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FLUCTUATIONS IN ONE-DIMENSIONAL DYNAMIC SYSTEMS

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We consider one-dimensional dynamic systems with random fluctuations that are encountered in applications, analyze solutions, and investigate the stability of stationary points.

In applications we frequently encounter dynamic systems with one-dimensional phase spaces. Examples are provided by coastal systems in geomorphology, autonomous populations in biology, autocatalytic reactions in chemistry, etc. [1, 2]. In general, such systems are described by equations of the form

$$\frac{dw}{dt} = F(w),$$

where

$$w \in [a, b]; \quad a, b \in R. \tag{1}$$

In such systems, an important role is often played by a variety of fluctuations that are as likely to be associated with fluctuations in macroscopic parameters as the stochastic nature of processes that occur in the systems themselves. Following [2], we will attempt to account for random fluctuations by replacing the deterministic equation (1) with the stochastic differential equation (a stochastic Ito equation)

$$dw = F(w)dt + g d\sigma(t), \tag{2}$$

where the second term describes the contribution of fluctuations to w, and g is the amplitude of the fluctuations (which we assume to be constant).

On $d\sigma(t)$ we impose the conditions

$$\begin{cases} \langle d\sigma(t)\rangle = 0, \\ \langle (d\sigma(t))^2\rangle = dt, \end{cases} \tag{3}$$

where $\langle \ldots \rangle = \int_0^\infty (\ldots)\rho(w, t|w_0, t_0)dw$ is the statistical average, and $\rho(w, t|w_0 t_0)$ is the probability that the system coordinate in the phase space at time t will have the value w if it has the value w_0 at time t_0.

Averaging (2), we obtain Ito's equation for the mean value [2]:

$$\frac{dw}{dt} = \langle F(w)\rangle. \tag{4}$$

From Eq. (2) and the statistical independence of $d\sigma(t)$ and w we obtain, for the probability distribution function, the Ito–Fokker–Planck equation [2]:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial w}(F(w)\rho) + \frac{g^2}{2}\frac{\partial^2 \rho}{\partial w^2}. \tag{5}$$

The formal solution of the Cauchy problem for this equation is of the form [3]

$$\rho(w, t) = \exp\left\{(t - t_0)\left[\frac{g^2}{2}\frac{d^2}{dw^2} - \frac{d}{dw}F(w)\right]\right\}\varphi(w), \tag{6}$$

where $\varphi(w) \equiv \rho(w, t_0)$ is the initial condition.

We now introduce the notation $p \equiv -i\frac{d}{dw}$, so the function $\rho(w, t)$ takes the form of the amplitude of the probability (Ψ-function) for a one-dimensional quantum particle with coordinate w and Hamiltonian $H = \frac{g^2}{2i}p^2 + pF(W)$ [4]. Thus, every one-dimensional stochastic system can be treated as a one-dimensional quantum particle with imaginary mass ig^{-2} that is subject to friction. It now follows, according to [4], that $\rho(w, t) = \langle w|t\rangle$, where $|w\rangle$ is the eigenvector of the coordinate operator of a quantum particle, and $|t\rangle$ is the state vector of the particle, which depends on time and is equal to

$$|t\rangle = \exp\{-iH(t - t_0)\}|t_0\rangle \equiv \exp\{-i(H_0 + H_1)(t - t_0)\}|t_0\rangle. \tag{7}$$

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where $|t_0\rangle$ is the state vector of the particle at time t_0.

$$H_0 = \frac{g^2}{2i}p^2, \quad H_1 = pF(w), \quad \langle w|t_0\rangle \equiv \varphi(w).$$

We represent $|t\rangle$ in the form

$$|t\rangle = \exp\{-iH_0(t-t_0)\}|t_0\rangle + U(t)|t_0\rangle; \tag{8}$$

now, upon differentiation of expression (8), we obtain, for the operator $U(t)$, the equation

$$\frac{dU}{dt} = -i(H_0+H_1)U - iH_1\exp\{-iH_0(t-t_0)\}, \tag{9}$$

from which it follows that

$$U(t) = \exp\{-iH_0(t-t_0)\}\int_{t_0}^{t}\exp\{-iH_0(\theta-t_0)\}$$
$$\times[-iH_1U(\theta) - iH_1\exp\{-iH_0(\theta-t_0)\}]\,d\theta; \tag{10}$$

