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RESIDUES OF LOGARITHMIC DIFFERENTIAL FORMS

A.G. Aleksandrov

Institute for Control Sciences, Russian Academy of Sciences,
Profsojuznaja str., 65, Moscow, 117997, Russian Federation, e-mail: ag_aleksandrov@mail.ru

Abstract. In this note we give an elementary introduction to the theory of logarithmic differential forms and their residues. In particular, we consider some properties of logarithmic differential forms related with properties of the torsion holomorphic differentials on singular hypersurfaces, briefly discuss the definitions of residues due to Poincaré, Leray and Saito, and then explain an elegant description of the modules of regular meromorphic differential forms in terms of residues of meromorhic differential forms logarithmic along a hypersurface with arbitrary singularities.

Keywords: logarithmic differential forms, residue-forms, residue map, regular meromorphic differential forms, torsion holomorphic differentials.

Introduction

From the historical point of view, the concept of logarithmic differential form had its origin in the classical theory of residues. The term "residue"(together with its formal definition) appeared for the first time in an article by A.Cauchy (1826), although one can find such a notion as implicit in Cauchy's prior work (1814) about the computation of particular integrals which were related with his research towards hydrodynamics. For the next three-four years, Cauchy developed residue calculus and applied it to the computation of integrals, the expansion of functions as series and infinite products, the analysis of differential equations, and so on ...

Though it was already transparent in the pioneer work of N.Abel, a major step towards the elaboration of the residue concept was made by H. Poincaré who introduced in 1887 the notion of differential residue 1-form attached to any rational differential 2-form in $\mathbf{C}^2$ with simple poles along a smooth complex curve. Subsequently É. Picard (1901), G. de Rham (1932/36), A.Weil (1947) obtained a series of similar results about residues of meromorphic forms of degree 1 and 2 on complex manifolds; such developments were associated with cohomological ideas, leading to the formulation of cohomological residue formulae. Such cohomological ideas were later pursued by G. de Rham (1954) and J.Leray (1959) who defined and studied residues of d-closed C^∞ regular differential q-forms on $S \setminus D$ with poles of the first order along a smooth hypersurface D in some complex manifold S, $q \geq 1$.

In 1972 J.-B. Poly [24] proved that Leray residue is well determined for any (not necessarily d-closed) *semi-meromorphic* differential forms ω as soon as ω and $d\omega$ have simple poles along a hypersurface.

In fact, for the first time these two conditions were considered by P.Deligne [11]; he introduced the notion of meromorphic differential forms with *logarithmic poles* along a divisor, normal crossings of smooth irreducible components. In such context this notion was extensively studied in algebraic geometry and in differential equations by many authors (for example, by Ph.Griffiths,

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J.Steenbrink, N.Katz). As a result in 1975, Kyoji Saito [25] considered *meromorphic* differential forms satisfied these conditions in the case of divisors with *arbitrary* singularities. Somewhat later, his note has been published in a volume [26] of the RIMS-publication series, which is not accessible to many of those interested in the subject. Saito established the basic properties of logarithmic differential forms and studied some applications to computing Gauss-Manin connection associated with the minimal versal deformations of simple hypersurface singularities of types A_2 and A_3. In 1980 a paper by Saito [27] was published; it contains an essential part of materials of the above mentioned works. Among other things, in this paper a general notion and important properties of residues of logarithmic differential forms are discussed in detail.

At present time the theory of logarithmic differential forms is exploited fruitfully in various fields of modern mathematics. Among them, one can mention the following:

complex algebraic geometry (the cohomology theory of algebraic varieties and Hodge theory [12], [10], [29], etc.),

topology and geometry (the theory of arrangements of real and complex hyperplanes [21], [7], the fundamental group of the complement of a singular hypersurface [19], etc.),

the theory of singularities, the deformation theory and the theory of Gauss-Manin connexion [26], [4], etc.,

the theory of $\mathcal{D}$-modules, the microlocal analysis, the theory of differential equations [11], [22], the theory of flat coordinate systems [28], etc.,

complex analysis (the theory of Abel's integrals [15], Torelli theorems, the theory of primitive forms and their periods [16], etc.),

the theory of special functions (generalized hypergeometric functions [12], etc.),

mathematical and theoretical physics (the theory of Frobenius varieties and the topological field theory [20], etc.)

Of course, this list is quite incomplete and can be easily extended by the specialists in related fields of mathematics.

Following our previous work [3] in this note we give an elementary introduction to the theory of logarithmic differential forms and their residues. In Section 1 we recall the basic notations, definitions and properties of logarithmic differential forms along a reduced hypersurface in a complex analytic manifold. In Section 2 we consider some relations of logarithmic differential forms and torsion holomorphic differentials on singular hypersurfaces. In the next sections we briefly discuss the definitions of Poincaré, de Rham, Leray and Saito residues, and apply the theory of regular meromorphic differential forms to the case of singular hypersurfaces. Among other things, we obtain a highly elegant description of these modules on an arbitrary singular hypersurface D in terms of residues of logarithmic differential forms.

1 Logarithmic differential forms

Let U be an open subset of $\mathbf{C}^m$, and let D be a hypersurface defined by an equation $h(z) = 0$, where $h(z) = h(z_1, \dots, z_m)$ is a holomorphic function in U, and $z_1, \dots, z_m$ is a system of coordinates. Suppose that D is *reduced*, that is, $h(z)$ has no multiple factors.

