UNIQUENESS THEOREMS OF MEROMORPHIC FUNCTIONS
IN SEVERAL COMPLEX VARIABLES

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Abstract. In the survey, results on the existence, growth, uniqueness, and value distribution of meromorphic (or entire) solutions of homogeneous linear partial differential equations of the second order with polynomial coefficients that are similar or different from that of meromorphic solutions of linear ordinary differential equations have been obtained. We have characterized those entire solutions of a special partial differential equation that relate to Bessel functions and prove in general that meromorphic solutions that grow much faster than the coefficient have zero Nevanlinna’s deficiency for each non-zero complex value. It’s well-know result that if a nonconstant meromorphic function \( f \) on \( \mathbb{C} \) and its \( k \)-th derivative \( f^{(k)} \) have no zeros for some \( k \geq 2 \), then \( f \) is of the form \( f(z) = \exp(Az + B) \) or \( f(z) = (Az + B)^{-m} \) for some constants \( A, B \). We have extended this result to meromorphic functions of several variables, by first extending the classic Tumura-Chinii theorem for meromorphic functions of one complex variable to that of meromorphic functions of several complex variables by utilizing Nevanlinna theory.

Keywords: meromorphic functions, homogeneous linear partial differential equation, holomorphic coefficients, Nevanlinna’s value distribution theory.

Analytic properties or characterizations of meromorphic (or entire) solutions of some partial differential equations (or system) of the first order have been exhibited clearly by several authors (cf. [2], [13], [18], [19]). In this survey, we introduce a few results on meromorphic solutions of homogeneous linear partial differential equations of the second order in two independent complex variables

\[
a_1 \frac{\partial^2 u}{\partial t^2} + 2a_2 \frac{\partial^2 u}{\partial t \partial z} + a_3 \frac{\partial^2 u}{\partial z^2} + a_4 \frac{\partial u}{\partial t} + a_5 \frac{\partial u}{\partial z} + a_6 u = 0, \tag{1.1}
\]

where \( a_k(t, z) \) are holomorphic functions for \( (t, z) \in \Sigma \), where \( \Sigma \) is a region on \( \mathbb{C}^2 \). Basic idea comes from S. N. Bernstein [3], H. Lewy [17], I. G. Petrovskii[19]. For more detail, see [15]. To prove these results, we used some methods in [5], [7], [11], [14], [21], [23] and [26].

First of all, we examine the following special differential equation:

\[
f^3 \frac{\partial^2 u}{\partial t^2} + s^3 \frac{\partial^2 u}{\partial z^2} + f \frac{\partial u}{\partial t} - z \frac{\partial u}{\partial z} + f^2 u = 0. \tag{1.2}
\]

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Theorem 1.1 The differential equation \((1.2)\) has an entire solution \(f(t, z)\) on \(\mathbb{C}^2\) if and only if \(f\) is an entire function expressed by the series
\[ f(t, z) = \sum_{n=0}^{\infty} m_n c_n J_n(t) z^n \] (1.3)
such that
\[ \limsup_{n \to \infty} |c_n|^{1/n} = 0, \] (1.4)
where \(J_n(t)\) is the first kind of Bessel's function of order \(n\). Moreover, the order \(\text{ord}(f)\) of the entire function \(f\) satisfies
\[ \rho \leq \text{ord}(f) \leq \max\{1, \rho\}, \] where
\[ \rho = \limsup_{n \to \infty} \frac{2 \log n}{\log |c_n|^{-1/n}}. \] (1.5)

By definition, the order of \(f\) is defined by
\[ \text{ord}(f) = \limsup_{r \to \infty} \frac{\log^{+} \log^{+} M(r, f)}{\log r}, \]
where
\[ \log^{+} x = \begin{cases} \log x, & \text{if } x \geq 1; \\ 0, & \text{if } x < 1; \end{cases} \]
and
\[ M(r, f) = \max_{|t| \leq r, |u| \leq r} |f(t, z)|. \]
G. Valiron [23] showed that each transcendental entire solution of a homogeneous linear ordinary differential equation with polynomial coefficients is of finite positive order. However, Theorem 1.1 shows that Valiron's theorem is not true for general partial differential equations. Here we exhibit another example that the following equation
\[ \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = 0 \]
has an entire solution \(\exp(te^t)\) of infinite order.