this last is clearly equivalent to the integral equation

$$\left\{1+i\int_{t_0}^{t}d\theta\exp\{iH_0(\theta-t)\}H_1(\theta\ldots\right\}U(t) = -i\int_{t_0}^{t}d\theta\exp\{iH_0(\theta-t)\}$$
$$\times H_1\exp\{-iH_0(\theta-t_0)\}, \tag{11}$$

whose solution is

$$U(t) = \sum_{n=0}^{\infty}(-i)^{n+1}\int_{t_0}^{t}d\theta_n\int_{t_0}^{\theta_n}d\theta_{n-1}\cdots\int_{t_0}^{\theta_2}d\theta_1\int_{t_0}^{\theta_1}d\theta_0\exp\{iH_0(\theta_n-t)\}H_1(\theta_n)$$
$$\times\exp\{iH_0(\theta_{n-1}-\theta_n)\}H_1(\theta_{n-1}\ldots\exp\{iH_0(\theta_0-\theta_1)\}H_1(\theta_0)\exp\{-iH_0(\theta_0-t_0)\}. \tag{12}$$

Here we have used the known expansion for the operators $(I-A)^{-1} = \sum_{n=0}^{\infty}A^n$, where A is an operator and I is the identity operator. As a result,

$$\rho(w,t) = \langle w|t\rangle = \langle w|\exp\{-iH_0)(t-t_0)\}|t_0\rangle + \langle w|U(t)|t_0\rangle$$
$$= \exp\left\{(t-t_0)\frac{g^2}{2}\frac{d^2}{dw^2}\right\}\varphi(w) + \sum_{n=0}^{\infty}(-i)^{n+1}\int_{t_0}^{t}d\theta_n\int_{t_0}^{\theta_n}d\theta_{n-1}\cdots\int_{t_0}^{\theta_2}d\theta_1\int_{t_0}^{\theta_1}d\theta_0$$
$$\times\int_0^{\infty}d\alpha_n\int_0^{\infty}d\beta_n\cdots\int_0^{\infty}d\alpha_1\int_0^{\infty}d\beta_1\int_0^{\infty}d\alpha_0\int_0^{\infty}d\beta_0 < \langle w|\exp\{iH_0(\theta_n-t)\}|\alpha_n\rangle$$
$$\langle\alpha_n|H_1(\theta_n)|\beta_n\rangle\langle\beta_n|\exp\{iH_0(\theta_{n-1}-\theta_n)\}|\alpha_{n-1}\rangle\langle\alpha_{n-1}|H_1(\theta_{n-1})|\beta_{n-1}\rangle$$
$$\langle\beta_{n-1}|\ldots\langle\beta_1|\exp\{iH_0(\theta_0-\theta_1)\}|\alpha_0\rangle\langle\alpha_0|H_1(\theta_0)|\beta_0\rangle$$
$$\times\langle\beta_0|\exp\{-iH_0(\theta_0-t_0)\}|t_0\rangle = \int_{-\infty}^{\infty}\frac{dy}{g\sqrt{2\Pi(t-t_0)}}\exp\left\{-\frac{(w-y)^2}{2g^2(t-t_0)}\right\}$$
$$\times\varphi(y) + \sum_{n=0}^{\infty}\int_{t_0}^{t}d\theta_n\int_{t_0}^{\theta_n}d\theta_{n-1}\cdots\int_{t_0}^{\theta_2}d\theta_1\int_{t_0}^{\theta_1}d\theta_0\int_0^{\infty}d\beta_n\cdots\int_0^{\infty}d\beta_0$$
$$\times\frac{\exp\left\{-\frac{(w-\beta_n)^2}{2g^2(t-\theta_n)}\right\}}{g^3\sqrt{2\Pi}(t-\theta_n)^{3/2}}[w-\beta_n]F(\beta_n)\frac{\exp\left\{-\frac{(\beta_n-\beta_{n-1})^2}{2g^2(\theta_n-\theta_{n-1})}\right\}}{g^3\sqrt{2\Pi}(\theta_n-\theta_{n-1})^{3/2}}$$
$$\times[\beta_n-\beta_{n-1}]F(\beta_{n-1})\ldots\frac{\exp\left\{-\frac{(\beta_1-\beta_0)^2}{2g^2(\theta_1-\theta_0)}\right\}}{g^3\sqrt{2\Pi}(\theta_1-\theta_0)^{3/2}}[\beta_1-\beta_0]F(\beta_0)$$

$$\times \int_{-\infty}^{\infty} \frac{dy}{g\sqrt{2\Pi(\theta - t_0)}} \exp\left\{-\frac{(\beta_0 - y)^2}{2g^2(\theta_0 - t_0)}\right\} \varphi(y).$$