Definition 1.1 ([25], [27]) A meromorphic differential q-form ω, $q \geq 0$, on U is called *logarithmic* (along a divisor D) if ω and its differential $d\omega$ have poles along D at worst of the first order. It means that $h\omega$ and $hd\omega$ are *holomorphic* differential forms on U.

Remark 1.2 In fact, for the first time the above two conditions appeared in a work by Deligne (see [11], Prop. 3.2, (i), p.72) who studied meromorphic differential forms with *logarithmic poles* along divisors with normal crossings (thus, such a divisor D is the union of its smooth irreducible components).

In practical computations, the second condition is usually replaced by the condition "$dh \wedge \omega$ is a *holomorphic* differential form on U"; both conditions are equivalent, in view of the identity $d(h\omega) = dh \wedge \omega + h \cdot d\omega$.

Let S be an m-dimensional complex manifold, and let $\Omega^\bullet_S = \left(\Omega^q_S, d\right)_{q=0,1,...}$ be the de Rham complex of germs of holomorphic differential forms on S, whose terms, locally at the point $x \in S$, are defined as follows:

$$\Omega^q_{S,x} = \mathcal{O}_{S,x}\langle \dots, dz_{i_1} \wedge \dots \wedge dz_{i_q}, \dots \rangle, \quad (i_1, \dots, i_q) \in [1, m].$$

Let D be a reduced hypersurface of S, and let $h = 0$ be an equation of D, locally at the point $x \in D$. A meromorphic q-form ω is logarithmic along D at x, if $h\omega$ and $hd\omega$ are holomorphic. We denote the $\mathcal{O}_{S,x}$-module of germs of logarithmic q-form at x and the corresponding *sheaf* of logarithmic differential q-form on S by $\Omega^q_{S,x}(\log D)$ and $\Omega^q_S(\log D)$, respectively. Thus, the $\mathcal{O}_S$-module $\Omega^q_S(\log D)$ is a submodule of $\Omega^q_S(\star D)$, consisting of all the "differential forms with polar singularities along D." Obviously, the sheaves $\Omega^q_S(\log D)$ and Ω^q_S coincide off the divisor D, for all $q \geq 0$. By definition,

$$\Omega^0_{S,x}(\log D) \cong \Omega^0_{S,x} \cong \mathcal{O}_{S,x}, \quad \Omega^m_{S,x}(\log D) \cong \frac{1}{h}\Omega^m_{S,x}.$$

In what follows, when we consider the local situation the point x will be taken to be 0 for simplicity. We shall also assume that U is an open subset of $\mathbf{C}^m$ containing the origin.

Example 1.3 Suppose $D \subset U$ be a hyperplane or, more generally, a smooth hypersurface defined by the equation $z_1 = 0$. Then

$$\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dz_1}{z_1}, dz_2, \dots, dz_m \right\rangle$$

is a *free* $\mathcal{O}_{S,0}$-module of rank m, generated by the forms $dz_1/z_1, dz_2, \dots, dz_m$. Moreover,

$$\Omega^q_{S,0}(\log D) \cong \bigwedge^q \Omega^1_{S,0}(\log D), \quad 1 \leq q \leq m.$$

Example 1.4 More generally, let us consider the case when D is the union of $k \leq m$ coordinates hyperplanes in $S = \mathbf{C}^m$. In other words, D is a *strong normal crossing*. This case considered in many works published before Saito's preprint [25]. Then the defining equation of D is written as follows: $h = z_1 \cdots z_k = 0$, and an easy calculation shows that

$$\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k}, dz_{k+1}, \dots, dz_m \right\rangle,$$

and for all $1 \leq q \leq m$ there are the following isomorphisms

$$\Omega^q_{S,0}(\log D) \cong \bigwedge^q \Omega^1_{S,0}(\log D).$$

Thus, $\Omega^q_{S,0}(\log D)$ is a *free* $\mathcal{O}_{S,0}$-module of rank $\binom{m}{q}$.

The following statement is a direct consequence of the basic definition (see [1], or [2], §1).

Claim 1.5 *Let $D \subset U$ be a reduced hypersurface defined by the equation $h = 0$. Then for any $q \geq 1$ there exists a natural isomorphism of $\mathcal{O}_{S,0}$-modules*

$$\Omega^q_{S,0} \bigcap \left((dh/h) \wedge \Omega^{q-1}_{S,0} \right) \cong dh \wedge \Omega^{q-1}_{S,0}(\log D).$$

Proof. Let us remark at first that there is a natural inclusion

$$\Omega^q_{S,0} \bigcap \left((dh/h) \wedge \Omega^{q-1}_{S,0} \right) \hookrightarrow dh \wedge \Omega^{q-1}_{S,0}(\log D).$$

If an element $\omega \in \Omega^q_{S,0}$ belongs to the $\mathcal{O}_{S,0}$-module on the left side, then it can be represented in the form $\omega = (dh/h) \wedge \eta$ for some $\eta \in \Omega^{q-1}_{S,0}$. Hence, by definition,

$$(\eta/h) \in \Omega^{q-1}_{S,0}(\log D) \Rightarrow \omega \in dh \wedge \Omega^{q-1}_{S,0}(\log D),$$

and we obtain the desirable inclusion. On the other hand, $h \cdot \Omega^{q-1}_{S,0}(\log D) \subseteq \Omega^{q-1}_{S,0}$. Multiplication by $\wedge dh$ induces the map

$$dh \wedge \Omega^{q-1}_{S,0}(\log D) \longrightarrow \frac{dh}{h} \wedge \Omega^{q-1}_{S,0}.$$

Obviously this gives us the inverse map to the first inclusion. This completes the proof of Claim.

