ON ASYMPTOTICS OF A PIECEWISE ANALYTIC FUNCTION SATISFYING CONTACT CONDITIONS

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Abstract: We examine a piecewise analytic function that is defined in sectors of a disk whose real and imaginary parts obey contact conditions on adjacent boundary parts. Under the assumption of the power behavior, the sharp asymptotics of this function is established at the center of the disk.

Keywords: analytic function, contact conjugation condition, power logarithmic asymptotics

Let the unit disk be divided into n open sectors $S_1, \dots, S_n$ of angle $2\pi/n$, with $n \geq 4$ an even number, which are enumerated counterclockwise. The radial segments, presenting the lateral sides of sectors, are oriented from $z = 0$ to the unit circle; their union is denoted by L. Given two positive numbers ν_1 and ν_2, compose the piecewise analytic function

$$\chi(z) = \begin{cases} \nu_1, & z \in S_1 \cup S_3 \cup \dots \cup S_{n-1}, \\ \nu_2, & z \in S_2 \cup S_4 \cup \dots \cup S_n. \end{cases} \tag{1}$$

Examine a piecewise analytic function $\phi(z)$ in $S = S_1 \cup \dots \cup S_n$ whose restriction ϕ_j to the sector S_j is continuous on $\overline{S}_j \setminus 0$. In particular, the pair of its boundary values $\phi^\pm$ is defined on the oriented curve L. Let these boundary values be subject to the so-called contact conditions

$$\mathrm{Re}(\phi^+ - \phi^-) = 0, \quad \mathrm{Im}[(\chi\phi)^+ - (\chi\phi)^-] = c, \tag{2}$$

where c is a constant function on each segment of L. Moreover, we assume that $\phi(z)$ satisfies the estimate

$$|\phi(z)| = O(|z|^\lambda) \quad \text{as } z \to 0 \tag{3}$$

for some $\lambda \in \mathbb{R}$. Note that, in the case of $\lambda > 0$, the piecewise constant function c must be equal to zero.

The question arises: What is the sharp asymptotics of $\phi(z)$ at $z = 0$? For $n = 6$, this question is closely connected with effective conductivity (see [1]) in a regular two-component system of regular triangles. Similar problems arise in many other applications, for example, in the inverse problem of scattering theory which is connected with the matrix Riemann problem [2]. In this article we solve the problem in the framework of the nonlocal Riemann problem studied in [3, 4].

To this end, we rewrite the statement (1), (2) of the problem in the form of the boundary conditions of the Riemann problem. Denote by $\partial^0 S_j$ and $\partial^1 S_j$ the lateral sides of S_j assuming that the rotation from $\partial^0 S_j$ to $\partial^1 S_j$ inside of the sector is realized counterclockwise. All lateral sides are enumerated similarly as follows:

$$\begin{matrix} \partial^1 S_1 & \partial^0 S_2 & \partial^1 S_2 & \partial^0 S_2 & \dots & \partial^1 S_n & \partial^0 S_1 \\ \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \dots & \Gamma_{2n-1} & \Gamma_{2n}; \end{matrix} \tag{4}$$

in this case the segments $L_1 = \Gamma_1 = \Gamma_2$, $L_2 = \Gamma_3 = \Gamma_4, \dots, L_n = \Gamma_{2n-1} = \Gamma_{2n}$ constitute L. The segment Γ_j is parametrized by the linear equation $\gamma_j(s) = \gamma_j'(0)s$, $0 \leq s \leq 1$, where $|\gamma_j'(0)| = 1$. Using these parametrizations, we can assume that the boundary values $\phi^\pm$ are defined on the segment $(0, 1]$ of the real axis. More exactly, we put

$$\phi_{\gamma,1} = \phi_1^- \circ \gamma_1,\ \phi_{\gamma,2} = \phi_1^+ \circ \gamma_2,\ \phi_{\gamma,3} = \phi_2^- \circ \gamma_3, \dots, \phi_{\gamma,2n} = \phi_1^+ \circ \gamma_{2n}.$$

In this notation, (2) can be rewritten as

$$\operatorname{Re} A\phi_\gamma = f, \tag{5}$$

where the matrix A is block-diagonal, i.e., $A = \operatorname{diag}(a_1, a_2, a_1, a_2, \dots, a_1, a_2)$ with the diagonal blocks

$$a_1 = \begin{pmatrix} 1 & -1 \\ i\nu_1 & -i\nu_2, \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & -1 \\ i\nu_2 & -i\nu_1, \end{pmatrix},$$

and similarly $f = (f_1, \dots, f_n)$ with $f_j = (0, c|_{L_j})$.

