left\{ \begin{matrix} (16-5s_1+8s_0)x_1^3 \\ +(40-50s_1+20s_0)x_2 \end{matrix} \right\} \end{pmatrix}$$

Remark 4.3 *In the case $s_0 = \frac{1}{2}$, $s_1 = 1$, the monodromy group of the system $V_j\vec{u} = B_{j+1}\vec{u}\,(j = 0,1,2)$ coincides with $W(H_3)$. This case is treated in Haraoka-Kato [6].*

We consider the system $V_j\vec{u} = B_{j+1}\vec{u}\ (j = 0,1,2)$ with $s_0 = -2$, $s_1 = 0$. Then we obtain

$$\left\{ \begin{array}{lcl} V_0v & = & -2v \\ V_1V_1v & = & 0 \\ V_2V_1v & = & 0 \\ V_2V_2v & = & -4x_1^2(3x_2^2+2x_1x_3)v + x_2(4x_1^3+5x_2)V_1v \end{array} \right. \qquad (4.7)$$

Theorem 4.4 *(cf. [15]) The function $v(x)$ defined by*

$$v(x) = \int_{\infty}^{-x_1} P(t)^{-1/2}dt$$

is a solution of (4.7).

The proof of this theorem is given by an argument similar to the case of type A_3.
If $u(x)$ is a solution of (4.7) such that $V_1u=0$, then u is a solution of

$$\begin{cases} V_0u=-2u \\ V_1u=0 \\ \{V_2^2+4x_1^2(3x_2^2+2x_1x_3)\}u=0 \end{cases} \tag{4.8}$$

Taking two paths C_1, C_2 appropriately and define

$$w_j(x)=\int_{C_j}\varphi_1(t)dt \quad (j=1,2)$$

Then each $w_j(x)$ is also a solution of (4.7) and in this manner we can construct solutions of (4.8). In this case it is not clear whether solutions of (4.8) are expressed by special functions or not. Moreover **PROBLEM 2** (the construction of the inverse mapping) is still open.

5 The discriminant of the group G_{336}, Shephard-Todd notation No.24.

In this case, we begin with defining the polynomial

$$\begin{aligned} P(t) &= t^7-\tfrac{7}{2}(c_1-1)x_2t^5-\tfrac{7}{2}(c_1-1)x_3t^4-7(c_1+4)x_2^2t^3-14(c_1+2)x_2x_3t^2 \\ &\quad +\tfrac{7}{2}\{(3c_1-7)x_2^3-(c_1+5)x_3^2\}t+\tfrac{1}{2}(7c_1-131)x_2^2x_3+x_7 \\ &\quad (c_1^2=-7) \end{aligned}$$

The discriminant of $P(t)$ is f_0^2 up to a constant factor, where

$$\begin{aligned} f_0 &= 2048x_2^9x_3-22016x_2^6x_3^3+60032x_2^3x_3^5-1728x_3^7+256x_2^7x_7-1088x_2^4x_3^2x_7 \\ &\quad -1008x_2x_3^4x_7+88x_2^2x_3x_7^2-x_7^3 \end{aligned}$$

and f_0 is the discriminant of the complex reflection group G_{336}. (The polynomial f_0 is same as the one shown in p.262 of the paper of A. Adler in the book "*The Eightfold Way*"by $x_2\to f$, $x_3\to\nabla$, $x_7\to C$. The polynomial $P(t)$ is given in p.406 of GMA of F. Klein, Band II.)

