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On Solvability of Linear Conjugation Problem in Weighted Holder Spaces

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Abstract. For analytic vector-functions from a family of Hölder spaces with a power weight at the nodes of an arbitrary piecewise-Lyapunov curve, a boundary-value problem for the linear conjugation on the curve is considered. Under the assumption that the boundary-value coefficient is a triangular matrix-function, we describe necessary and sufficient solvability conditions for that problem and provide its efficient solution.

Consider the classical linear conjugation problem

$$\phi^+ - G\phi^- = g \tag{1}$$

for analytic l-vector-functions $\phi = (\phi_1, \ldots, \phi_l)$, where the matrix coefficient $G = (G_{ij})_1^l$ is defined on a piecewise-smooth curve Γ such that the set of its nodes is denoted by F. In a neighborhood of infinity, components of such a function admit the expansion

$$\phi_k(z) = \sum_{j \le n_k - 1} c_{k,j} z^j,$$

where n_k are integers. Express that fact as follows:

$$\deg \phi_k \le n_k - 1, \quad 1 \le k \le l. \tag{2}$$

It is assumed that the curve Γ consists of orientable smooth arcs $\Gamma_1, \ldots, \Gamma_m$ such that their pairwise intersections are at most their endpoints forming a finite set F of points. It is assumed that the function $g(t)$ belongs to the Hölder class $C^\mu(\Gamma_0)$ for any arc Γ_0 such that $\Gamma_0 \subseteq \Gamma \setminus F$. The class of all such functions is denoted by $C^\mu_{loc}(\Gamma, F)$. Respectively, the desired function ϕ, which is analytic in $D = \mathbb{C} \setminus \Gamma$, belongs to the class $C^\mu_{loc}(D, F)$ defined by the following condition: $\phi \in C^\mu(\overline{D}_0)$ for any subdomain D_0 of D, bounded by a piecewise-smooth contour such that it does not contain points τ from F. In particular, one-sided boundary-values $\phi^\pm$ exist and belong to $C^\mu_{loc}(\Gamma, F)$.

In a small neighborhood $\{|z - \tau| < r\}$ of a point τ from F, the curve Γ consists of several smooth arcs $\Gamma_{\tau,j},\ 1 \le j \le n_\tau$, such that they have a joint endpoint τ. Therefore, in that neighborhood, the open set D is decomposed into curvilinear sectors $D_{\tau,j},\ 1 \le j \le n_\tau$, with a joint vertex τ. The sum of n_τ with respect to all τ from F is equal to $2m$. Any Γ_j includes two end-arcs of the family $\Gamma_{\tau,j},\ 1 \le j \le n_\tau,\ \tau \in F$; denote them by Γ_j^0 and Γ_j^1 respectively. Let τ_j^0 and τ_j^1 be the left-hand and right-hand endpoints of the arc Γ_j respectively (with respect to its orientation). In the above notation, let τ_j^p be the marked point of $\Gamma_j^p,\ p = 0, 1$. Thus, the arc Γ_j^0 leaves τ_j^0 and the arc Γ_j^1 enters τ_j^1. Assign $\sigma_{\tau,j} = -1$ if the arc $\Gamma_{\tau,j}$ leaves the point τ and assign $\sigma_{\tau,j} = 1$ otherwise.

It is assumed that the matrix coefficient G is piecewise-continuous (more exactly, is continuous on each arc Γ_j) and each value of the scalar function $\det G$, including its one-sided limit values at nodes τ, is different from zero. The family of $2m$ one-sided limit values

$$G(\tau, j) = \lim G(t) \quad \text{as} \quad t \to \tau,\ t \in \Gamma_{\tau,j}, \tag{3}$$

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can be represented by the family $G(\tau_j^0+0),\ G(\tau_j^1-0),\ 1\le j\le m$, of the corresponding limit values at endpoints of any arc Γ_j.

Problem (1) is considered in weight classes if the function g from $C^\mu_{loc}(\Gamma,F)$ behaves as a power at nodes τ from F, i. e., $g(t)=O(|t-\tau|^{\lambda_\tau})$ as $t\to\tau\in F$, where the weight order λ_τ is arbitrary. More exactly, on each arc $\Gamma_{\tau,j}$, it is representable in the form $g(t)=(t-\tau)^{\lambda_\tau-\mu}g_{\tau,j}(t)$, where the function $g_{\tau,j}$ belongs to $C^\mu(\Gamma_{\tau,j})$ and vanishes at the point τ. Denote the class of such functions by $C^\mu_\lambda(\Gamma,F)$. Its various properties (with slightly else notation) are described in [1] (see [2] as well). Note that if $\lambda>0$, then all functions of that class belong to $C^\varepsilon(\Gamma)$, $\varepsilon=\min(\mu,\lambda_\tau,\ \tau\in F)$, and vanish at points τ of F.