In accord with (4), Γ_1 and Γ_{2n} are the lateral sides of S_1 and the former segment is oriented negatively and the latter positively. Similarly, Γ_3 is oriented negatively and Γ_2 positively relative to S_2, and so on. In view of this, the signature of the orientation is as follows: $\sigma = (\sigma_j)_1^{2n}$, with $\sigma_j = 1$ for an even j and $\sigma_j = -1$ for an odd j. Let the matrix A^σ be obtained from A by the complex conjugation of jth column if $\sigma_j = -1$ and by preservation of this column if $\sigma_j = 1$. In this case the matrix $B = (A^\sigma)^{-1}A^{-\sigma}$ takes the block-diagonal form $B = \operatorname{diag}(b_1, b_2, b_1, b_2, \dots, b_1, b_2)$, where

$$b_1 = \begin{pmatrix} 1 & -1 \\ -i\nu_1 & -i\nu_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ i\nu_1 & i\nu_2, \end{pmatrix} = \frac{1}{\nu_1 + \nu_2} \begin{pmatrix} -\nu_1 + \nu_2 & -2\nu_2 \\ -2\nu_1 & \nu_1 - \nu_2 \end{pmatrix}$$

and b_2 is defined similarly by the rearrangement of ν_1 and ν_2. Putting $s = (\nu_1 + \nu_2)/2$, $\delta = (\nu_1 - \nu_2)/2$, we can write

$$b_1 = -\frac{1}{s} \begin{pmatrix} \delta & \nu_2 \\ \nu_1 & -\delta \end{pmatrix}, \quad b_2 = -\frac{1}{s} \begin{pmatrix} -\delta & \nu_1 \\ \nu_2 & \delta \end{pmatrix}. \tag{6}$$

The diagonal block of the end symbol [3] relating to the boundary conditions (5) on the lateral sides of the sectors $S_1, \dots, S_n$ is the $(2n \times 2n)$-matrix

$$X(\zeta) = B + z(\zeta)T_\alpha, \quad z(\zeta) = e^{\pi i\zeta/n}, \tag{7}$$

where $T_\alpha = (\delta_{\alpha i,j})_1^{2n}$ is the permutation matrix

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \dots & 2n-2 & 2n-1 & 2n \\ 2n & 3 & 2 & \dots & 2n-1 & 2n-2 & 1 \end{pmatrix},$$

defined by the lateral sides of the sectors S_j in (4).

The results of [3] as applied to the problem (1)–(3) are stated as follows.

Theorem 1. *In every strip $\lambda_1 < \operatorname{Re}\zeta < \lambda_2$, the function $\det X(\zeta)$ has finitely many zeros. Let Δ be the projection of the set of these zeros onto the real axis and let the condition (3) with $\lambda \notin \Delta$ hold. Let also the point $\delta \in \Delta$ be such that $\delta > \lambda$ and $\Delta \cap (\lambda, \delta) = \varnothing$. Assume that $\zeta_1, \dots, \zeta_n$ are the zeros of the function $\det X(\zeta)$ on the line $\operatorname{Re}\zeta = \delta$ and $r_1, \dots, r_n$ are the orders of poles of the matrix function $X^{-1}(\zeta)$ at these points, respectively. Then, for $\delta \neq 0$, at every sector S_i the function $\phi(z)$ is representable as*

$$\phi(z) = \sum_{j=1}^{n} \sum_{k=0}^{r_j - 1} c_{jk} (\log z)^k z^{\zeta_j} + O(|z|^{\delta+\varepsilon}), \tag{8}$$

with some $c_{jk} \in \mathbb{C}$ and $\varepsilon > 0$.