Lemma 1.6 ([27], $(1.1), iii)$**)** *Let ω be a meromorphic q-form on U, $q \geq 0$, and let $D \subset U$ be a hypersurface as above. Then ω is logarithmic along D if and only if there exist a holomorphic function g defining a hypersurface $V \subset U$, a holomorphic $(q-1)$-form ξ and a holomorphic q-form η on U such that*

a) $\dim_{\mathbf{C}} D \cap V \leq m - 2$,

b) $g\omega = \dfrac{dh}{h} \wedge \xi + \eta$.

Proof. For simplicity let us consider the case $m = 2$. Suppose that ω is a logarithmic q-form. Then we have

$$\omega = \frac{a_1 dz_1 + a_2 dz_2}{h}, \quad dh \wedge \omega = \frac{h'_1 a_2 - h'_2 a_1}{h} dz_1 \wedge dz_2 \stackrel{\text{def}}{=} b(z) dz_1 \wedge dz_2,$$

where a_1, a_2 and $b(z)$ are holomorphic, and $h'_i = \partial h / \partial z_i$, $i = 1, 2$. Further,

$$h'_1 \omega = \frac{h'_1 a_1 dz_1 + h'_1 a_2 dz_2}{h} =$$

$$\frac{h'_1 a_1 dz_1 + h'_2 a_1 dz_2}{h} + \frac{h'_1 a_2 - h'_2 a_1}{h} dz_2 = \frac{dh}{h} \wedge a_1 + b(z) dz_2.$$

It means that

$$h'_1 \omega = \frac{dh}{h} \wedge a_1 + b(z) dz_2.$$

There is analogous representation for $h_2'\omega$, and hence for any $g\omega$, where $g \in J(h) = (h_1', h_2')$, the Jacobian ideal of h. Since D is reduced, there is a function $g \in J(h)$ defining a non-zero divisor in $\mathcal{O}_D/(h)$ as required in the condition a).

Conversely, the relation b) implies that

$$h\omega = dh \wedge \frac{\xi}{g} + \frac{\eta}{g},$$

that is, $h\omega$ and $dh \wedge \omega$ are holomorphic in codimension ≥ 2, hence, in virtue of the Riemann extension theorem, they are holomorphic everywhere. This completes the proof when $m = 2$. The general case can be considered analogously.

Corollary 1.7 ([25]) *With the preceding notations the following conditions are equivalent:*

1) $\omega \in \Omega^q_S(\log D)$,
2) $\dfrac{\partial h}{\partial z_i}\omega \in \dfrac{dh}{h} \wedge \Omega^{q-1}_U + \Omega^q_U$ *for all* $i = 1, \ldots, m$.

Corollary 1.8 *The sheaves* $\Omega^q_S(\log D)$, $q = 0, 1, \ldots, m$, *are* $\mathcal{O}_S$*-modules of finite type; the direct sum* $\oplus_{q=0}^m \Omega^q_S(\log D)$ *is an* $\mathcal{O}_S$*-exterior algebra closed under the exterior differentiation* d.

As a consequence, $\Omega^q_S(\log D)$ are *coherent* sheaves of $\mathcal{O}_S$-modules for all $q \geq 0$.

2 Torsion differentials

In this section we consider simple relations between logarithmic differential forms and torsion holomorphic differentials on hypersurfaces with singularities. By definition, $\mathcal{O}_{D,0} = \mathcal{O}_{S,0}/(h)\mathcal{O}_{S,0}$, and

$$\Omega^q_{D,0} = \Omega^q_{S,0}/\big(h \cdot \Omega^q_{S,0} + dh \wedge \Omega^{q-1}_{S,0}\big), \quad q \geq 1.$$

Thus, $\Omega^q_{D,0}$ is the $\mathcal{O}_{D,0}$-module of germs of *holomorphic* differential forms on the *hypersurface* D at the point $0 \in D$. The module $\Omega^1_{D,0}$ is usually called the module of Kähler regular differentials. The standard differentiation d induces the action on $\Omega^q_{D,0}$ denoted by the same symbol. Thus, the de Rham complex of sheaves of germs of holomorphic differential forms on D is well defined:

$$\Omega^\bullet_D = \big(\Omega^q_D, d\big)_{q=0,1,\ldots}.$$

For completeness, recall the notion of torsion. Given a commutative ring A with the total ring of fractions F, and an A-module N of finite type, we consider the kernel of the canonical map $\iota\colon N \to N \otimes_A F$, the *torsion* submodule of N, and denote it by $\operatorname{Tors} N$; it consists of all the elements of N which are killed by non-zero divisors of A.

