On a Nonlocal Problem in Function Theory

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Abstract—We consider nonlocal boundary value problems for three harmonic functions each of which is defined in its own domain. A contact condition is posed on the common part of the boundaries of these domains, and the Dirichlet or Neumann data (or mixed boundary conditions) are given on the remaining parts of the boundary. We prove the unique solvability of these problems.

1. STATEMENT OF THE PROBLEMS

Let D^p, $1 \le p \le 3$, be a finite domain bounded by a simple piecewise smooth contour on the plane. The set of endpoints of smooth arcs comprising this contour will be denoted by F^p. We assume that the interior angles of the domain D^p at the points $\tau \in F^p$ are not equal to 0 or 2π; i.e., the points τ are not cusps of the contour ∂D^p. Let all contours ∂D^p have a common smooth arc $\Gamma_0^p = \Gamma_0$ with endpoints τ_1 and τ_2. This arc is directed from τ_1 to τ_2, and all domains D^p lie to the left of this arc. The complement $\partial D^p \backslash \Gamma_0$ is denoted by Γ_+^p, $1 \le p \le 3$. Consider three problems arising in the theory of stratified sets [1].

Problem D. Find a family of three functions $u^p \in C(\overline{D}^p \backslash F^p)$, $1 \le p \le 3$, that are harmonic in D^p and satisfy the boundary conditions

$$u^1 = u^2 = u^3, \quad \frac{\partial u^1}{\partial n} + \frac{\partial u^2}{\partial n} + \frac{\partial u^3}{\partial n} = 0 \quad \text{on} \quad \Gamma_0, \tag{1}$$

$$u^p|_{\Gamma_+^p} = g^p, \qquad 1 \le p \le 3, \tag{2_{D}}$$

where n is the unit outward normal. At the points $\tau \in F^p$, the functions $u^p(z) = u^p(x, y)$, where $z = x + iy$, can have singularities of logarithmic type; i.e., $u^p(z)|z - \tau|^\varepsilon \to 0$ for any $\varepsilon > 0$ as $z \to \tau$.

Problem N is stated in a similar way with conditions (2_{D}) replaced by the conditions

$$\left.\frac{\partial u^p}{\partial n}\right|_{\Gamma_+^p} = (h^p)', \qquad 1 \le p \le 3, \tag{2_{N}}$$

where the prime stands for the derivative with respect to the arc length parameter.

Let the curve Γ_+^p be divided into two parts Γ_k^p, $k = 0, 1$.

Problem DN is stated in the same way as above with conditions (2_{D}) and (2_{N}) replaced by the boundary conditions

$$u^p|_{\Gamma_1^p} = g^p, \qquad \left.\frac{\partial u^p}{\partial n}\right|_{\Gamma_2^p} = (h^p)', \qquad 1 \le p \le 3. \tag{2_{DN}}$$

Throughout the following, in Problems D and N, we assume that Γ_+^p is a smooth arc with endpoints τ_1 and τ_2. In Problem DN, the curves Γ_k^p with endpoints τ_k, τ^p, $k = 1, 2$, which comprise Γ_+^p, are assumed to be smooth arcs.

Obviously, Problem D, N, and DN can be restated for analytic functions ϕ^p whose real parts coincide with the harmonic functions u^p. Since the imaginary parts of the functions ϕ^p are defined up to a constant, it follows in view of the Cauchy–Riemann conditions that the boundary conditions acquire the form

$$\operatorname{Re}(\phi^1-\phi^2)|_{\Gamma_0} = \operatorname{Re}(\phi^2-\phi^3)|_{\Gamma_0} = 0, \qquad \operatorname{Im}(\phi^1+\phi^2+\phi^3)|_{\Gamma_0} = C, \tag{3}$$

$$\operatorname{Re}\phi^p|_{\Gamma^p_+} = g^p, \tag{4_D}$$

$$\operatorname{Im}\phi^p|_{\Gamma^p_+} = h^p, \tag{4_N}$$

$$\operatorname{Re}\phi^p|_{\Gamma^p_1} = g^p, \qquad \operatorname{Im}\phi^p|_{\Gamma^p_2} = h^p, \tag{4_DN}$$

with some constant $C \in \mathbb{R}$. Just as above, the corresponding problems are referred to as Problems D, N, and DN.

2. MAIN RESULTS FOR THE GENERAL BOUNDARY VALUE PROBLEM

Problems D, N, and DN are nonlocal Riemann boundary value problems, which were studied in detail in [2, 3]. Let us state the main results for these problems. We start from the case of a single domain D bounded by a piecewise smooth contour ∂D without cusps. This contour consists of m smooth arcs $\Gamma_1, \dots, \Gamma_m$, and the set of endpoints of these arcs is denoted by F. (Obviously, it consists of m elements.) Each arc Γ_j is assumed to be oriented, so that the domain D lies either to the left or to the right of Γ_j. Accordingly, we set $\sigma_j = 1$ and $\sigma_j = -1$, and as a result we obtain the orientation signature $\sigma = (\sigma_i)^m_1$.