If $\delta = 0$ and $0 \notin \{\zeta_1, \dots, \zeta_n\}$ then we should replace the last summand in this decomposition with $c + O(|z|^\varepsilon)$, $c \in \mathbb{C}$. Finally, assume that $\delta = 0$ and, for instance, $\zeta_1 = 0$. Then (8) is valid, where r_1 is substituted for $r_1 - 1$.

In accord with this theorem, we need to describe the zeros of $\det X$ and the orders of the poles of X^{-1}. It is convenient to introduce the notation

$$\delta = \frac{\nu_1 - \nu_2}{2}, \quad s = \frac{\nu_1 + \nu_2}{2}, \quad \nu = \sqrt{\nu_1 \nu_2}, \quad q = \frac{\nu}{s}; \tag{9}$$

part of which has been already used in (6).

Lemma 1. *All zeros of* $\det X(\zeta)$ *are simple and can be described by the inequalities*

$$\zeta = \pm\frac{\cos^{-1}(1-2q^2\sin^2 k\theta)}{2\theta} + \frac{\pi s}{\theta}, \quad k = 0, 1, \dots, \frac{n}{2}-1, \ s = 0, \pm1, \pm2, \dots, \tag{10}$$

where $\theta = 2\pi/n$.

Proof. Given the class of entire $(n_j \times n_j)$-matrix-functions $X_j(\zeta)$, $j = 1, 2$, for convenience, introduce the following equivalence relation: $X_1 \sim X_2$ whenever there exist entire $(n_1 \times n_1)$-matrix-functions $Y(\zeta)$ and $Z(\zeta)$ such that the product of their determinants is constant and does not vanish and

$$YX_1Z = \mathrm{diag}(1, X_2),$$

where, for definiteness, $n_1 \geq n_2$ and the symbol 1 stands for the identity $((n_1-n_2)\times(n_1-n_2))$-matrix (for $n_1 = n_2$, the right-hand side of this relation is replaced with X_2).

Proceed with (7). Let T_β be the matrix of some permutation β. The product BT_β is the matrix with the entries $(BT_\beta)_{ij} = B_{i,\beta^{-1}j}$. Hence, the right multiplication of B by T_β is equivalent to the permutation β^{-1} of its columns $B_1, \dots, B_n$, i.e., the jth column coincides with $(BT_\beta)_j = B_{\beta^{-1}j}$. Moreover, $T_\alpha T_\beta = T_{\beta\circ\alpha}$. In particular, the permutation matrix of (7) meets the equality $T_\alpha^2 = 1$ and so $X \sim BT_\alpha + z \sim BT_\alpha T_\beta + zT_\beta$. Choose the cyclic permutation $1 \to 2 \to \cdots \to 2n \to 1$ as β. If we rewrite T_β in the form of a block $(n\times n)$-matrix grouping the numbers $\{1,2\}, \{3,4\}, \dots, \{2n-1, 2n\}$ then

$$T_\beta = \begin{pmatrix} e_1 & e_2 & 0 & 0 & \cdots & 0 \\ 0 & e_1 & e_2 & 0 & \cdots & 0 \\ 0 & 0 & e_1 & e_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_2 & 0 & 0 & 0 & \cdots & e_1 \end{pmatrix}$$

with the (2×2)-matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Consider the matrix $BT_\alpha T_\beta = BT_{\beta\circ\alpha}$. As is easily seen, the permutation $(\beta\circ\alpha)^{-1} = \alpha^{-1}\circ\beta^{-1} = \alpha\circ\beta^{-1}$ leaves the odd numbers immovable and realizes the cyclic permutation $2 \to 2n \to 2n-2 \to \cdots \to 4 \to 2$ of even numbers. Hence, the columns of $\widetilde{B} = BT_\alpha T_\beta$ can be written as

$$\widetilde{B}_i = \begin{cases} B_i, & i = 1, 3, \dots, 2n-1, \\ B_{2n}, & i = 2, \\ B_{i-2}, & i = 4, 6, \dots, 2n. \end{cases}$$