If \(0 < \lambda = \text{ord}(f) < \infty\), we define the type of \(f\) by
\[ \text{typ}(f) = \limsup_{r \to \infty} \frac{\log^{+} M(r, f)}{r^\lambda}. \]
For the type of entire solutions of the equation \((1.2)\), we have an analogue of Lindelöf-Pringsheim theorem: its proof is essentially the same as that of the determining of the type for Taylor series of entire functions of one complex variable.

Theorem 1.2 If \(f(t, z)\) is an entire solution of \((1.2)\) defined by (1.3) and (1.4) such that
\(1 < \lambda = \text{ord}(f) < \infty\), then the type \(\sigma = \text{typ}(f)\) satisfies
\[ e^{\lambda \sigma} = 2^{\lambda/2} \limsup_{n \to \infty} 2n |c_n|^{\lambda/(2n)}. \]
Brosch [4] proved that if two nonconstant meromorphic functions \(f\) and \(g\) on \(\mathbb{C}\) share three distinct values \(c_1, c_2, \ldots, c_3\) counting multiplicities, and if \(f\) is a solution of the differential equation

\[
\left( \frac{dw}{dz} \right)^n = \sum_{j=0}^{2n} b_j(z) w^j := P(z, w)
\]

such that \(b_0, b_1, \ldots, b_{2n} (b_{2n} \neq 0)\) are small functions of \(f\) (grow slower than \(f\)), furthermore if \(P(z, c_i) \neq 0\) for \(i = 1, 2, 3\), then \(f = g\). To state a generalization of Broschi's result to PDE, we abbreviate

\[
u_t = \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}, \quad u_{ttt} = \frac{\partial^3 u}{\partial t^3},
\]

and so on, and set

\[
Du = a_0 u_t^2 + 2a_1 u_t u_{tt} + a_2 u_{ttt},
\]

\[
Lu = a_0 u_{tt} + 2a_1 u_{ttt} + a_2 u_{tttt} + a_3 u_{tttt} + a_4 u_{ttttt}.
\]

We make the following assumption:

(A) All coefficients \(a_i\) in (1.1) are polynomials and when \(a_{6} = 0\) there are no nonconstant polynomials \(u\) satisfying the system

\[
\begin{cases}
Du = 0, \\
Lu = 0.
\end{cases}
\]

For technical reason, here we study only meromorphic functions of finite orders. The order of a meromorphic function of several variables may be defined by using its Nevanlinna's characteristic function (cf. [12], [22]).

**Theorem 1.3** Assume that the assumption (A) holds. Let \(f(t, z)\) be a nonconstant meromorphic solution of (1.1) such that \(\text{ord}(f) < \infty\) and let \(g\) be a nonconstant meromorphic function of finite order on \(\mathbb{C}^2\). If \(f\) and \(g\) share \(0, 1, \infty\) counting multiplicity, one of the following five cases is occurred:

(a) \(g = f\);
(b) \(gf = 1\);
(c) \(a_6 = 0, \ gf = f + g\);
(d) \(a_6 = 0, \) and there exist a constant \(b \notin \{0, 1\}\) and a polynomial \(\beta\) such that

\[
f = \frac{1}{b-1} (e^\beta - 1), \quad g = \frac{b}{b-1} (1 - e^{-\beta});
\]

(e) \(a_6 \neq 0, \) \(f^2 g^2 = 3fg - f - g.\)
When \( a_0 \neq 0 \), the case (b) may happen. For example, we consider the differential equation
\[
\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - u = 0,
\] (1.6)
which has an entire solution of order 1
\[
f(t, z) = e^{t+z}.
\]
Let's compare \( f \) with the following entire function of order 1
\[
g(t, z) = e^{t-z}.
\]
Obviously, \( f \) and \( g \) share 0, 1, \( -1, \infty \) counting multiplicity, but \( g \neq f, gf = 1 \). Now the differential equation
\[
Lu + Du + a_0 = 0
\]
has a nonconstant polynomial solution
\[
u(t, z) = t + z.
\]
The condition (A) is meaningful. For example, Theorem 1.1 shows that the differential equation (1.2) has a lot of entire solutions of finite orders. Obviously, the condition (A) associated to the differential equation (2) holds, and hence we can obtain the fact:

**Corollary 1.4** Let \( f(t, z) \) be a nonconstant meromorphic solution of (1.2) such that \( \text{ord}(f) < \infty \) and let \( g \) be a nonconstant meromorphic function of finite order on \( \mathbb{C}^2 \). If \( f \) and \( g \) share 0, 1, \( -1, \infty \) counting multiplicity, then we have either \( g = f \) or \( gf = 1 \) or \( f^2g^2 = 3fg - f - g \).

