On elliptic boundary value problems on upper half-plane†

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The boundary value problems for elliptic systems of the second order with leading and constant coefficients are considered in a half-plane. The investigation is based on the Bitsadze formula which represents a general solution of this system through a vector-valued analytic function. The transformation of this formula is studied in Holder spaces with weight. As a consequence, the explicit formulas of the solution to the problems are received. The applications to anisotropic elasticity are also given.

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1. Elliptic systems of the second order and *J*-analytic functions

Let us consider the elliptic system

$$a_{11}u_{xx} + (a_{12} + a_{21})u_{xy} + a_{22}u_{yy} = 0 \tag{1}$$

with constant coefficients $a_{ij} \in \mathbb{R}^{l\times l}$ for vector-valued function $u = (u_1, \dots, u_l)$. The ellipticity condition means that $\det a_{22} \neq 0$ and roots of the characteristic polynomial

$$\chi = \det P, \quad P(z) = a_{11} + (a_{12} + a_{21})z + a_{22}z^2$$

are not real. Let $\nu_1, \ldots, \nu_m$ be all different roots in the half-plane $\operatorname{Im} \nu > 0$ and $l_1, \ldots, l_m$ be their multiplicities, $l_1 + \cdots + l_m = l$.

LEMMA 1 *There exist matrixes $b, J \in \mathbb{C}^{l \times l}$ such that*

$$\det B \neq 0, \quad B = \begin{pmatrix} b & \overline{b} \\ bJ & \overline{bJ} \end{pmatrix},$$
$$J = \operatorname{diag}(J_1, \ldots, J_m), \quad \sigma(J_i) = \{\nu_i\}, \tag{2}$$
$$a_{11} b + (a_{12} + a_{21}) bJ + a_{22} bJ^2 = 0.$$

The matrix J is defined uniquely to an accuracy of similarity and can be chosen in the Jordan form. The multiplicity and order of a eigenvalue ν of this matrix coincide with the multiplicity of the characteristic polynomial $\chi(x)$ and the order of pole of the matrix-valued function $P^{-1}(z)$ at the point ν, respectively.

Proof Let us consider the block matrix

$$A = \begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix} \in \mathbb{C}^{2l \times 2l},$$

where $a_0 = -a_{22}^{-1} a_{11}, \ a_1 = -a_{22}^{-1}(a_{12} + a_{21})$ and 0 (1) means the null (unit) matrix. The identity

$$(z - A)\begin{pmatrix} 1 & 1 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z & 0 \\ -a_0 & z^2 - a_1 z - a_0 \end{pmatrix}$$

shows that the characteristic polynomial of A coincides with $\chi(z)$ to accuracy of a constant multiplier. Let us introduce an invertible block-matrix $B = (b_{ij})_1^2 \in \mathbb{C}^{2l \times 2l}$ such that $B_{j2} = \overline{B}_{j1}, \ j = 1, 2,$ and $AB = B \operatorname{diag}(J, \overline{J})$. From the last equality it follows that $B_{11} J = B_{12}$ and $B_{21} J = a_0 B_{11} + a_1 B_{21}$. Putting $B_{11} = B,$ we receive (2).

Obviously we can choose the matrix J here in the Jordan form. If the pair (b_1, J_1) is the other solution of equation (2) then the matrixes $\operatorname{diag}(J, \overline{J})$ and $\operatorname{diag}(J_1, \overline{J}_1)$ are similar. Since $\sigma(J) = \sigma(J_1) \subseteq \{\operatorname{Im} z > 0\}$, the matrixes J and J_1 have common Jordan form and therefore are similar. The last assertion of the lemma follows from the above matrixes identity.