Define vector fields V_0, V_1, V_2 by

$${}^t(V_0,V_1,V_2)=M^t(\partial_{x_2},\partial_{x_3},\partial_{x_7})$$

Then V_0, V_1, V_2 form the generators of logarithmic vector fields along $f_0=0$. Here

$$M=\begin{pmatrix} 2x_2 & 3x_3 & 7x_7 \\ x_3^2 & -\frac{1}{12}x_7 & -\frac{4}{3}x_2(28x_2^3x_3-128x_3^3+3x_2x_7) \\ 7x_7 & -56x_2(2x_2^3-13x_3^2) & 28(32x_2^6-40x_2^3x_3^2-84x_3^4+59x_2x_3x_7) \end{pmatrix}$$

Put

```
A0={{s0,0,0},{0,s0+4,0},{0,0,s0+5}};
A1={{0,1,0},{1/162*x2*(4*(-1+c4-s0)*(8+c4+2*s0)*x2^3-
    3*(24+43*c4+5*c4^2+24*s0+
     19*c4*s0)*x3^2),1/9*(-10+c4-4*s0)*x2^2,c4*x3/504},
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    {-7/54*(8*(-152+37*c4+7*c4^2-172*s0-14*c4*s0-38*s0^2)*x2^3*x3-
    18*(8*c4+c4^2+76*s0)*x3^3+3*(8+c4+38*s0)*x2*x7),
    -14/3*(-20+5*c4-38*s0)*x2*x3,-1/9*(8+c4+2*s0)*x2^2}};
A2={{0,0,1},
    {-7/54*(8*(-152+37*c4+7*c4^2-190*s0-14*c4*s0-38*s0^2)*x2^3*x3-18*c4*(8+
     c4)*x3^3+3*(8+c4+8*s0)*x2*x7),-14/3*(-152+5*c4-38*s0)*x2*x3,
     -1/9*(2+c4+2*s0)*x2^2},
    {98/9*(48*(-24+5*c4+c4^2-36*s0-c4*s0)*x2^5+4*(-440+97*c4+19*c4^2-658*s0
    -89*c4*s0-722*s0^2)*x2^2*x3^2+3*(8+c4+38*s0)*x3*x7),
    -1176*(-2+c4)*(2*x2^3-x3^2),14/3*(190+5*c4+76*s0)*x2*x3}};
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There is a system of differential equation of rank three defined by

$$V_j \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} = A_j \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} \quad (j = 0, 1, 2)$$

This system has two parameters s_0, c_4.

Substituting $s_0 = -1, c_4 = 0$ in A_j, we obtain $A_j^{(0)}$;

$$A_0^{(0)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad A_1^{(0)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{2}{3}x_2^2 & 0 \\ \frac{7}{3}(8x_2^3 x_3 - 76x_3^3 + 5x_2 x_7) & -84x_2 x_3 & -\frac{2}{3}x_2^2 \end{pmatrix},$$

$$A_2^{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 532x_2 x_3 & 0 \\ 196(32x_2^5 - 112x_2^2 x_3^2 - 5x_3 x_7) & 2352(2x_2^3 - x_3^2) & 532x_2 x_3 \end{pmatrix}$$

The system

$$V_j \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} = A_j^{(0)} \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} \quad (j = 0, 1, 2)$$

has a quotient which is defined by $V_1 u = 0$. Assuming $V_1 u = 0$, the system for $\begin{pmatrix} u \\ V_2 u \end{pmatrix}$ turns out to be

$$\begin{cases} V_0 \begin{pmatrix} u \\ V_2 u \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \\ V_1 \begin{pmatrix} u \\ V_2 u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{7}{3}(8x_2^3 x_3 - 76x_3^3 + 5x_2 x_7) & -\frac{2}{3}x_2^2 \end{pmatrix} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \\ V_2 \begin{pmatrix} u \\ V_2 u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 196(32x_2^5 - 112x_2^2 x_3^2 - 5x_3 x_7) & 532x_2 x_3 \end{pmatrix} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \end{cases} \quad (5.1)$$

We now study the restriction of the system (5.1) to the hyperplane $x_2 = 0$. Then we obtain an ordinary differential equation

$$\left(\partial_{x_7}^2 + \frac{18x_7^2}{7(1728x_3^7 + x_7^3)} \partial_{x_7} + \frac{10x_7}{49(1728x_3^7 + x_7^3)} \right) u = 0$$

One of its solutions is

$$x_3^{-1/3} F\left(\frac{1}{21}, \frac{10}{21}; \frac{2}{3}; -\frac{x_7^3}{1728 x_3^7}\right)$$

Similarly as the restriction to $x_3 = 0$ of (5.1), we obtain an ordinary differential equation

$$\left(\partial_{x_7}^2 - \frac{256 x_2^7 + 11 x_7^2}{7 x_7 (256 x_2^7 - x_7^2)} \partial_{x_7} + \frac{3}{49(-256 x_2^7 + x_7^2)}\right) u = 0$$

One of its solutions is

$$x_2^{-1/2} F\left(\frac{1}{14}, \frac{3}{14}; \frac{3}{7}; \frac{x_7^2}{256 x_2^7}\right)$$

Remark 5.1 *We note that, in the case $c_4 = -9$, $s_0 = 1/2$, the system of differential equations has a monodromy group isomorphic to G_{336}. This case is treated in [6].*

6 The discriminant of the group G_{2160}, Shephard-Todd notation No.27.

Consider the polynomial

$$P(t) = t^6 + y_1 t^5 + y_2 t^4 + y_3 t^3 + y_4 t^2 + y_5 t + y_6.$$

Substitute y_j $(j = 1, 2, \ldots, 6)$ by x_j $(j = 1, 2, \ldots, 6)$;