The case (b) in Theorem 1.3 may really happen for \( a_0 = 0 \). For example, we consider the differential equation
\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial z} = 0,
\] (1.7)
which has an entire solution \( f(t, z) = e^{t+z} \) of order 1 such that the assumption (A) holds obviously. The entire solution \( f \) and the function \( g = e^{t-z} \) share 0, 1, \( -1, \infty \) counting multiplicity, and satisfy \( gf = 1 \), that is, the case (b) in Theorem 1.3 happens for the case \( a_0 = 0 \).

For a real number \( x \), let \( [x] \) denote the maximal integer \( \leq x \). We give the following result that is an analogue of A. Anastassiou's theorem [1] on uniqueness of entire functions of one variable.

**Theorem 1.5** Let \( f(t, z) \) and \( g(t, z) \) be transcendental entire solutions of (1.2) such that \( \text{ord}(f) < \infty, \text{ord}(g) < \infty \), and
\[
\frac{\partial^j f}{\partial t^j \partial z}(0, 0) = \frac{\partial^j g}{\partial t^j \partial z}(0, 0), \quad j = 0, 1, \ldots, q,
\]
where
\[
q = \max\{[\text{ord}(f)], [\text{ord}(g)]\}.
\]
If there exists a complex number \( a \) with \( (a, f(0, 0)) \neq (0, 0) \) such that \( f \) and \( g \) share a counting multiplicity, then we have \( f = g \).
Theorem 1.3 shows that when \( a_0 = 0 \), global solutions of the equation (1.1) can be quite complicated, however, when \( a_0 \neq 0 \), these solutions have normal properties. Next result also supports this view. Theorem 1.6 extends a theorem (cf. Theorem 5.8 of [10]) on meromorphic solutions of linear ordinary differential equations.

**Theorem 1.6** Assume that all \( a_k \) in (1.1) are entire functions on \( \mathbb{C}^2 \) which grow slower than a meromorphic solution of equations (1.1) on \( \mathbb{C}^2 \). If \( a_0 \neq 0 \), then the deficiency of the solution for each non-zero complex number is zero.

For example, the telegraph equation

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + 2a_0 \frac{\partial u}{\partial x} + \alpha^2 \psi = 0
\]

has entire solutions

\[
u(t, z) = e^{-\alpha t} \{ f(z + ct) + g(z - ct) \},
\]

where \( f \) and \( g \) are entire functions on \( \mathbb{C} \). If \( \alpha \neq 0 \), Theorem 1.6 shows that the deficiency of a non-constant \( u(t, z) \) for each non-zero complex number \( \alpha \) is zero, which means that the equation

\[
f(z + ct) + g(z - ct) - \alpha e^{\alpha t} = 0
\]

has zeros.

Let \( \mathbb{Z}^m \) denote the set of non-negative integers. For \( \tau = (\tau_1, \ldots, \tau_m) \in \mathbb{C}^m \), \( 1 = (i_1, \ldots, i_m) \in \mathbb{Z}^m \), we write

\[
\tau_k = \frac{\partial}{\partial \tau_k}, \quad k = 1, \ldots, m; \quad \tau^l = \tau_1^l \cdots \tau_m^l; \quad |l| = i_1 + \cdots + i_m.
\]

We have interesting in the following problem:

**Conjecture 1.7** If \( f \) is a meromorphic function in \( \mathbb{C}^m \) such that \( f \) and \( \tau f \) have no zeros for some \( l = (l_1, \ldots, l_m) \in \mathbb{Z}^m \) with \( l_k \geq 2 \) (\( 1 \leq k \leq m \)) and such that the set of poles of \( f \) is algebraic, then there exists a partition

\[
\{1, \ldots, m\} = I_0 \cup I_1 \cup \cdots \cup I_k
\]

such that \( I_i \cap I_j = \emptyset \) (\( i \neq j \)), and

\[
f(z_1, \ldots, z_m) = \exp \left( \sum_{i \in I_0} A_i z_i + B_0 \right) \prod_{j=1}^{k} \left( \sum_{i \in I_j} A_i z_i + B_j \right)^{-n_j},
\]

where \( A_i, B_j \) are constants with \( A_i \neq 0 \), and \( n_j \) are positive integers.