It is easy to show [1] that the condition $\det b \neq 0$ is equivalent to

$$\det\left(\int_{\mathbb{R}} P^{-1}(t)\, dt\right) \neq 0.$$

According to Bitsadze [2] the systems satisfying the condition $\det b \neq 0$ ($\det b = 0$) are called weak (strong) connected. For example the Bitsadze system

$$u_{xx} + 2e u_{xy} - u_{yy} = 0, \quad e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is strong connected and for it

$$b = \begin{pmatrix} 1 & 0 \\ -i & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}.$$

The Lame system of plane elasticity (in the isotropic case) [3]

$$\mu(u_{xx} + u_{yy}) + (\lambda + \mu)\operatorname{grad} \operatorname{Div} u = 0,$$

with positive coefficients $\lambda,\ \mu$ is weak connected and for it

$$b = \begin{pmatrix} 1 & 0 \\ i & -æ \end{pmatrix}, \qquad J = \begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}, \qquad æ = \frac{\lambda + 3\mu}{\lambda + \mu}.$$

The strong elliptic systems by Vishik [4] are defined by the condition $P(t) > 0,\ t \in \mathbb{R}$ of positive definiteness of the matrix $P(t)$. More restricted requirement of positive definiteness of the block matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is strong ellipticity by Somigliano [5]. The intermediate class corresponds to strengthened ellipticity [6], it consists of elliptic systems (1) for which $a \geq 0$.

Let us consider the elliptic system of the first-order

$$\frac{\partial \phi}{\partial y} - J\frac{\partial \phi}{\partial x} = 0 \tag{3}$$

which corresponds to the Cauchy–Riemann system for $J = i$. The solution $\phi = (\phi_1, \ldots, \phi_l)$ of this system is a real analytic function and in a neighborhood of each point $z_0 = x_0 + iy_0$ it expands in a generalized Taylor series

$$\phi(z) = \sum_{k=0}^{\infty} \frac{1}{k!}[z - z_0]^k \phi^{(k)}(z_0), \quad \phi^{(k)} = \frac{\partial^k \phi}{\partial x^k},$$

where, hereinafter we use, the matrix notation $[x + iy] = x + Jy,\ x,\ y \in \mathbb{R}$. For Toeplitz matrix J, the system (3) was first investigated by Douglis [7] in the frame of so called hypercomplex numbers. So the solutions ϕ of the system (3) we call the function analytic in the Douglis sense or shortly J-analytic functions.

For these functions the generalized Cauchy type integral

$$(I\varphi)(z) = \frac{1}{\pi i}\int_{\Gamma} [t - z]^{-1}[dt]\varphi(t), \quad z \notin \Gamma,$$

plays the same role as in the usual analytic theory. In particular, under the same assumption as in the classical case $J = i$, we have the Cauchy formula $2\phi = I\phi^+$ and the Sohotski–Plemel formula $(I\varphi)^{\pm} = \pm\varphi + I^*\varphi$, where $I^*\varphi$ means the corresponding singular integral.

The canonical system (3) is closely connected with the second-order system (1). From (2) and (3) it immediately follows that the function

$$u = \operatorname{Re} b\phi \tag{4}$$

satisfies the system (1). Conversely the J-analytic function ϕ here is restored by the formula

$$\phi' = 2(b_1 u_x + c_1 u_y), \qquad \begin{pmatrix} b_1 & c_1 \\ \overline{b}_1 & \overline{c}_1 \end{pmatrix} = B^{-1}, \tag{5}$$

with block matrix B from (2).

There exists a simple connection between J-analytic and usual analytic functions. Suppose firstly $\sigma(J) = \{\nu\}$. Then the direct check shows that the transformation

$$(E\psi)(x+iy) = \sum_{j=0}^{l-1} \frac{y^j}{j!}(J-\nu)^j \psi^{(j)}(x+\nu y), \tag{6}$$

moves the analytic vector-valued functions ψ of complex variable $x+\nu y$ into J-analytic functions $\phi = E\psi$. This correspondence between the functions ψ and $\phi = E\psi$ is invertible in the class C^∞. More exactly

$$\psi(\xi + i\eta) = \sum_{j=0}^{l-1} \frac{1}{j!}\left(\frac{-\eta}{\operatorname{Im}\nu}\right)^j (J-\nu)^j \frac{\partial^j \phi}{\partial x^j}(\xi + \tilde{\nu}\eta),$$

where $\tilde{\nu} = (i - \operatorname{Re}\nu)/\operatorname{Im}\nu$.