It is well-known that torsion differentials $\operatorname{Tors}\Omega^q_{D,0}$ play a key role in analysis of topology and geometry of singular varieties. In the case of an isolated n-dimensional singularity $(D, 0)$, the torsion modules $\operatorname{Tors}\Omega^q_{D,0}$ are trivial for all $q = 1, \ldots, n-1$, while $\operatorname{Tors}\Omega^n_{D,0}$ is a finite dimensional vector space. Furthermore, if D is the quasi-homogeneous germ of a hypersurface or complete intersection with isolated singularities then $\dim_{\mathbb{C}} \operatorname{Tors}\Omega^n_{D,0} = \mu$, where μ is the Milnor number of D; it is a very important topological invariant of the singularity.

The following examples show that generators of the module of logarithmic differential forms are naturally expressed through torsion differentials on D.

Example 2.1 Suppose $S = \mathbf{C}^2$ and consider the hypersurface D given by the equation $h = xy = 0$. It is a plane curve with a node. In other words, it is an A_1-singularity, a very particular case of strong normal crossing from Example 1.4. Then

$$\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dx}{x}, \frac{dy}{y} \right\rangle, \ \Omega^2_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dx \wedge dy}{xy} \right\rangle$$

are *free* $\mathcal{O}_{S,0}$-modules of rank 2 and 1, respectively. In this case there is also the following representation: $\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\langle dh/h, \theta/h\rangle$, where $\theta = ydx - xdy$. It is not difficult to verify that $\theta \in \operatorname{Tors}\Omega^1_{D,0}$. Indeed, taking a non-zero divisor $(x+y) \in \mathcal{O}_{D,0}$ one obtains the following identities in $\Omega^1_{D,0}$:

$$(x+y)\cdot\theta = xydx - x^2dy + y^2dx - xydy = -(x-y)dh + 2h(dx - dy) \equiv 0.$$

Moreover, in this case, $\operatorname{Tors}\Omega^1_{D,0} \cong \mathcal{O}_{D,0}\langle\theta\rangle \cong \mathbf{C}\langle\theta\rangle, \ \mu = 1.$

Example 2.2 (cf. [30]) With the preceding notations let $D \subset S$ be a plane curve with a cusp given by the equation $h = x^2 - y^3 = 0$. In other words, it is an A_2-singularity. Easy calculations show that

$$\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dh}{h}, \frac{2ydx - 3xdy}{h} \right\rangle, \ \Omega^2_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dx \wedge dy}{h} \right\rangle$$

are again free $\mathcal{O}_{S,0}$-modules of rank 2 and 1, respectively. Notice that the numerator of the second generator of $\Omega^1_{S,0}(\log D)$, the differential 1-form $\theta = 2ydx - 3xdy$, represents an element of the *torsion* submodule $\operatorname{Tors}\Omega^1_{D,0} \subset \Omega^1_{D,0}$. Indeed, in our case $A = \mathcal{O}_{D,0} \cong \mathbf{C}\langle t^2, t^3\rangle$, $N = \Omega^1_{D,0}$, $F = \mathbf{C}(t)$, and the mapping ι is given by the normalization of D, that is, $x = t^3$, $y = t^2$. Thus, $\iota(\theta) = \iota(2ydx - 3xdy) = 2t^2dt^3 - 3t^3dt^2 = 0$, that is, $\theta \in \operatorname{Ker}(\iota) \cong \operatorname{Tors}\Omega^1_{D,0}$. Equivalently, take a non-zero divisor $x \in \mathcal{O}_{D,0}$. One then obtains $x\cdot\theta = 2xydx - 3x^2dy = 5hdx - 3xdh \equiv 0$ in $\Omega^1_{D,0} = \Omega^1_{S,0}/(h\cdot\Omega^1_{S,0} + dh\wedge\mathcal{O}_{S,0})$. Further calculations show (cf. [30]) that $\operatorname{Tors}\Omega^1_{D,0} \cong \mathcal{O}_{D,0}\langle\theta\rangle \cong \mathbf{C}\langle\theta, y\cdot\theta\rangle$, that is, $\mu = 2$.

Proposition 2.3 **([1])** *For $q = 1, \ldots, m$, there are exact sequences of $\mathcal{O}_{S,0}$-modules*

$$0 \longrightarrow \Omega^{q-1}_{S,0}/h\cdot\Omega^{q-1}_{S,0}(\log D) \xrightarrow{\wedge dh} \Omega^q_{S,0}/h\cdot\Omega^q_{S,0} \longrightarrow \Omega^q_{D,0} \longrightarrow 0,$$

$$0 \longrightarrow \Omega^q_{S,0}/dh\wedge\Omega^{q-1}_{S,0}(\log D) \xrightarrow{\cdot h} \Omega^q_{S,0}/dh\wedge\Omega^{q-1}_{S,0} \longrightarrow \Omega^q_{D,0} \longrightarrow 0,$$

$$0 \longrightarrow \Omega^q_{S,0} + \frac{dh}{h}\wedge\Omega^{q-1}_{S,0} \longrightarrow \Omega^q_{S,0}(\log D) \xrightarrow{\cdot h} \operatorname{Tors}\Omega^q_{D,0} \longrightarrow 0,$$

where the homomorphisms of exterior and usual multiplication are denoted by $\wedge dh$ and by $\cdot h$, respectively.