This is open if \( m > 1 \). For detail discussion, see [16]. When \( m = 1 \), the conclusion of Conjecture 1.7 was obtained by Tumura [24], and Hayman [8] gave a proof for the case \( l = l_m = 2 \). Later, as a correction of the gap in Tumura’s proof, Clunie [6] gave a valid proof of the assertion for any \( l > 1 \).
Let $f$ be a meromorphic function in $\mathbb{C}^n$ which we shall assume to be not constant. We shall be concerned largely with meromorphic functions $h$ which are polynomials in $f$ and the partial derivatives of $f$ with coefficients $a$ of the form

$$|| \ T(r, a) = o(T(r, f)), \ (1.8)$$

where $T(r, f)$ is the Nevanlinna's characteristic function of $f$, and where the symbol "\|" means that the relation holds outside a set of $r$ of finite linear measure. Such functions $h$ will be called differential polynomials in $f$. To study Conjecture 1.7, the following result will play a crucial role.

**Theorem 1.8** Suppose that $f$ is meromorphic and not constant in $\mathbb{C}^n$, that

$$g = f^n + P_{n-1}(f), \ (1.9)$$

where $P_{n-1}(f)$ is a differential polynomial of degree at most $n - 1$ in $f$, and that

$$|| \ N(r, f) + N \left( r, \frac{1}{g} \right) = o(T(r, f)), \ (1.8)$$

where $N(r, f)$ is the Nevanlinna's valence function of $f$ for poles. Then

$$g = \left( f + \frac{a}{n} \right)^n,$$

where $a$ is a meromorphic function of the form (1.8) in $\mathbb{C}^n$ determined by the terms of degree $n - 1$ in $P_{n-1}(f)$ and by $g$.

When $m = 1$, Theorem 1.8 is due to Hayman ([9], Theorem 3.9, p.69). By using Theorem 1.8, we can give a proof of Conjecture 1.7, under a condition on non-vanishing of the partial derivatives of order $> 1$ that differs from the one posed in the conjecture, as follows:

**Theorem 1.9** If $f$ is a meromorphic function in $\mathbb{C}^n$ such that $f, \partial^k f, \ldots, \partial^m f$ have no zeros for some $k \geq 2$ ($1 \leq k \leq m$) and such that the set of poles of $f$ is algebraic, then there exists a partition

$$\{1, \ldots, m\} = I_0 \cup I_1 \cup \cdots \cup I_k$$

such that $I_i \cap I_j = \emptyset$ ($i \neq j$), and

$$f(z_1, \ldots, z_m) = \exp \left( \sum_{i \in I_0} A_i z_i + B_0 \right) \prod_{j=1}^k \left( \sum_{i \in I_j} A_i z_i + B_j \right)^{-n_j},$$

where $A_i, B_j$ are constants with $A_i \neq 0$, and $n_j$ are positive integers.

In particular, if $f$ is entire, the function $f$ in Theorem 1.9 has only an exponential form

$$f(z_1, \ldots, z_m) = \exp \left( A_1 z_1 + \cdots + A_m z_m + B_0 \right).$$

We shall utilize the methods developed in [9], [12] and [13] and generalized Clunie lemma to prove the main results.
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ТЕОРЕМЫ ЕДИНИСТВЕННОСТИ МЕРОМОРФНЫХ ФУНКЦИЙ НЕСКОЛЬКИХ КОМПЛЕКСНЫХ ПЕРЕМЕННЫХ

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Аннотация. В работе исследуются вопросы существования, единственности и распределения значений мероморфных (или целых) решений линейных дифференциальных уравнений в частных производных второго порядка с полиномиальными коэффициентами.

Ключевые слова: мероморфные функции, однородные линейные дифференциальные уравнения в частных производных, теория Неваннина распределения непрерывности.