We can also consider the operation E for scalar functions ψ. In this case $E\psi$ is $l \times l$-matrix-valued function satisfying the equation (3). Thus for scalar function $\psi(z) = z^{-1}$ we have the equality $E\psi = [z]^{-1}$.

In a general case if the vector ψ is written in the block form $\psi = (\psi_1, \ldots, \psi_m)$ (analogously the block diagonal matrix $J = \operatorname{diag}(J_1, \ldots, J_m)$ in (2)), then under definition we have $(E\psi)_i = E_i\psi_i$, $i = 1, \ldots, m$, where E_i is defined by (6) with respect to $\nu = \nu_i$.

The transformation E together with (4) gives the known Bitsadze formula [2]

$$u = \operatorname{Re} bE\psi, \tag{7}$$

which describes a general solution u of (1) through analytic function ψ.

2. Bitsadze transformation on the upper half-plane

There are two difficulties in using the transformation E. Firstly, the vectors ϕ and ψ have different domains of their definition, secondly, there are derivatives of the function ψ in the formula (6).

In the case of the half-plane $D = \{y > 0\}$ considered below, both these obstacles are absent. Really the affine transformation $x + iy \to x + \nu y$ is invariant in D and how the transformation E is invariant in the Holder class with weight is shown next.

Let us remember the definition of these spaces. Let $C^\mu(\overline{D})$ be the usual Holder space with the finite norm

$$|\varphi| = |\varphi|_0 + \{\varphi\}_\mu = \sup_D |\varphi(z)| + \sup_{z_1 \neq z_2} \frac{|\varphi(z_1) - \varphi(z_2)|}{|z_1 - z_2|^\mu}.$$

Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and $F \subseteq \overline{\mathbb{R}}$ be a finite set containing the point $\tau = \infty$. The family $\lambda = (\lambda_\tau, \tau \in F\}$ of real numbers is called weighted order, these orders are equipped with natural operations. For $\lambda_\tau = \nu,\ \ \tau \in F,$ we write $\lambda = \nu \in \mathbb{R}$. Let us put

$$\sigma_\tau = \begin{cases} 1, & \tau \neq \infty, \\ -1, & \tau = \infty. \end{cases} \tag{8}$$

The weighted function $\rho_\lambda(z)$ is the smooth function on $\mathbb{C}\backslash F$ which does not turn into 0 everywhere and which is equal identically to $|z - \tau|^{\lambda_\tau}, \tau \neq \infty$, and $|z|^{-\lambda_\tau}, \tau = \infty$, in a neighborhood of F. Then the space $C^\mu_\lambda(\overline{D}; F)$ is defined by the norm

$$|\varphi| = |\rho_{-\lambda}\varphi|_0 + \{\rho_{\sigma\mu-\lambda}\varphi\}_\mu.$$

The space $C^{1,\mu}_{(\lambda)},\ 0 < \lambda < 1,$ of differential functions is described by the conditions $\varphi_x, \varphi_y \in C^\mu_{\lambda-\sigma}$. Below, the main interest represents the space

$$\left\{\varphi \in C^{1,\mu}_{(\lambda)}(\overline{D}; F), \varphi(\infty) = 0\right\}, \quad 0 < \lambda < 1. \tag{9}$$

THEOREM 1 *The transformation E is invertible in the class* (9). *Accordingly the Bitsadze formula* (7) *establishes an isomorphism between the solutions u of* (1) *and analytic vector-valued function ψ in this class. Nevertheless, $u_x(x,0) = \mathrm{Re}\, b\psi'(x),\ u_y(x,0) = \mathrm{Re}\, bJ\psi'(x)$ and therefore $\psi'(x) = 2(b_1 u_x + c_1 u_y)(x,0)$.*