```
y1=x1
y2=(5/16)*(9 + sr)*x2,
y3=(5/64)*(11 + 3*sr)*x1*x2,
y4=(5/512)*(37 + 45*sr)*x2^2,
y5=(61 + 5*sr)*(-64*x1^3*x2 + 373*x1*x2^2 + 15*sr*x1*x2^2 + 2*x3))/12288,
y6=(-279 + 145*sr)*(-512*x1^4*x2 + 2864*x1^2*x2^2 +
 1425*x2^3 + 135*sr*x2^3 +16*x1*x3)/3538944,
```

where $sr^2 = -15$. Then f_0^2 the discriminant of the polynomial $P(t)$, where

$$\begin{aligned} f_0 &= 65536 x_1^{11} x_2^2 - 1765376 x_1^9 x_2^3 + 17406016 x_1^7 x_2^4 - 73887360 x_1^5 x_2^5 + 107371008 x_1^3 x_2^6 \\ &\quad + 34338816 x_1 x_2^7 - 4096 x_1^8 x_2 x_3 + 96640 x_1^6 x_2^2 x_3 - 707952 x_1^4 x_2^3 x_3 + 1622592 x_1^2 x_2^4 x_3 \\ &\quad + 186624 x_2^5 x_3 + 64 x_1^5 x_3^2 - 1584 x_1^3 x_2 x_3^2 + 7128 x_1 x_2^2 x_3^2 + 9 x_3^3 \end{aligned}$$

up to a constant factor.

Remark 6.1 *By direct computation, we find that*

$$P\left(\frac{(3 - 5sr)}{72} x_1\right) = \frac{5(-45 + 11sr)}{1152} \left\{x_2 - \frac{(39 - sr)}{216} x_1^2\right\}^3$$

This means that

$$\left(\frac{(3 - 5sr)}{72} x_1, \left(\frac{5(-45 + 11sr)}{1152}\right)^{1/3} \left\{x_2 - \frac{(39 - sr)}{216} x_1^2\right\}\right)$$

is a point on the trielliptic curve $s^3 = P(t)$ on (s, t) plane.

The polynomial f_0 is regarded as the discriminant of the complex reflection group No.27. In particular, f_0 is obtained as the determinant of the matrix

$$M = \begin{pmatrix} x_1 & 2x_2 & 5x_3 \\ x_2^2 & \frac{1}{432}(144x_1x_2^2 - x_3) & \begin{matrix}\frac{1}{108}(640x_1^6x_2 - 9388x_1^4x_2^2 + 36600x_1^2x_2^3 \\ -19872x_2^4 - 28x_1^3x_3 + 307x_1x_2x_3)\end{matrix} \\ x_3 & \begin{matrix}\frac{1}{135}(-1920x_1^4x_2 + 8724x_1^2x_2^2 \\ +16416x_2^3 + 139x_1x_3)\end{matrix} & \begin{matrix}-\frac{4}{135}x_1(65920x_1^6x_2 - 887092x_1^4x_2^2 + 2886120x_1^2x_2^3 \\ +367632x_2^4 - 2692x_1^3x_3 + 20533x_1x_2x_3)\end{matrix} \end{pmatrix}.$$

It can be shown that f_0 coincides with the polynomial gk13 in my notation by a weight preserving coordinate change in the notation of my note. We define vector fields V_0, V_1, V_2 by

$$\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = M \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}$$

Then V_0, V_1, V_2 form generators of the logarithmic vector fields along the set $f_0 = 0$ in the (x_1, x_2, x_3)-space. By direct computation, we have

$$[V_1, V_2] = \frac{1}{540}(3200x_1^5x_2 - 16412x_1^3x_2^2 - 18056x_1x_2^3 - 80x_1^2x_3 - 307x_2x_3)V_0$$

$$-\frac{8}{135}(474x_1^4 - 4102x_1^2x_2 + 7209x_2^2)V_1 - \frac{1}{54}x_1(6x_1^2 - 73x_2)V_2$$

We consider the system of differential equations

$$V_j \begin{pmatrix} u \\ V_1u \\ V_2u \end{pmatrix} = A_j \begin{pmatrix} u \\ V_1u \\ V_2u \end{pmatrix} \quad (j = 0, 1, 2)$$

where A_0, A_1, A_2 are matrices of rank three defined as follows.