Let us choose smooth parametrizations $\gamma_j :\ [0,1] \to \Gamma_j$ coordinated with the sense of the arcs Γ_j. To a function ϕ analytic in the domain D and continuous in $\overline{D}\backslash F$, we assign the family $\phi^+_\gamma = (\phi^+_{\gamma,j})^m_1$ of its boundary values transferred with the use of these parametrization to the interval $(0,1)$ of the real line:

$$\phi^+_\gamma = (\phi^+_{\gamma,j})^m_1, \qquad \phi^+_{\gamma,j} = \phi^+ \circ \gamma_j. \tag{5}$$

Consider the nonlocal Riemann problem

$$\operatorname{Re} a\phi^+_\gamma = f, \tag{6}$$

where $a(s) = \{a_{ij}(s)\} \in C[0,1]$, is an $m \times m$ matrix function and f is a real m-vector function defined and continuous on the interval $(0,1)$. In coordinates, this boundary condition can be represented in the form of m linear relations

$$\operatorname{Re}\sum_{j=1}^m a_{ij}\ \phi^+_{\gamma,j} = f_i, \qquad 1 \le i \le m.$$

Let us describe the Hölder classes in which the problem will be considered. Let $C^\mu(\overline{D})$ be the space of functions satisfying the Hölder condition with exponent $0 < \mu < 1$ in the closed domain $\overline{D}$. By $C^\mu_{-0}(\overline{D}, F)$ we denote the class of functions $\phi(z)$ that are analytic in the domain D and satisfy the condition

$$\phi_\varepsilon(z) = [\Pi_{\tau\in F}(z-\tau)^{\mu+\varepsilon}]\phi(z) \in C^\mu(\overline{D}^p), \qquad \phi_\varepsilon|_F = 0,$$

for each $\varepsilon > 0$. Obviously, a function $\phi(z)$ of this class satisfies the Hölder condition with exponent μ outside any neighborhood of the points $\tau \in F$ and admits singularities of logarithmic type at these points. On the contrary, let us introduce the class $C^\mu_{+0}(\overline{D}, F)$ of functions $\phi(z)$ analytic in D and satisfying the condition

$$\phi_{-\varepsilon}(z) = [\Pi_{\tau\in F^p}(z-\tau)^{\mu-\varepsilon}]\phi(z) \in C^\mu(\overline{D}^p), \qquad \phi_{-\varepsilon}|_F = 0,$$

for some $\varepsilon > 0$. The functions of this class satisfy the Hölder condition (with some small exponent) in the entire closed domain $\overline{D}$ and vanish at the points $\tau \in F$. Finally, let $C^\mu_{(+0)}$ be the class obtained as the extension of the class C^μ_{+0} by polynomials. (One can restrict considerations to polynomials of degree $\le m-1$, where m is the number of points in the set F.) In a similar way, one can introduce the classes $C^{\mu}_{\pm 0}([0,1];0,1)$ for vector functions $f(t)$ on the interval $(0,1)$ with respect to the weight function $[t(t-1)]^{\mu\pm\varepsilon}$.

We consider problem (6) in the classes $C^{\mu}_{\pm 0}(\overline{D},F)$ and in the class $C^{\mu}_{(+0)}(\overline{D},F)$ of functions that are continuous in the closed domain $\overline{D}$; we assume that the right-hand side belongs to the corresponding class. In the case of the class $C^{\mu}_{(+0)}(D,F)$, the values $f(0)$ and $f(1)$ of the right-hand side $f\in C^{\mu}_{(+0)}([0,1];0,1)$ should be subjected to necessary matching conditions. They are given by some linear relations that provide that, for a given function analytic in D, there exists a function $\psi\in C(\overline{D})$ such that the corresponding right-hand sides in the boundary condition (6) for the difference $\phi^p-\psi^p$ are zero at the endpoints of the interval $[0,1]$.

For the smoothness of the data of our problems, we assume that the arcs Γ_j belong to the class $C^{\mu+0}$ (i.e., admit parametrizations γ whose derivatives γ_j' belong to the class $C^{\mu+\varepsilon}[0,1]$ with some $\varepsilon>0$). In a similar way, we assume that the matrix function $a(s)$ occurring in the boundary condition (6) belongs to the class $C^{\mu+0}[0,1]$.