Proof For the analytic vector-value function ψ in the class (9) we can write the Cauchy formula

$$\psi(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} (t - z)^{-1}\psi^+(t)\, dt$$

in the half-plane D. Since $E(t-z)^{-1} = [t-z]^{-1},\ t \in \mathbb{R},$ we have:

$$(E\psi)(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} [t-z]^{-1}\psi^{+}(t)\,dt.$$

It may be shown analogously [8] that the above integrals (9), (10) define boundary operators from the space $\{\psi \in C_{\lambda}^{1,\mu}(\mathbb{R}, F), \psi(\infty) = 0\}$ to the class (8). Using the Sohotski–Plemel formula for these integrals, we have

$$2\psi^{+}(t_0) = \psi^{+}(t_0) + \frac{1}{\pi i}\int_{\mathbb{R}} (t-t_0)^{-1}\psi^{+}(t)\,dt = 2(E\psi)^{+}(t_0).$$

So together with (4) and (5) we complete the proof.

3. The boundary value problems

Let us consider the problem

$$(p_1 u_x + p_2 u_y + p_0 u)\big|_R = g \tag{10}$$

for the equation (1) in the class (9), where the real $l \times l$-matrix-valued functions p_j are continuous on $\overline{\mathbb{R}}\backslash F$, and the right hand function $g \in C^{\mu}_{\lambda-\sigma}(\overline{\mathbb{R}}; F)$. More exactly the functions p_j, $j = 1, 2$ permit one side limits $p_j(\tau \pm 0)$ at the points $\tau \in F$ (for $\tau = \infty$ they correspond $p_j(\mp\infty)$) and $p_j(t) - p_j(\tau \pm 0) \in C^{\mu+\varepsilon}_{\varepsilon}(\Gamma_{\tau\pm 0}, \tau)$ with some $\varepsilon > 0$ for the segment $\Gamma_{\tau\pm 0}\backslash\tau \subseteq \mathbb{R}\backslash F$ with the end $\tau \in F$. As for the function p_0, it belongs to the class $C^{\mu+\varepsilon}_{\lambda-\sigma+\varepsilon}$, $\varepsilon > 0$.

Because of Theorem 1 this problem reduces equivalently to the problem

$$\operatorname{Re}(G\psi' + G_0\psi)\big|_R = g \tag{11}$$

with the coefficients $G = p_1 b + p_2 bJ,\ G_0 = p_0 b$ for the analytic vector-valued function $\psi(z)$. For the latter, one can use the result of classical theory [9–11]. In particular, with the help of the canonical function $X(z)$, their solution can be represented by the explicit formula.

Using the Bitsadze transformation, we can reformulate these results to the original problem (10). Under the definition, this problem belongs to normal type if the matrix $G = p_1 b + p_2 bJ$ satisfies the conditions

$$\det G(t) \neq 0,\ t \in \mathbb{R}\backslash F; \qquad \det G(\tau \pm 0) \neq 0,\ \tau \in F.$$

In this case, one can introduce the increment

$$\arg(\det G)\big|_{\mathbb{R}} = \sum_{\tau \in F} \mp \arg\det G(\tau \pm 0), \tag{12}$$

which does not depend on the choice of arg and the family of invertible matrixes

$$Q_\tau = (G\overline{G}^{-1})(\tau+0)(\overline{G}G^{-1})(\tau-0), \quad \tau \in F,$$

the eigenvalues $w \in \sigma(Q_\tau)$ of which lie on the unit circle $|w| = 1$.

With the matrix Q of this type, we can connect the piecewise constant function

$$\chi(\delta, Q) = \sum_{w \in \sigma(Q)} l(w)\Big(\Big[\delta - \frac{\arg w}{2\pi}\Big] + \frac{\arg w}{2\pi}\Big), \tag{13}$$

which does not depend on the choice of arg. Here the notation [ ] means the integer part of number and $l(w)$ is a multiplicity of the eigenvalue w.