Proof. The exactness of the first and second sequences follows directly from the basic Definition 1.1. Let us consider a differential q-form $\omega \in \Omega^{q-1}_{S,0}$ represented an element of the quotient $\Omega^{q-1}_{S,0}/h\cdot\Omega^{q-1}_{S,0}(\log D)$. Suppose $dh\wedge\omega = h\cdot\eta$, $\eta \in \Omega^q_{S,0}$, and set $\tilde{\omega} = \omega/h$. It is obvious that $h\tilde{\omega}$ and $dh\wedge\tilde{\omega}$ are holomorphic, hence $\tilde{\omega} \in \Omega^{q-1}_{S,0}(\log D)$ by definition. Thus the first sequence is exact from the left. Evidently it is exact from the right too. In the same way, one can easily prove the exactness of the second sequence. The exactness from the left of the third sequence follows from definition. In view of Lemma 1.6, it is clear that $\mathrm{Im}\,(\cdot h) \subseteq \mathrm{Tors}\,\Omega^q_{D,0}$ because for a non-zero divisor g one has the following chain of implications:

$$g\omega = \frac{dh}{h} \wedge \xi + \eta \quad \Rightarrow g(h\omega) = dh \wedge \xi + h\eta \equiv 0 \quad \Rightarrow h\omega \in \mathrm{Tors}\,\Omega^q_{D,0}.$$

Now let take an element $\omega \in \mathrm{Tors}\,\Omega^q_{D,0}$. By definition, there is a non-zero divisor $g \in \mathcal{O}_{D,0}$ such that $g\,\omega = 0$. We will denote by g and ω their representatives in $\mathcal{O}_{S,0}$ and $\Omega^q_{S,0}$, respectively. Then one has $g{\cdot}\omega = dh \wedge \xi + h{\cdot}\eta$, $\xi \in \Omega^{q-1}_{S,0}$, $\eta \in \Omega^q_{S,0}$. Since g is a non-zero divisor, the condition $b)$ of Lemma 1.6 is satisfied. This implies that $\omega/h = \tilde{\omega} \in \Omega^q_{S,0}(\log D)$, that is, $\omega \in \mathrm{Im}\,(\cdot h)$.

Remark 2.4 It is well-known [14] that $\mathrm{Tors}\,\Omega^q_{D,0} = 0$, $0 < q < c$, where $c = \mathrm{codim}\,(\mathrm{Sing}\,D, D)$ and $\mathrm{Sing}\,D$ is the singular locus of D. On the other side, any reduced hypersurface (or complete intersection) D is normal if and only if $c \geq 2$ by virtue of Serre's criterion ("R_1 and S_2 conditions imply normality"). Hence, when D is *normal* then the exact sequence of Proposition 2.3 implies the following isomorphisms

$$\Omega^q_{S,0}(\log D) \cong \Omega^q_{S,0} + \frac{dh}{h} \wedge \Omega^{q-1}_{S,0}, \quad 1 \leq q < c.$$

It is not difficult to see that the support of $\mathrm{Tors}\,\Omega^1_D$ is contained in the singular locus $\mathrm{Sing}\,D$ of the hypersurface D. Moreover, there is a system of generators of $\mathcal{O}_D$-module $\mathrm{Tors}\,\Omega^1_D$ containing at least $m-1$ elements.

Corollary 2.5 *There are the following long exact sequences of* $\mathcal{O}_{S,0}$*-modules*

$$0 \to \Omega^q_{S,0} + \tfrac{dh}{h} \wedge \Omega^{q-1}_{S,0} \to \Omega^q_{S,0}(\log D) \xrightarrow{\cdot h} \Omega^q_{D,0} \to \Omega^q_{D,0}/\mathrm{Tors}\,\Omega^q_{D,0} \to 0,$$
$$0 \to dh \wedge \Omega^{q-1}_{S,0}(\log D) \to \Omega^q_{S,0} \oplus \tfrac{dh}{h} \wedge \Omega^{q-1}_{S,0} \to \Omega^q_{S,0}(\log D) \xrightarrow{\cdot h} \mathrm{Tors}\,\Omega^q_{D,0} \to 0.$$

Proof. This is an immediate consequence of Proposition 2.3 and Claim 1.5.

Remark 2.6 The last sequence is very useful in computing the torsion modules in the case when $\Omega^q_{S,0}(\log D)$ is a *free* $\mathcal{O}_{S,0}$-module; it gives us an $\mathcal{O}_{S,0}$-free *resolution* of the torsion module. Following P.Cartier [9] a hypersurface $D \subset S$ is called *Saito divisor* or, more often, *Saito free* divisor if for some $q \geq 1$ and, consequently, for all q, the $\mathcal{O}_S$-module $\Omega^q_S(\log D)$ is locally free. For example, the discriminants of the minimal versal deformations of isolated hypersurface or complete intersection singularities are Saito free divisors.

3 Poincaré residue

The following construction [15] is a direct generalization of the original Poinacaré definition of the residue 1-form associated with any rational 2-form in $\mathbf{C}^2$.