```
A0={{s0,0,0},{0,3+s0,0},{0,0,4+s0}};
A1={{0,1,0},{1/2099520*(320*(1+1728*h1-4*s0)*(3+864*h1+s0)*x1^6-
    120*(47+58752*h1-280*s0)*(3+864*h1+s0)*x1^4*x2+36*(2115+
    2967840*h1+679311360*h1^2-15870*s0-2515104*h1*s0-
    6125*s0^2)*x1^2*x2^2-3888*(15+30240*h1+7464960*h1^2-175*s0+
    28512*h1*s0)*x2^3+45*(3+864*h1-35*s0)*x1*x3),-1/810*x1*((55+
    42768*h1+40*s0)*x1^2-3*(255+83376*h1+175*s0)*x2),-1/6*h1*(x1^2-
    6*x2)},{1/656100*(-25280*(1+1728*h1-4*s0)*(3+864*h1+s0)*x1^7
    +120*(9879+16458336*h1+3920596992*h1^2-42907*s0-16232832*h1*s0-
    17560*s0^2)*x1^5*x2-36*(123660+210094560*h1+50250378240*h1^2-
    721860*s0-210720096*h1*s0-308135*s0^2)*x1^3*x2^2+1944*(345+3546720*h1+
    992839680*h1^2-14815*s0+6258816*h1*s0-5775*s0^2)*x1*x2^3-45*(147+
    42336*h1-1625*s0)*x1^2*x3-3645*(5+1440*h1+44*s0)*x2*x3),(4*(316*(-15+
    25704*h1+10*s0)*x1^4-3*(-16645+15303168*h1+8125*s0)*x1^2*x2-2430*(56+
    1440*h1-11*s0)*x2^2))/2025,(x1*(2*(-85+42768*h1-20*s0)*x1^2-3*(-715
    +166752*h1-175*s0)*x2))/1620}};
A2={{0,0,1},{1/656100*(-25280*(1+1728*h1-4*s0)*(3+864*h1+s0)*x1^7+120*(9879+
```