THEOREM 2 *The problem* (1), (10) *in the class* (9) *is Fredholm if and only if it belongs to normal type and* $e^{2\pi i\lambda_\tau} \notin \sigma(Q_\tau)$, $\tau \in F$. *Its index is given by the formula*

$$æ = -\frac{1}{\pi}\arg\det G\big|_{\mathbb{R}} - \sum_{\tau \in F} \chi(\lambda_\tau, Q_\tau) - l. \tag{14}$$

From the definition (13) it follows that $\chi(\delta, Q) = (2\pi)^{-1}\arg\det Q + \text{integer}$. So the right hand part of (14) is an integer.

Proof The problem (1), (10) is equivalent to (11) in the class (9). So it is sufficient to prove the theorem for the problem (11).

The operator of differentiation $\psi \to \psi'$ is invertible from the class (9) to $C^\mu_{\lambda-\sigma}$. For $\psi_1 = \psi' \in C^\mu_{\lambda-\sigma}$ let us put $T\psi_1 = \operatorname{Re} G_0\psi^+$. Under accepted assumptions with respect to p_j this operator is bounded $C^\mu_{\lambda-\sigma}(\overline{D};\overline{F}) \to C^{\mu+\varepsilon}_{\lambda-\sigma+\varepsilon}(\mathbb{R};F)$ with some $\varepsilon > 0$. Since the embedding $C^{\mu+\varepsilon}_{\nu+\varepsilon} \subseteq C^\mu_\nu$ is compact, the operator T is also compact $C^\mu_{\lambda-\sigma}(\overline{D};F) \to C^\mu_{\lambda-\sigma}(\mathbb{R};F)$. So the problem (11) in the class (9) is the Fredholm equivalent to problem

$$\operatorname{Re} G\psi_1 = g_1 \tag{15}$$

in the class $C^\mu_{\lambda-\sigma}$ and their indexes coincide.

With the help of substitution $\tilde\psi_1(\tilde z) = \psi_1(z)$, $\tilde z = (z-i)/(z+i)$ the problem (15) reduces to analogous one

$$\operatorname{Re}\,\tilde G\tilde\psi_1 = \tilde g_1 \tag{15~}$$

in the unit circle $\tilde D = \{|\tilde z| < 1\}$. By this substitution the space $C^\mu_{\lambda-\sigma}(D;F)$ transforms into $C^\mu_{\lambda-\sigma}(\tilde D;\tilde F)$ and $\tau = \infty$ corresponds to $\tilde\tau = 1$.

The arguments from the proof of Theorem 3 in [12] for scalar case are also applicable to the problem (15~) with the matrix coefficient $\tilde G$. So the Fredholm condition of this problem is given by (12) and its index formula in the class $C^\mu_{\lambda-\sigma}$ is the following:

$$\tilde æ = l - \frac{1}{\pi}\arg\det\tilde G\big|_{\partial\tilde D} - \sum_{\tau \in F}\chi(\lambda_\tau - \sigma_\tau, Q_\tau) - ml,$$

where m is a number of elements F. Taking into account (8) and the relation $\chi(\delta+1) = \chi(\delta) + l$, we can rewrite this equality in the form (14) that completes the proof.

4. The problems with constant coefficients

Let us consider the problem

$$(p_1 u_x + p_2 u_y)\big|_{\partial D} = g, \quad p_j \in \mathbb{R}^{l\times l} \tag{16}$$

for which the matrix $G = p_1 b + p_2 bJ \in \mathbb{C}^{l\times l}$. In this case we can essentially complete the Theorem 2.

THEOREM 3 *If* $\det G = 0$, *then the homogeneous problem* (1), (16) *in the class* (9) *has an infinite number of linear independent solutions. For example if* $G\eta = 0$ *for some* $\eta \in \mathbb{C}^l$, $\eta \neq 0$, *then the function* $u = \operatorname{Re} b(E\psi_0)\eta$ *is a solution of the homogeneous problem* (1), (15) *for each scalar analytic function* ψ_0 *belonging to the class* (9).