Recalling the definition of B in (7) and using (6), we can rewrite $\widetilde{B}$ in the similar $(n\times n)$-block form

$$-sBT_\alpha T_\beta = \begin{pmatrix} c_1 & d_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & c_1 & d_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ d_2 & 0 & 0 & 0 & \cdots & c_2 \end{pmatrix},$$

with the (2×2)-matrices

$$c_1 = \begin{pmatrix} \delta & 0 \\ \nu_1 & 0 \end{pmatrix}, \quad d_1 = \begin{pmatrix} 0 & \nu_2 \\ 0 & -\delta \end{pmatrix}, \quad c_2 = \begin{pmatrix} -\delta & 0 \\ \nu_2 & 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 0 & \nu_1 \\ 0 & \delta \end{pmatrix}.$$

Therefore,

$$X \sim -sBT_\alpha T_\beta - szT_\beta \sim \begin{pmatrix} p_1 & q_1 & 0 & 0 & \cdots & 0 \\ 0 & p_2 & q_2 & 0 & \cdots & 0 \\ 0 & 0 & p_1 & q_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ q_2 & 0 & 0 & 0 & \cdots & p_2 \end{pmatrix} \tag{11}$$

with the (2×2)-matrices

$$p_1=\begin{pmatrix}\delta & -sz\\ \nu_1 & 0\end{pmatrix},\quad q_1=\begin{pmatrix}0 & \nu_2\\ -sz & -\delta\end{pmatrix},$$

$$p_2=\begin{pmatrix}-\delta & -sz\\ \nu_2 & 0\end{pmatrix},\quad q_2=\begin{pmatrix}0 & \nu_1\\ -sz & \delta\end{pmatrix}.$$

Assigning $P=\operatorname{diag}(p_1,p_2,p_1,p_2,p_1,p_2)$ and $x_j=p_j^{-1}q_j$, we can thus write

$$P^{-1}X\sim\begin{pmatrix}1 & x_1 & 0 & 0 & \cdots & 0\\ 0 & 1 & x_2 & 0 & \cdots & 0\\ 0 & 0 & 1 & x_1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ x_2 & 0 & 0 & 0 & \cdots & 1\end{pmatrix}. \tag{12}$$

Subtracting the first column multiplied by x_1 on the right from the second on the right-hand side of this relation, we obtain the equivalent matrix

$$P^{-1}X\sim\begin{pmatrix}1 & 0 & 0 & 0 & \cdots & 0\\ 0 & 1 & x_2 & 0 & \cdots & 0\\ 0 & 0 & 1 & x_1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ x_2 & -x_2x_1 & 0 & 0 & \cdots & 1\end{pmatrix}.$$

Subtracting the second column multiplied by x_2 on the right from the third in the matrix obtained and continuing this process, we arrive at the relation $P^{-1}X\sim 1+Y$, where all elements of the block $(n\times n)$-matrix Y except for the elements of the last row vanish and the last row consists of the elements

$$x_2,-(x_2x_1),(x_2x_1)x_2,-(x_2x_1)^2,(x_2x_1)^2x_2,\dots,-(x_2x_1)^{n/2}.$$

Thus,

$$P^{-1}X\sim\begin{pmatrix}1_{2n-2} & 0\\ \cdots & 1-(x_2x_1)^{n/2}\end{pmatrix}, \tag{13}$$

where 1_k designates the identity $(k\times k)$-matrix. It is immediate from (11) that

$$p_1\sim\begin{pmatrix}-sz & \delta\\ 0 & \nu_1\end{pmatrix}\sim\begin{pmatrix}-sz & 0\\ 0 & \nu_1\end{pmatrix},\quad p_2\sim\begin{pmatrix}-sz & 0\\ 0 & \nu_2\end{pmatrix};$$

and so $P\sim\operatorname{diag}(1_n,z1_n)$. Taking (13) into account, we infer

$$X\sim\operatorname{diag}(1_n,z1_n)\operatorname{diag}[1_{2n-2},1-(x_2x_1)^{n/2}]\sim z\operatorname{diag}[1_{n-2},1-(x_2x_1)^{n/2}]. \tag{14}$$