```
16458336*h1+3920596992*h1^2-75307*s0-16232832*h1*s0-17560*s0^2)*x1^5*x2-
36*(123660+210094560*h1+50250378240*h1^2-1275765*s0-210720096*h1*s0-
308135*s0^2)*x1^3*x2^2+1944*(345+3546720*h1+992839680*h1^2-3530*s0+
6258816*h1*s0-5775*s0^2)*x1*x2^3-45*(147+42336*h1-3785*s0)*x1^2*x3-
6075*(3+864*h1-35*s0)*x2*x3),4*(632*(15+12852*h1+5*s0)*x1^4-3*(24375+
15303168*h1+8125*s0)*x1^2*x2-2430*(-33+1440*h1-11*s0)*x2^2)/2025,x1*(2*
(5+42768*h1-20*s0)*x1^2-3*(15+166752*h1-175*s0)*x2)/1620},{1/820125*(4*
(1997120*(1+1728*h1-4*s0)*(3+864*h1+s0)*x1^8-120*(680901+1246968864*h1+
302650380288*h1^2-3612193*s0-1421635968*h1*s0-1027000*s0^2)*x1^6*x2+
36*(6514065+14746523040*h1+3706696028160*h1^2-67397805*s0-17324851104*h1*
s0-16957205*s0^2)*x1^4*x2^2-972*(-235065+392096160*h1+132420925440*h1^2
-3712385*s0+1286813088*h1*s0-1072500*s0^2)*x1^2*x2^3+1574640*(5+1440*h1
-11*s0)*(-5+8640*h1+33*s0)*x2^4+45*(4503+1296864*h1-144155*s0)*x1^3*x3+
3645*(740+213120*h1+5891*s0)*x1*x2*x3)),-16*(49928*(-25+60048*h1)*x1^5-
192*(-44815+84795282*h1)*x1^3*x2-1620*(4469+1982880*h1)*x1*x2^2
+180225*x3)/10125,-8*(632*(-10+6426*h1-5*s0)*x1^4-3*(-16250+7651584*h1-
8125*s0)*x1^2*x2-2430*(22+720*h1+11*s0)*x2^2)/2025}};
```

Remark 6.2 *A_1, A_2, A_3 contain parameters s_0, h_1. The determination of A_1, A_2, A_3 was accomplished by Masayuki Noro (Kobe Univ.).*

<u>The case $s_0 = \frac{1}{6}$, $h_1 = -\frac{19}{5184}$</u>

In this case the monodromy group of the system of differential equations becomes G_{2160}. This case is treated in [6].

<u>The case $s_0 = -3$, $h_1 = 0$</u>

In this case there is a quotient of the system above. In fact,

$$\left\{ \begin{array}{ccc} V_1 u & = & \frac{1}{162} x_1 (13x_1^2 - 162x_2) u \\ V_j \begin{pmatrix} u \\ V_2 u \end{pmatrix} & = & B_{j+1} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \quad (j = 0, 1, 2) \end{array} \right.$$

is a quotient of the system $V_j \vec{u} = A_j \vec{u}$ $(j = 0, 1, 2)$ defined above, where B_j $(j = 0, 1, 2)$ are matrices of rank two defined below:

```
B0={{-3,0},{0,1}};
B1={{1/162*x1*(13*x1^2-162*x2),0},
   {(-98592*x1^7+1926304*x1^5*x2-10970316*x1^3*x2^2+17754552*x1*x2^3
   -15066*x1^2*x3+30861*x2*x3)/43740,(-1350*x1^3+15390*x1*x2)/43740}};
B2={{0,1},{-1/164025*(4*(-6490640*x1^8+180214176*x1^6*x2-999084132*x1^4*x2^2+
    712058040*x1^2*x2^3+1244595456*x2^4-2995542*x1^3*x3+665577*x1*x2*x3)),
    -4*(511920*x1^4-3948750*x1^2*x2+4330260*x2^2)/164025}};
```

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ГРУППЫ ОТРАЖЕНИЙ РАНГА ТРИ И СИСТЕМЫ УРАВНЕНИЙ ДИФОРМАЛИЗАЦИИ

Джиро Секигучи

Токийский университет сельского хозяйства и технологии,
Коганей, Токио, 184-8588, Япония, e-mail: sekiguti@cc.tuat.ac.jp

Аннотация. В работе приведены результаты, описывающие дискриминант неприводимых комплексных групп отражения ранга три.

Ключевые слова: неприводимые уравнения, дискриминант комплексных групп отражения.