If $\det G \neq 0$, *then the condition*

$$\int_{\mathbb{R}} g(t)\,dt = 0 \tag{17}$$

is necessary and sufficient for one-to-one solvability of the problem in the considered class and its solution is given by the formula

$$u(x,y) = \operatorname{Re}\frac{1}{\pi}\int_{\mathbb{R}} b(t - x - Jy)^{-1} G^{-1} f(t)\,dt, \quad f(t) = \int_{-\infty}^{t} g(s)\,ds. \tag{18}$$

Note that by virtue of (17) the function $f \in C^{1,\mu}_{(\lambda)}$ has values $f(\pm\infty) = 0$ and therefore in (18) the integral exists.

Proof In the case considered, the problem (11) transforms to

$$\operatorname{Re} G\psi' = g. \tag{19}$$

If $\det G = 0$, $G\eta = 0$ then the function $\psi(z) = \psi_0(z)\eta$ with scalar and analytic ψ_0 satisfies the homogeneous problem (19).

Let $\det G \neq 0$. Since the analytic function $G\psi'$ belongs to $C^{\mu}_{\lambda-\sigma}(D;F)$, $0 < \lambda < 1$, we can write the Cauchy formula for this problem in half-plane D. In particular, the condition (17) is necessary for solvability of the problem (19) and so for solvability of (16). From this condition it follows that $f(+\infty) = f(-\infty) = 0$ and we can assume $\psi(\infty) = f(\infty) = 0$. Using the Schwarz formula [9] for the function $G\psi$, we get the representation

$$\psi(z) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{G^{-1} f(t)\,dt}{t - z}.$$

By virtue of (7) and the equality $E(t-z)^{-1} = [t-z]^{-1}, \ t \in \mathbb{R}$, we come to the formula (18).

For the given system (1) we can always find the matrixes $p_j \in \mathbb{R}^{l\times l}$ such that $\det(p_1 b_1 + p_2 bJ) \neq 0$ i.e. the matrixes for which the problem (16) is correct. For example in the case of Bitsadze system it is sufficient to put $p_1 = p, \ p_2 = 1$, where $p \neq s + te$ for each $s, t \in \mathbb{R}$. On the other hand, for each of the matrixes $p_1, p_2 \in \mathbb{R}^{l\times l}$ there always exists the elliptic system (1) for which $\det(p_1 b + p_2 bJ) = 0$. So a "universal" problem of type (16) which is correct for each elliptic system (1) does not exist. But for nonlocal problems the situation is quite different.

THEOREM 4 *The problem of Carleman type*

$$u(x,0) + u(-x,0) = f(x), \quad u_y(x,0) - u_y(-x,0) = g'(x) \tag{20}$$

is uniquely solvable for each elliptic system (1) *in the class* (9) *and its solution is given by the formula*

$$u(x,y) = \operatorname{Re}\frac{2}{\pi}\int_{\mathbb{R}} b(t - x - Jy)^{-1}(b_1 f + c_1 g)(t)\,dt, \tag{21}$$

where b_1 and c_1 are defined in (5).

Proof According to Theorem 1 we can represent the equalities (20) in the form

$$\operatorname{Re} b\{\psi(x) + \psi(-x)\} = f, \quad \operatorname{Re} bJ\{\psi'(x) - \psi'(-x)\} = g'(x).$$

Integrating the last relation and using notation (5) we have

$$\psi(x) + \psi(-x) = \tilde{f}(x), \quad \tilde{f} = 2(b_1 f + c_1 g).$$

Putting

$$\tilde{\psi}(z) = \begin{cases} \psi(z), & \operatorname{Im} z > 0, \\ -\psi(-z), & \operatorname{Im} z > 0, \end{cases}$$

we can rewrite the above boundary condition in the linear conjugation form $\tilde{\psi}^{+} - \tilde{\psi}^{-} = f$. So based on the Sohotski–Plemel formula, we can give the solution of this problem in the explicit form

$$\psi(z) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(t)\,dt}{t - z}, \quad z \in D.$$

Substituting it into (7) we get (21).