Let ω be a meromorphic differential m-form on an m-dimensional complex analytic *manifold* S with a polar divisor $D \subset S$. Thus, locally we have a representation:

$$\omega = \frac{f(z)dz_1 \wedge \ldots \wedge dz_m}{h(z)},$$

where f and h are germs of holomorphic functions, and h is a local equation of D. By definition, the Poincaré residue $\text{rés}_D(\omega)$ is a meromorphic $(m-1)$-form on D whose singularities are contained in the singular locus $\operatorname{Sing} D \subset D$. To define this form explicitly, let us note that at each point $x \in D \setminus \operatorname{Sing} D$ at least one of the derivatives of h does not vanish:

$$\frac{\partial h}{\partial z_i}\Big|_{z=x} \neq 0.$$

Then the Poincaré residue of ω in a small neighbourhood of x is defined as follows:

$$\text{rés}_D(\omega) = (-1)^{m-i} \frac{f(z) dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_m}{\partial h(z)/\partial z_i}\Big|_D .$$

It is not difficult to verify that this restriction depends neither on the index i nor on the local coordinates and on defining equations of D. Moreover, the Poincaré residue is holomorphic on the complement $S \setminus D$. When D is *smooth*, one can take $h(z) = z_m$, and then, as usually,

$$\text{rés}_D\left(\frac{f(z) dz_1 \wedge \ldots \wedge dz_m}{z_m}\right) = f(z) dz_1 \wedge \ldots \wedge dz_{m-1},$$

that is, $\text{rés}_D(\omega)$ is holomorphic on D. As a result one has the following sequence of sheaves

$$0 \longrightarrow \Omega_S^m \longrightarrow \Omega_S^m(D) \xrightarrow{\text{rés}} \Omega_D^{m-1} \longrightarrow 0 ,$$

where $\Omega_S^m(D)$ denotes the sheaf of meromorphic forms on S having a simple pole along the divisor D. In particular, one concludes that the germ of every holomorphic $(m-1)$-form on the nonsingular divisor D is a Poincaré residue. It is obvious that this is true globally when the first cohomology group vanishes: $H^1(S, \Omega_S^m) = 0$.

4 Leray residue-form

As remarked in Introduction De Rham and Leray considered d-closed C^∞ regular differential forms on $S \setminus D$ having simple poles on D, where D is a submanifold of codimension 1 in a smooth manifold S. In particular, they proved that locally for such a form there is the following represenation:

$$(*) \qquad \omega = \frac{dh}{h} \wedge \xi + \eta,$$

where ξ and η are germs of regular differential forms on S. In fact, $\xi_{|D}$ is globally and uniquely determined; it is closed on D. If ω is holomorphic on $S \setminus D$ then the form $\xi_{|D}$ is holomorphic on D. The form $\xi_{|D}$ is called the Leray *residue-form* on D; it is denoted by $\text{res}[\omega]$. It is not difficult to see that the definition of the Leray residue-form generalizes the Ponacaré residue described above.

Similarly to the construction from the end of the previous section, making use of local representation $(*)$, for any $q = 1, \ldots, m$ one gets (see [23]) the exact sequence

$$0 \longrightarrow \Omega_S^q \longrightarrow \Omega_S^q(D) \xrightarrow{\text{res}} \Omega_D^{q-1} \longrightarrow 0,$$

which is equivalent, since the divisor D is a *smooth* hypersurface, to the sequence

$$0 \longrightarrow \Omega^q_S \longrightarrow \Omega^q_S(\log D) \xrightarrow{\text{res}} \Omega^{q-1}_D \longrightarrow 0\ .$$

Below we show that a generalization of this sequence to the case when the divisor D has arbitrary singularities requires more delicate considerations.

5 Saito residue map

In fact, Leray considered d-closed forms on $S \setminus D$ in order to construct a natural homomorphism of cohomology spaces $H^p(S \setminus D) \to H^{p-1}(D)$, and then the co-boundary homomorphisms of homology groups $H_{p-1}(D) \to H_p(S \setminus D)$, the main ingredient of his famous residue-formula.

Furthermore, in 1972 J.-B. Poly [24] proved that the representation $(*)$ are valid for any *semi-meromorphic* differential form ω as soon as ω and $d\omega$ have simple poles along a smooth hypersurface $D \subset S$. By definition, a differential form ω is called semi-meromorphic when locally all its coefficients can be represented as quotient of smooth and holomorphic functions. Hence, the Leray residue is also well determined for such forms without assumption on their d-closedness.

Following Saito [27] we describe a natural generalization of the Leray residue for *meromorphic* differential forms satisfying the above two conditions for a divisor D with arbitrary singularities, that is, for *logarithmic* differential forms in the sense of Definition 1.1.

Let $D \subset S$ be a hypersurface, and let the sheaf $\mathcal{M}_D$ be the $\mathcal{O}_D$-module of germs of meromorphic functions on D.

Definition 5.1 (see [27], (2.2)) The (logarithmic) residue morphism is a homomorphism of $\mathcal{O}_S$-modules

$$\text{res.}:\ \Omega^q_S(\log D) \longrightarrow \mathcal{M}_D \otimes_{\mathcal{O}_D} \Omega^{q-1}_D,$$

defined locally as follows: taking the representation of the basic Lemma 1.6, for any $\omega \in \Omega^q_{S,0}(\log D)$ we set

$$\text{res.}\,\omega = \frac{1}{g}\cdot\xi.$$

Thus, the residue res. ω is the germ of the meromorphic $(q-1)$-form in the module $\mathcal{M}_{D,0} \otimes_{\mathcal{O}_{D,0}} \Omega^{q-1}_{D,0}$.