In accord with (9) and (11), we derive that

$$x_1=-\frac{1}{\nu_1z}\begin{pmatrix}sz^2 & \delta z\\ \delta z & s\end{pmatrix},\quad x_2=-\frac{1}{\nu_2z}\begin{pmatrix}sz^2 & -\delta z\\ -\delta z & s\end{pmatrix}$$

and

$$x=x_2x_1=\frac{1}{\nu^2z^2}\begin{pmatrix}s^2z^4-\delta^2z^2 & \delta sz^3-s\delta z\\ -\delta sz^3+s\delta z & -\delta^2z^2+s^2\end{pmatrix}. \tag{15}$$

We have

$$x^{n/2}-1=\prod_{k=0}^{n/2-1}(x-\varepsilon_k),\quad \varepsilon_k=e^{4\pi ik/n}.$$

In this case (14) yields

$$\det X \sim z^n \prod_{k=0}^{n/2-1} \det(x - \varepsilon_k). \tag{16}$$

By (15),

$$x - \varepsilon = \frac{1}{\nu^2 z^2} \begin{pmatrix} s^2 z^4 - a^2 z^2 & \delta s z(z^2 - 1) \\ -\delta s z(z^2 - 1) & s^2 - a^2 z^2 \end{pmatrix}, \quad a^2 = \delta^2 + \varepsilon_k \nu^2,$$

and simple calculations imply that

$$\det(x - \varepsilon_k) = \frac{\varepsilon_k}{\nu^2 z^2} \left[s^2 z^4 - \left(\frac{\nu^2}{\varepsilon_k} + \nu^2 \varepsilon_k + 2\delta^2 \right) z^2 + s^2 \right] = \frac{\varepsilon_k h_k(z^2)}{q^2 z^2}, \tag{17}$$

where

$$h_k(t) = t^2 - 2(1 - q^2 + q^2 \cos 2k\theta)t + 1 = t^2 - 2(1 - 2q^2 \sin^2 k\theta)t + 1,$$

and we put $\theta = 2\pi/n$ for brevity. The roots of this polynomial are two complex conjugate roots

$$1 - 2q^2 \sin^2 k\theta \pm 2iq(\sin 2k\theta)\sqrt{1 - q^2 \sin^2 k\theta} = e^{\pm i\varphi_k}, \quad \varphi_k = \cos^{-1}(1 - 2q^2 \sin^2 k\theta).$$

Using the last relation together with (16) and (17), we finally obtain

$$\det X \sim \prod_{k=0}^{n/2-1} [(z^2 - e^{i\varphi_k})(z^2 - e^{-i\varphi_k})].$$

Recalling that $z^2 = e^{2i\theta\zeta}$, we conclude that the zeros of $\det X(\zeta)$ are simple and listed in (10).

Lemma 1 and Theorem 1 lead to the following result.

Theorem 2. *Let $\delta > \lambda$ be a point of the form* (10) *nearest to λ. Then $\phi(z)$ in each sector S_j is representable as*

$$\phi(z) = cz^\delta + O(|z|^{\delta+\varepsilon}), \quad c \neq 0, \tag{18}$$

for $\delta \neq 0$ and as

$$\phi(z) = c_0 + c_1 \log z + O(|z|^\varepsilon), \quad |c_0| + |c_1| \neq 0, \tag{19}$$

for $\delta = 0$.

In conclusion, we examine the case of $\nu_1 = \nu_2$. In this case $q = 1$ and the points (10) are integers lying on the line. On the other hand, if $\nu_1 = \nu_2$ and $c = 0$, then (1) and (2) ensure the analyticity of $\phi(z)$ in the punctured unit disk and thereby $\phi(z)$ admits poles only which agrees with (18). If $-1 < \lambda < 0$ and $c \neq 0$ in (2) then $\phi(z)$ has the logarithmic decomposition (19).

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