The class $C^\mu_\lambda(D,F)\subseteq C^\mu_{loc}(D,F)$ is defined with respect to curvilinear sectors $D_{\tau,j}$ in the same way. If $n_\tau=1$, then S_τ is a disk with the cut along Γ_τ and this sector is decomposed (along the specified direction) into two sectors such that the opening of each one is less than 2π.

It is assumed that the matrix coefficient G belongs to $C^\mu(\Gamma_0)$ for each arc Γ_0 such that $\Gamma_0\subseteq\Gamma\setminus F$ and there exists a (small) positive ε depending on G such that

$$G_{\tau,j}(t)=G(t)-G(\tau,j)\in C^\mu_\varepsilon(\Gamma_{\tau,j},\tau),\quad 1\le j\le n_\tau,\ \tau\in F,$$

in the notation of (3). The class of such piecewise-continuous matrices is denoted by $C^\mu_{(+0)}(\Gamma,F)$.

If the Hölder exponent μ is not specified and $-1<\lambda\le 0$, then, for curves such that $n_\tau\le 2$, problem (1) is thoroughly investigated for the scalar case (i. e., for the case where $l=1$, see [3, 4]) and for the vector case (see [5]). In the general case, its Fredholm solvability is considered in [6] in detail. However, in the general case, the kernel and cokernel of that problem (their dimensions are determined by partial indices of the matrix G) are unknown. The situation changes if the matrix $G=(G_{ij})_1^l$ is triangular. For definiteness, let

$$G_{ij}=0,\quad i>j, \tag{4}$$

where the diagonal elements G_{kk} are different from zero at each point of Γ, including their limit values at points τ of F. Then the resolving of problem (1) is reduced to the sequential resolving of scalar problems

$$\phi_k^+-G_{kk}\phi_k^-=f_k,\quad k=1,\dots,l, \tag{5}$$

in the class of functions ϕ_k belonging to $C^\mu_\lambda(D,F)$ and obeying Condition (2) at infinity. Solutions of those problems are easily described by means of the Cauchy integrals and canonical functions (see [7]).

Assign

$$\frac{1}{2\pi}\arg\prod_{j=1}^{n_\tau}[G_k(\tau,j)]^{\sigma_{\tau,j}}=\alpha_{k,\tau}+i\beta_{k,\tau},\quad 0\le\alpha_{k,\tau}<1, \tag{6}$$

and assume that

$$\lambda_\tau\notin\Delta_{k,\tau}=\{\alpha_{k,\tau}+s,\ s\in\mathbb{Z}\}. \tag{7}$$

Then the weight order $\delta_k=(\delta_{k,\tau},\tau\in F)$ can be introduced according to the condition

$$\delta_{k,\tau}\in\Delta_{k,\tau},\quad \lambda_\tau<\delta_{k,\tau}<\lambda_\tau+1. \tag{8}$$

Consider the branch of the logarithm $\ln G_{kk}$, continuous on $\Gamma\setminus F$ and such that it and G_{kk} itself belong to the class $C^\mu_{(+0)}(\Gamma,F)$. Consider the Cauchy index $\operatorname{Ind}G_{kk}$ of the function G_{kk}, i. e., the sum of the increments of the continuous branch of $\ln G_{kk}$ on arcs Γ_j taken according to their orientation, divided by $2\pi i$:

$$\operatorname{Ind}G_{kk}=\frac{1}{2\pi i}\sum_{j=1}^m[(\ln G_{kk})(\tau_j^1-0)-(\ln G_{kk})(\tau_j^0+0)].$$

By virtue of (6), we have the relation

$$\frac{1}{2\pi i}\sum_{s=1}^{n_\tau}\sigma_{\tau,s}(\ln G_{kk})(\tau,s)=\alpha_{k,\tau}+i\beta_{k,\tau}+s_{k,\tau},\quad s_{k,\tau}\in\mathbb{Z}. \tag{9}$$

Note that the sum of the left-hand parts of that relation, taken over all τ from F, coincides with the Cauchy index $\operatorname{Ind}G_{kk}$. Therefore,

$$\operatorname{Ind}G_{kk}=\sum_\tau(\alpha_{k,\tau}+i\beta_{k,\tau}+s_{k,\tau}).$$