Claim 5.2 ([27], (2.5)) *Let $D \subset S$ be a hypersurface. Then for any $q \geq 1$ there exists the following exact sequence of $\mathcal{O}_S$-modules*

$$0 \longrightarrow \Omega^q_S \longrightarrow \Omega^q_S(\log D) \xrightarrow{\text{res.}} \mathcal{M}_D \otimes_{\mathcal{O}_D} \Omega^{q-1}_D.$$

Proof. Making use of the representation of logarithmic forms as in the definition of the symbol res. above, one obtains

$$\text{res.}\,\omega = 0 \Leftrightarrow g\omega \in \Omega^q_{S,0} \Leftrightarrow \omega \in \Omega^q_{S,0}.$$

This completes the proof.

Remark 5.3 In particular, for $q = 1$ one has

$$0 \longrightarrow \Omega^1_S \longrightarrow \Omega^1_S(\log D) \xrightarrow{\text{res.}} \mathcal{M}_D \cong \mathcal{M}_{\widetilde{D}},$$

where $\widetilde{D}$ is the normalization of D. Moreover (see [27], Lemma (2.8)), if $\pi\colon \widetilde{D} \longrightarrow D$ is the morphism of normalization, then the image Im (res.) contains $\pi_*(\mathcal{O}_{\widetilde{D}})$ consisting of the so-called *weakly holomorphic* function on D, that is, of meromorphic functions, whose preimage becomes holomorphic on the normalization.

Remark 5.4 By this way we can consider the image of the logarithmic residue res. $\Omega^q_S(\log D)$ as an $\mathcal{O}_D$-module. Indeed, the definition of logarithmic forms implies that $h \cdot \left(\Omega^q_{S,0}(\log D)/\Omega^q_{S,0}\right) = 0$. Hence, the multiplication by $h\cdot$ annihilates Im (res.).

6 Regular meromorphic forms and Saito residue map

We are going to describe the image of the Saito residue map in terms of regular meromorphic forms for logarithmic differential forms with poles along a divisor $D \subset S$ with arbitrary singularities together with a generalization of the exact sequences from Section 3 and Section 4.

Now we consider the sheaves of $\mathcal{O}_D$-modules ω^q_D, $q \geq 0$, called *regular meromorphic* differential q-forms on the hypersurface D. So let X, $\dim X = n \geq 1$, be the germ of an analytic subspace of an m-dimensional complex manifold S, and let $\omega^n_X = \mathrm{Ext}^{m-n}_{\mathcal{O}_S}(\mathcal{O}_X,\ \Omega^m_S)$ be the Grothendieck *dualizing module* of X.

Definition 6.1 ([18], [8]) The sheaves ω^q_X, $q = 0, 1, \ldots, n$, of *regular meromorphic* differential q-forms on X are defined as follows: ω^q_X consists of all meromorphic differential forms of order q on X such that $\omega \wedge \eta \in \omega^n_X$ for any $\eta \in \Omega^{n-q}_X$ or, equivalently, $\omega^q_X \cong \mathrm{Hom}_{\mathcal{O}_X}\left(\Omega^{n-q}_X,\ \omega^n_X\right)$.

Let us apply this Definition in the particular case when $X = D$ is a hypersurface, that is, $n = m - 1$.

Claim 6.2 *Let $D \subset U$ be a reduced hypersurface. Then* res. $\Omega^{q+1}_S(\log D) \subseteq \omega^q_D$ *for all* $q = 0, 1, \ldots, m-1$.

Proof. Set $dz = dz_1 \wedge \ldots \wedge dz_m$. Then with preceding notations one has a natural isomorphism $\omega^n_D \cong \mathcal{O}_D(dz/dh)$. That is, $\omega^q_D \cong \mathrm{Hom}_{\mathcal{O}_D}\left(\Omega^{n-q}_D,\ \mathcal{O}_D\big(dz/dh\big)\right)$ for all $q = 0, 1, \ldots, n$. Then Corollary 1.7 implies that $\frac{\partial h}{\partial z_i} \cdot \text{res.}\, \Omega^q_S(\log D)\big|_U \subset \Omega^{q-1}_D\big|_{D\cap U}$ for all $i = 1, \ldots, m$, or, equivalently, $dh \wedge \text{res.}\, \Omega^q_S(\log D)\big|_U \subset \Omega^q_D\big|_{D\cap U}$. This completes the proof.

Below we use an equivalent description of the regular meromorphic differential forms ω^q_D, $q \geq 0$, on the hypersurface D obtained by D.Barlet in a more general context (see [8], Lemma 4). In fact, there is the following exact sequence of $\mathcal{O}_{D,0}$-modules:

$$0 \longrightarrow \omega^q_{D,0} \xrightarrow{\mathcal{C}} \mathrm{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega^{q+1}_{S,0}) \xrightarrow{\wedge dh} \mathrm{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega^{q+2}_{S,0}), \quad q \geq 0,$$

where $\omega^q_{D,0} \subset j_* j^* \Omega^q_{D,0}$ and $\mathcal{C}$ is induced by the multiplication by the fundamental class of D in S. Thus, $\mathcal{C}(v)$ corresponds to the Čech cocycle w/h such that $w = v \wedge dh$.