The boundary condition (16) we can integrate and write in a simpler form. Let us introduce the conjugate function v to the solution u of the equation (1) by the rule

$$v_x = a_{21}u_x + a_{22}u_y, \quad v_y = -(a_{11}u_x + a_{12}u_y).$$

Substituting the representation (4) we get

$$v = \operatorname{Re} cE\psi, \qquad c = a_{21}b + a_{22}bJ. \tag{22}$$

Since $u_y = a_{22}^{-1}(v_x - a_{21}u_x)$, from (15) we can move to the boundary condition

$$(pu + bv)\big|_R = f, \quad f(t) = \int_{-\infty}^{t} g(s)\,ds,$$

with the coefficients $p = p_1 - p_2a_{22}^{-1}a_{21},\ q = p_2a_{22}^{-1}$. In this case $G = pb + qc$. For $G = b$ ($G = c$) we have the first (second) boundary value problem i.e. the Dirichlet (Neumann) problem.

It is convenient to reformulate the Theorem 4 for these problems.

THEOREM 5 (a) *Let the system* (1) *be weak connected, i.e.* $\det b \neq 0$. *Then the Dirichlet problem is uniquely solvable in the class* (9) *and its solution u and the conjugate function v are defined by the formulas*

$$u(x,y) = \operatorname{Re} \frac{1}{\pi i}\int_{\mathbb{R}} b(t - x - Jy)^{-1}b^{-1}f(t)\,dt,$$
$$v(x,y) = \operatorname{Re} \frac{1}{\pi i}\int_{\mathbb{R}} c(t - x - Jy)^{-1}b^{-1}f(t)\,dt.$$

If this system is strong connected then the homogeneous problem has an infinite number of linear independent solutions.

(b) *In notations* (20) *let* $\det c \neq 0$. *Then the Neumann problem is uniquely solvable in the class* (9) *and its solution u and the conjugate function v are defined by the formulas*

$$u(x,y) = \operatorname{Re} \frac{1}{\pi i}\int_{\mathbb{R}} b(t - x - Jy)^{-1}c^{-1}f(t)\,dt,$$
$$v(x,y) = \operatorname{Re} \frac{1}{\pi i}\int_{\mathbb{R}} c(t - x - Jy)^{-1}c^{-1}f(t)\,dt.$$

If $\det c = 0$, *then the homogeneous problem has an infinite number of linear independent solutions.*

This effect connected with Dirichlet problem was firstly opened by Bitsadze [13] in 1948 (for his system in the unit circle). In particular, for strong elliptic systems the Dirichlet problem is always uniquely solvable, but for Neumann problem it cannot be asserted even for strengthened elliptic systems. For example, the system

$$u_{xx} - (q + q^{\mathrm{T}})u_{xy} + u_{yy} = 0,$$

where the matrix $q \in \mathbb{R}^{l\times l}$ is orthogonal and $\det(1 \pm q) \neq 0$, is strengthened elliptic but the correspondence matrix $c = qb - bJ$ is not invertible.

It may be shown [1] that the condition $\det c \neq 0$ is fulfilled for strengthened system (1) with additional requirement rang $a \geq 2l - 1,\ a = \{a_{ij}\}_1^2$.