Hence, one can introduce the integer

$$\widetilde{\operatorname{Ind}}\, G_{kk} = \operatorname{Ind} G_{kk} - \sum\nolimits_{\tau} (\alpha_{k,\tau} + i\beta_{k,\tau})$$

such that

$$\sum\nolimits_{\tau} s_{k,\tau} = \widetilde{\operatorname{Ind}}\, G_{kk}. \tag{10}$$

It is well known (see [1, 2]) that the Cauchy-type integral

$$\phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(t)dt}{t - z}, \quad z \in D,$$

defines a bounded operator $I: \ C^{\mu}_{\lambda}(\Gamma, F) \to C^{\mu}_{\lambda}(D, F)$ provided that $-1 < \lambda < 0$. Assigning

$$h_k(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\ln G_{kk})(t)dt}{t - z}, \quad z \in D,$$

introduce the function

$$X_k(z) = e^{h_k(z)} \prod\nolimits_{\tau \in F} (z - \tau)^{-s_{\tau}}, \tag{11}$$

where s_τ are integers. Under a suitable choice of s_τ, this function is canonical for problem (5).

In other words, it satisfies the boundary-value condition

$$X^{+}_{k} = G_{kk}\, X^{-}_{k} \tag{12}$$

and has the corresponding power-like behavior at points τ of F and at infinity.

More exactly, in the sectors $D_{\tau,j}$, the function h_k is representable as follows (see [7]):

$$h_k(z) = \frac{1}{2\pi i} \left[\sum\nolimits_{s=1}^{n_\tau} \sigma_{\tau,s} (\ln G_{kk})(\tau, s) \right] \ln(z - \tau) + h_{k,\tau,j}(z), \ \ h_{k,\tau,j} \in C^{\mu}_{(+0)}(D_{\tau,j}, \tau),$$

where $\sigma_{\tau,s} = -1$ if the arc $\Gamma_{\tau,s}$ leaves the point τ and $\sigma_{\tau,s} = 1$ otherwise.

By virtue of (9), we have the relation

$$X_k(z) = A_{k,\tau,j}(z)(z - \tau)^{-s_\tau + s_{k,\tau} + \alpha_{k,\tau} + i\beta_{k,\tau}}, \quad z \in D_{\tau,j},$$

where the functions $A_{k,\tau,j}$ and $1/A_{k,\tau,j}$ belong to the class $C^{\mu}_{(+0)}(D_{\tau,j}, \tau)$.

Select the integer s_τ in (11) to satisfy the relation $-s_\tau + s_{k,\tau} + \alpha_{k,\tau} = \delta_{k,\tau}$ in notation of (8). Then the previous relation takes the form

$$X_k(z) = A_{k,\tau,j}(z)(z - \tau)^{\delta_{k,\tau} + i\beta_{k,\tau}}, \quad z \in D_{\tau,j}. \tag{13}$$

From (8), we deduce that $s_\tau - s_{k,\tau} < \alpha_{k,\tau} - \lambda_\tau < s_\tau - s_{k,\tau} + 1$, which means that $s_\tau - s_{k,\tau} = [\alpha_{k,\tau} - \lambda_\tau]$, where $[x]$ denotes the integer part of the number x. Taking into account (10), this implies that

$$\sum\nolimits_{\tau} s_\tau = \sum\nolimits_{\tau} [\alpha_{k,\tau} - \lambda_\tau] + \widetilde{\operatorname{Ind}}\, G_{kk}.$$

Since the function $h_k = I(\ln G_{kk})$ from relation (11) vanishes at infinity, we have the relation

$$\lim_{z \to \infty} z^{\kappa_k} X_k(z) = 1, \quad \kappa_k = \sum\nolimits_{\tau} [\alpha_{k,\tau} - \lambda_\tau] + \widetilde{\operatorname{Ind}}\, G_{kk}. \tag{14}$$

It is easy to describe the solvability of problem (5) by means of properties (12), (13), and (15) of the canonical function (see [7]). By virtue of the choice of δ_k in (8), the weight Cauchy-type operator

$$(I_k\varphi)(t_0) = \frac{X^{+}_{k}(t_0)}{\pi i} \int_{\Gamma} \frac{\varphi(t)dt}{X^{+}_{k}(t)(t - t_0)}, \quad t_0 \in \Gamma, \tag{15}$$

boundedly acts from $C^{\mu}_{\lambda}(\Gamma, F)$ to $C^{\mu}_{\lambda}(D, F)$; taking into account (14), we see that the degree at infinity satisfies the inequality $\deg\, (I_k\varphi) \le \kappa_k - 1$.