In the computation above we used the known formulas

$$\exp\left\{A\frac{d^2}{dw^2}\right\}\varphi(w) = \int_{-\infty}^{\infty} \frac{dy}{2\sqrt{\Pi A}} \exp\left\{-\frac{(w-y)^2}{4A}\right\}\varphi(y);$$

$$\langle\alpha|H_1(\theta)|\beta\rangle \equiv \langle\alpha|pF(w)|\beta\rangle = -i\delta'(\alpha - \beta)F(\beta). \tag{13}$$

where $\langle\alpha|H_1(\theta)|\beta\rangle$ is the operator H_I in the w-representation [4]. In the zero-th order theory of perturbations we have

$$\rho_0(w,t) = \int_{-\infty}^{\infty} \frac{dy}{g\sqrt{2\Pi(t-t_0)}} \exp\left\{-\frac{(w-y)^2}{2g^2(t-t_0)}\right\}\varphi(y)$$

$$+ \int_{t_0}^{t} d\theta \int_{0}^{\infty} d\gamma \frac{\exp\left\{-\frac{(w-\gamma)^2}{2g^2(t-\theta)}\right\}}{g^3\sqrt{2\Pi}(t-\theta)^{3/2}}[w-\gamma]F(\gamma)$$

$$\times \int_{-\infty}^{\infty} \frac{dy}{g\sqrt{2\Pi(\theta - t_0)}} \exp\left\{-\frac{(\gamma - y^2)}{2g^2(\theta - t_0)}\right\}\varphi(y). \tag{14}$$

As we can see, the zero-th approximation (14) corresponds to expansion in the amplitude of the fluctuations to $1/g^4$, inclusive. Thus, this approximation is better for sufficiently large fluctuation amplitudes.

We now consider the stationary case of Eq. (5),

$$-\frac{d}{dw}[F(w)\rho(w)] + \frac{g^2}{2}\frac{d^2}{dw^2}\rho(w) = 0. \tag{15}$$

Integration of Eq. (15) yields

$$\rho' - \frac{2}{g^2}F\rho = const. \tag{16}$$

It follows from the physical meaning of $\rho(w)$ that

$$\lim_{w\to\infty} \rho'(w) = \lim_{w\to\infty} \rho(w) = 0. \tag{17}$$

It follows that Eq. (16) now takes the form

$$\rho' - \frac{2}{g^2}F(w)\rho = 0, \tag{18}$$

whose solution is

$$\rho = C\exp\left\{\frac{2}{g^2}\int F(w)\,dw\right\}. \tag{19}$$

where the constant C is found from the normalization conditions.

The extrema of the function (19) are found from the condition $F(w) = 0$.

Thus, the extrema of the function (19) correspond to the fixed point of the dynamic system (1). Let the function (19) have a maximum or minimum at the point $\tilde{w}$. Now

$$F(w) = F'(\tilde{w})(w - \tilde{w}) + O(|w - \tilde{w}|^2), \tag{20}$$

where O is the Landau symbol. If $\tilde{w}$ is a maximum point, $F'(w) < 0$ and it is clear from (20) that $\tilde{w}$ is a stable stationary point. If, however, $\tilde{w}$ is a minimum point, $F'(w) > 0$ and $\tilde{w}$ is an unstable stationary point. Thus, in measurements we most probably observe a system at a stable stationary point and least probably detect an unstable point, with agrees with the case of the dynamic system (1) without fluctuations. If, however, the right side of (1) depends on a parameter and the system loses stability at a stationary point when the parameter is changed, in the case of stochastic dynamics, a change in the system parameter will cause the system to leave the neighborhood of a stationary point when it becomes unstable to enter the neighborhood of another stable stationary point.

References

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