5. Application to anisotropic plane elasticity

The plane elastic medium is characterized by the displacement vector $u = (u_1, u_2)$ and by stress and deformation tensors

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_2 \end{pmatrix}, \qquad \varepsilon = \begin{pmatrix} \varepsilon_1 & \varepsilon_3 \\ \varepsilon_3 & \varepsilon_2 \end{pmatrix},$$

where $\varepsilon_1 = u_{1x}$, $\varepsilon_2 = u_{2y}$, $2\varepsilon_3 = u_{1y} + u_{2x}$. The vectors $\tilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\tilde{\varepsilon} = (\varepsilon_1, \varepsilon_2, 2\varepsilon_3)$ are connected by Hooke law i.e. by linear relation [14,15]

$$\tilde{\sigma} = \alpha\tilde{\varepsilon}, \qquad \alpha = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_5 \\ \alpha_4 & \alpha_2 & \alpha_6 \\ \alpha_5 & \alpha_6 & \alpha_3 \end{pmatrix} > 0.$$

If the external forces are absent then the equilibrium equations have the form $\sigma_{(1)x} + \sigma_{(2)y} = 0$, where $\sigma_{(j)}$ means jth column of the matrix σ. Using the Hooke law we receive the Lame system (1) for the replacement vector u with the coefficients defined from the matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_6 & \alpha_6 & \alpha_4 \\ \alpha_6 & \alpha_3 & \alpha_3 & \alpha_5 \\ \alpha_6 & \alpha_3 & \alpha_3 & \alpha_5 \\ \alpha_4 & \alpha_5 & \alpha_5 & \alpha_2 \end{pmatrix}.$$

This system is strengthened elliptic and rang $a = 3$. In terms of conjugate function v we can write the expression of the stress tensor σ in the form $\sigma_{(1)} = -v_y$, $\sigma_{(2)} = v_x$.

So the Neumann problem on the upper half-plane corresponds to setting up the normal component $\sigma_{(2)}$ of the tensor σ. The solutions of the Dirichlet and Neumann problems are given by the above formulae (D) and (N) respectively.

For the matrix J taken in the Jordan form only the two following cases

$$\text{(i) } J = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \ \nu_1 \neq \nu_2; \quad \text{(ii) } J = \begin{pmatrix} \nu & 1 \\ 0 & \nu \end{pmatrix}$$

are possible. According to these cases the matrixes b and c can be described explicitly [16].

Let scalar polynomials p_i, q_i, $i = 1, 2, 3$ be defined from the equalities

$$P = \begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix}, \qquad Q = \begin{pmatrix} p_2 & -p_3 \\ -p_3 & p_1 \end{pmatrix},$$

$$R(z) = (a_{21} + a_{22}z)Q(z) = \begin{pmatrix} zq_2 & q_1 \\ -q_2 & q_3 \end{pmatrix}.$$

(i) *Columns of matrixes b and c are determined by equalities*

$$b_{(i)} = k_i \begin{cases} Q_{(1)}(\nu_i) \\ Q_{(2)}(\nu_i), \end{cases} \qquad c_{(i)} = k_i \begin{cases} R_{(1)}(\nu_i) \\ R_{(2)}(\nu_i), \end{cases} \qquad i = 1, 2,$$

where $k_i \neq 0$ *and upper (below) equality is chosen under condition* $q_2(\nu_i) \neq 0$ ($q_1(\nu_i) \neq 0$).

(ii) *Columns of the matrix b are given by the equalities*

$$b_{(1)} = k_1 Q_{(1)}(\nu), \qquad b_{(2)} = k_1 Q'_{(1)}(\nu) + k_2 Q_{(1)}(\nu)$$

with arbitrary $k_j \in \mathbb{C},\ k_1 \neq 0$. *The matrix c is defined analogously with respect to* $R(z)$.

In the orthotropic medium when $\alpha_5 = \alpha_6 = 0,$ the characteristic equation $\det P = 0$ is biquadratic and its roots are described explicitly. In particular for isotropic medium we have the case (ii) with $\nu = i$ and expressions

$$b = \begin{pmatrix} 1 & 0 \\ i & -æ \end{pmatrix}, \qquad c = \mu \begin{pmatrix} 2i & æ - 1 \\ 2 & i(æ + 2) \end{pmatrix}, \qquad æ = \frac{\lambda + 3\mu}{\lambda + \mu}.$$

With respect to another function theoretical approaches to orthotropic elasticity see [17–19].

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