Introduce the following finite-dimensional function families on Γ:

$$P_k = \{X_k^+(t) \sum\nolimits_{0\le i\le n_k+\kappa_k-1} c_i t^i, \quad c_i \in \mathbb{C}\}$$

$$Q_k = \{(X_k^+)^{-1}(t) \sum\nolimits_{0\le i\le -n_k-\kappa_k-1} c_i t^i, \quad c_i \in \mathbb{C}\},$$

where each sum over the empty set of indices is assumed to be equal to zero. Thus, $P_k = 0$ if $n_k + \kappa_k \le 0$ and, conversely, $Q_k = 0$ if $n_k + \kappa_k \ge 0$.

In this notation, the following classical result holds (see [3, 7]).

Theorem 1 *Problem (5) is solvable if and only if*

$$\int_\Gamma f_k(t) q_k(t) dt = 0, \quad q_k \in Q_k.$$

If this condition is satisfied, then its solution is represented as follows:

$$\phi_k = p_k + I_k f, \quad p_k \in P_k.$$

Consider the original vector problem posed by (1). Introduce the singular integral operator $I_k^- \varphi = (I_k\varphi)^-$; due to (12) and the Sochocki–Plemelj relation applied to integral (15), the specified operator acts as follows:

$$(I_k\varphi)^-(t_0) = \frac{1}{2G_{kk}(t_0)}\left[-\varphi(t_0) + \frac{X_k^+(t_0)}{\pi i}\int_\Gamma \frac{\varphi(t)dt}{X_k^+(t)(t-t_0)}\right], \quad t_0 \in \Gamma.$$

Further, introduce the operators R_{kj}, $1 \le k \le j-1 \le l-1$, defined recurrently with respect to $j = k+1, \dots, l$ by the relations

$$R_{kj} = \begin{cases} G_{kj}, & j = k+1, \\ G_{kj} + \sum_{k+1\le s\le j-1} G_{ks} I_s^- R_{sj}, & j \ge k+2. \end{cases} \tag{16}$$

In particular,

$$R_{l-1,l} = G_{l-1,l}, \quad R_{l-2,l-1} = G_{l-2,l-1},$$

$$R_{l-2,l} = G_{l-2,l-1} + G_{l-2,l-1} I_{l-1}^- G_{l-1,l},$$

and so on.

Finally, in the space of l-vector-functions $f = (f_1, \dots, f_l)$, consider the operators $(If)_k = I_k f_k$,

$$(Af)_k = \sum\nolimits_{j\ge k+1} R_{kj} f_j, \ \text{ and } \ (Bf)_k = f_k + \sum\nolimits_{j\ge k+1} R_{kj} I_j^- f_j, \quad 1 \le k \le l. \tag{17}$$

Introduce the bilinear form

$$(f, g) = \sum_{k=1}^{l} \int_\Gamma f_k(t) g_k(t) dt;$$

using it, in the finite-dimensional spaces $P = P_1 \times \dots \times P_l$ and $Q = Q_1 \times \dots \times Q_l$, select the subspaces

$$P^0 = \{p \in P \mid (Ap, q) = 0, \ q \in Q\} \ \text{ and } \ Q^0 = \{q \in Q \mid (Ap, q) = 0, \ p \in P\}. \tag{18}$$

Theorem 2 *Problem (1)-(2) is solvable if and only if*

$$(Bg, q) = 0, \quad q \in Q^0. \tag{19}$$

If this condition is satisfied, then its solution is represented as follows:

$$\phi = p + I(Ap + Bg), \quad p \in P^0. \tag{20}$$

Proof. Let us use the component-wise form (6) with the right-hand part

$$f_k = g_k + \sum\nolimits_{j=k+1}^{l} G_{kj}\phi_j^-, \quad 1 \le k \le l,$$

for the boundary-value condition posed by (1). As usual, the sum over the empty set of indices is assumed to be equal to zero, i. e., $f_l = g_l$. To show that this relation can be represented in the form

$$f_k = g_k + \sum_{j \ge k+1} R_{kj}(p_j + I_j^- g_j), \quad p_j \in P_j, \tag{21}$$

in notation of (16), we sequentially use Theorem 1. For $k = l$, this relation is obvious. Using the induction and taking into account relations (21) with numbers $k, k+1, \ldots, l$, let us establish the same relation with number $k-1$. Due to (20), we have the relation

$$f_{k-1} = g_{k-1} + \sum\nolimits_{s \ge k} G_{k-1,s}\phi_s^-. \tag{22}$$

For brevity, assign $\widetilde{g}_j = p_j + I_j^- g_j$. From the induction assumption and Theorem 1, we conclude that

$$\phi_s^- = p_s + I_s^-[g_s + \sum\nolimits_{j \ge s+1} R_{sj}\widetilde{g}_j] = \widetilde{g}_s + I_s^- \sum\nolimits_{j \ge s+1} R_{sj}\widetilde{g}_j.$$