Theorem 6.3 ([2], §4) *Let $D \subset S$ be a reduced hypersurface. Then for any $q \geq 1$ there is the following exact sequence*

$$0 \to \Omega_S^{q+1} \longrightarrow \Omega_S^{q+1}(\log D) \xrightarrow{\text{res.}} \omega_D^q \to 0.$$

In particular, ω_D^q and $\operatorname{res.}\Omega_S^{q+1}(\log D)$ are isomorphic $\mathcal{O}_D$-modules.

Proof. It is sufficient to verify the statement locally. In view of Claim 6.2 it remains to prove that any element of ω_D^q can be represented as the residue of a logarithmic q-form.

Let $\mathcal{K}.(h)$ be the ordinary Koszul complex associated with h, that is,

$$0 \to \mathcal{O}_{S,0}e_0 \xrightarrow{d_0} \mathcal{O}_{S,0} \xrightarrow{d_{-1}} \mathcal{O}_{D,0} \to 0,$$

where $\mathcal{K}_1(h) = \mathcal{O}_{S,0}e_0$, $\mathcal{K}_0(h) = \mathcal{O}_{S,0}$ and $d_0(e_0) = h$, $d_{-1}(1) = 1$. Then we have the following piece of the dual exact sequence

$$\cdots \longrightarrow \operatorname{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_0(h),\ \Omega_{S,0}^{q+1}) \xrightarrow{d^0} \operatorname{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_1(h),\ \Omega_{S,0}^{q+1}) \longrightarrow$$
$$\longrightarrow \operatorname{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega_{S,0}^{q+1}) \to 0.$$

Hence, any element of $\operatorname{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega_{S,0}^{q+1})$ can be represented as a Čech 0-cochain (more explicitly, a 0-cocycle) in the following way

$$\nu/h \in \operatorname{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_1(h),\ \Omega_{S,0}^{q+1}) \cong C^0_{(1)}(\Omega_{S,0}^{q+1}),$$

where $\nu \in \Omega_{S,0}^{q+1}$. Choose now an element $\nu \in \Omega_{S,0}^{q+1}$ such that

$$\frac{\nu}{h} \wedge dh \in \operatorname{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0}, \Omega_{S,0}^{q+2}),$$

corresponds to the trivial element. That is, $\nu \wedge dh/h$ is defined by an element of $d^0\big(\operatorname{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_0(h), \Omega_{S,0}^{q+2})\big)$. This means that $\nu \wedge dh = h \cdot \eta$ for some form $\eta \in \Omega_{S,0}^{q+2}$. The first exact sequence of Proposition 2.3 implies that $\nu \in h \cdot \Omega_{S,0}^{q+1}(\log D)$. Set $\tilde{\nu} = \mathcal{C}^{-1}(\nu/h)$. By definition, $\mathcal{C}(\tilde{\nu})$ corresponds to a Čech cocycle ν/h such that $\nu = \tilde{\nu} \wedge dh$ (take $v = \tilde{\nu}$, $w = \nu$ in the above description of ω_D^q with the help of multiplication by the fundamental class). This yields $\mathcal{C}(\tilde{\nu}) = \nu/h = \tilde{\nu} \wedge dh/h$, and $\operatorname{res.}(\nu/h) = \tilde{\nu}$. Thus, for any element $\tilde{\nu} \in \omega_D^q$ there is a preimage under the logarithmic residue map represented by ν/h. This completes the proof.

Remark 6.4 In fact, the representation $(*)$ implies directly that $\operatorname{res.}\Omega_S^m(\log D) \cong \omega_D^n \cong \mathcal{O}_D(dz/dh)$, in view of the formal decomposition $dz/h = (dh/h) \wedge (dz/dh)$. Further, it is not difficult to verify that in the case of plane node of Example 2.1 there is natural isomorphisms $\operatorname{res.}\Omega_S^1(\log D) \cong \pi_*(\mathcal{O}_{\tilde{D}}) \cong \omega_D^0$ (cf. Remark 5.3). A similar result is also valid in a more general situation (see [27], Theorem (2.9)).

Remark 6.5 It should also be underlined that there is a far reaching generalization of main results of this section to the case of complete intersections. In papers [5] and [6] it was developed the theory of *multi-logarithmic* differential forms and their residues with applications to the general theory of multidimensional residue and residue currents on complex spaces.

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ВЫЧЕТЫ ЛОГАРИФМИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ

А.Г. Александров
Институт проблем управления РАН,
ул. Профсоюзная, 65, Москва, 117997, Россия, e-mail: ag_aleksandrov@mail.ru

Аннотация. В этой заметке излагается элементарное введение в теорию логарифмических дифференциальных форм и их вычетов. В частности, рассматриваются некоторые свойства логарифмических форм, связанные с кручением голоморфных дифференциалов на особых гиперповерхностях, кратко обсуждаются понятия вычета, данные Пуанкаре, Лерэ и Саито, а затем приводится красивое описание регулярных мероморфных дифференциалов в терминах вычетов мероморфных дифференциальных форм, логарифмических вдоль гиперповерхности с произвольными особенностями.

Ключевые слова: логарифмические дифференциальные формы, форма-вычет, регулярные мероморфные дифференциальные формы, кручение голоморфных дифференциалов.