Substituting this expression in (22), we obtain the relation

$$f_{k-1} = g_{k-1} + \sum\nolimits_{s \ge k} G_{k-1,s}\widetilde{g}_s + \sum\nolimits_{j \ge k+1} \Big[\sum\nolimits_{k \le s \le j-1} G_{k-1,s} I_s^- R_{sj}\Big]\widetilde{g}_j.$$

This relation can be represented in the form

$$f_{k-1} = g_{k-1} + \sum\nolimits_{j \ge k} R_{k-1,j}\widetilde{g}_j,$$

where the operator $R_{k-1,j}$ acts as follows:

$$R_{k-1,j} = \begin{cases} G_{k-1,j}, & j = k, \\ G_{k-1,j} + \sum_{k \le s \le j-1} G_{k-1,s} I_s^- R_{sj}, & j \ge k+1; \end{cases}$$

this is coordinated with the recurrent definition given by (16).

In notation (17), relations (21) are represented in the operator form as follows:

$$f = Ap + Bg, \quad p \in P. \tag{23}$$

Due to Theorem 1, the solvability conditions for the considered problem have the following brief form: there exists a vector p from P such that

$$(Ap + Bg, q) = 0, \quad q \in Q, \tag{24}$$

where g is the right-hand part.

For brevity, let $m = \dim P$, $n = \dim Q$, and the vectors p^i, $1 \le i \le m$, and q^j, $1 \le j \le n$, form bases in P and Q respectively. Then, with respect to $p = \sum_i \xi_i p^i$, the orthogonality conditions posed by (24) are equivalent to the relations

$$\sum_{i=1}^{m} (Ap^i, q^j)\xi_i = -(Bg, q^j), \quad 1 \le j \le n.$$

They can be treated as a system od linear equations with respect to ξ. It is solvable if and only if

$$\sum_{j=1}^{m} (Bg, q^j)\eta_j = 0 \tag{25}$$

for all solutions η of the adjoint homogeneous system

$$\sum_{j=1}^{m} (Ap^i, q^j)\eta_j = 0, \quad 1 \le i \le n.$$

Obviously, for those solutions, vectors $q = \sum_j \eta_j q^j$ describe the subspace Q^0 of Q from (18). Then (25) takes the form of Conditions (19). They are orthogonality conditions necessary and sufficient for the solvability of problem (1). If they are satisfied, then the solution is given by relation (20), which completes the proof of the theorem.

Due to (20), the dimension of the space of solutions of the homogeneous problem is equal to $\dim P^0$. From the definition of A given by (17), it is clear that A is a one-to-one operator. Then, by virtue of (19), the codimension of problem (1) is equal to $\dim Q^0$. In particular, its index κ is equal to

$$\kappa = \dim P^0 - \dim Q^0. \tag{26}$$

To prove that

$$\dim P^0 - \dim Q^0 = \dim P - \dim Q, \tag{27}$$

we note that $P = P^0 \oplus P^1$ and $Q = Q^0 \oplus Q^1$, and consider the bilinear form $\langle p, q\rangle = (Ap, q)$ on the product $P^1 \times Q^1$. From Definition (18), it follows that this form is nondegenerate, i. e., if $p \in P^1$, then the validity of the relation $\langle p, q\rangle = 0$ for each q from Q^1 implies that $p = 0$ and, conversely, if $q \in Q^1$, then the validity of the relation $\langle p, q\rangle = 0$ for each p from P^1 implies that $q = 0$. Hence, the dimensions of the spaces P^1 and Q^1 coincide each other, which proves relation (27).

If s is integer, then assign

$$s^+ = \begin{cases} s, & s \geq 0, \\ 0, & s < 0, \end{cases} \qquad s^- = \begin{cases} 0, & s \geq 0, \\ -s, & s < 0. \end{cases}$$

Thus, $s^+ - s^- = s$ for each s. In this notation, we have

$$\dim P = \sum\nolimits_k (n_k + \kappa_k)^+ \quad \text{and} \quad \dim Q = \sum\nolimits_k (n_k + \kappa_k)^-.$$

Combining this with (26)-(27), we obtain the relation

$$\kappa = \sum\nolimits_{k=1}^{l} (n_k + \kappa_k)$$

for the index of problem (1). For $l = 1$, this relation is coordinated with Theorem 1.

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