The above-described statement includes the case of several domains D as well. Let domains D^p, $p=1,\dots,s$, be unrelated and satisfy all above-mentioned conditions. Let the arcs Γ_i^p, $1\le i\le m^p$, the set F^p of their endpoints, the terminal arcs $\Gamma_i^p(\tau)$, and the curvilinear sectors $S^p(\tau)$, $\tau\in F^p$, be defined on the basis of D^p just as above. The set of m^p arcs Γ_i^p is denoted by $\{\partial D^p\}$. We set $m=m^1+\dots+m^s$ and number the disjoint union $\bigcup_p\{\partial D^p\}$ in a unified manner as Γ_j, $1\le j\le m$. Thus, we write $\Gamma_j\in\{\partial D^p\}$ meaning that $\Gamma_j=\Gamma_i^p$ for some p and i.

The signature of the orientation σ is also defined in a unified way: $\sigma_j=1$ if the arc $\Gamma_j\in\{\partial D^p\}$ has the positive sense with respect to the domain D^p (i.e., leaves this domain on the left), and $\sigma_j=-1$ otherwise. Let the parametrizations γ_j of the arcs Γ_j have the same meaning as above. Starting from a family $\phi=(\phi^p)$ of functions ϕ^p analytic in D^p, we form an m-vector function ϕ_γ of their boundary values on the interval $(0,1)$ by setting $\phi_{\gamma,j}=(\phi^p)^+\circ\gamma_j$ for $\Gamma_j\in\{\partial D^p\}$. Then problem (6) for the considered family ϕ can be posed in a similar way. The notation of the above-introduced classes remains the same for these families.

Let the matrix a^σ be obtained from a by the rule

$$(a^\sigma)_{ij}=\begin{cases} a_{ij} & \text{for}\quad \sigma_j=1\\ \overline{a}_{ij} & \text{for}\quad \sigma_j=-1.\end{cases}$$

In other words, the jth columns of the matrices a^σ and a coincide if $\sigma_j=1$ and are complex conjugate if $\sigma_j=-1$. In particular, $\overline{a}^\sigma=a^{-\sigma}$.

We say that problem (6) has normal type if $\det a^\sigma(t)\ne 0$, $0\le t\le 1$. We assume that this condition is satisfied and set

$$b=(a^\sigma)^{-1}\overline{a}^\sigma. \tag{7}$$

There exists an arbitrarily small $r>0$ such that the disks with centers $\tau\in F^p$ and radius r are pairwise disjoint and intersect the domain D^p over curvilinear sectors $S(\tau)=S^p(\tau)$ with vertex τ. If τ is an endpoint of the arc $\Gamma_i\in\{\partial D^p\}$, then the intersection of the above-mentioned disk with Γ_i is given by the arc $\Gamma_i(\tau)$ with endpoint τ. It is convenient to set

$$\Gamma_i(\tau)=\begin{cases}\Gamma_i^{(0)} & \text{if}\quad \tau \text{ is the left endpoint of } \Gamma_i\\ \Gamma_i^{(1)} & \text{otherwise.}\end{cases}$$

We denote the lateral sides of the sector $S=S(\tau)$ by $\partial^\pm S(\tau)$ and assume that the rotation from ∂^+S to ∂^-S around the vertex τ inside the sector is counterclockwise. Therefore, the set of $2m$ lateral edges of all m sectors $S^p(\tau)$, $\tau\in F^p$, $1\le p\le s$, coincides with the set $\{\Gamma_i^{(k)}$, $1\le i\le m,\ k=0,1\}$. We number these terminal arcs in a unified manner as $\Gamma_{(k)}$, $1\le k\le 2m$. Consider the $2m\times 2m$ matrix function $V(\zeta)$ of the complex variable ζ with entries

$$V_{kr}(\zeta)=\begin{cases} e^{i\theta\zeta} & \text{if}\quad \{\Gamma_{(k)},\Gamma_{(r)}\}=\{\partial^+S,\partial^-S\}\\ 0 & \text{otherwise,}\end{cases} \tag{8}$$

where S is one of the m sectors $S^p(\tau)$, $\tau\in F^p$, $1\le p\le s$, and θ is the opening angle of that sector.

This matrix is entirely determined by the geometric properties of the domains D^p that form the set D. Next, we introduce the constant $2m \times 2m$ matrix B with entries

$$B_{kr} = \begin{cases} \overline{b_{ij}(0)} & \text{if} \quad \Gamma_{(k)} = \Gamma_i^{(0)},\, \Gamma_{(r)} = \Gamma_j^{(0)} \\ b_{ij}(1) & \text{if} \quad \Gamma_{(k)} = \Gamma_i^{(1)},\, \Gamma_{(r)} = \Gamma_j^{(1)} \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

The matrix function $(V+B)(\zeta)$ analytic on the entire plane is referred to as the *terminal symbol* of the problem. The meromorphic function

$$\frac{\det(V+B)(\zeta)}{\det(V+1)(\zeta)}$$

tends to nonzero limits as $\operatorname{Im}\zeta \to \pm\infty$, $\operatorname{Re}\zeta = \text{const}$; thus, the projection of the zeros of the function $\det(V+B)(\zeta)$ onto the real axis is a discrete set. Let $\alpha > 0$ be small enough to ensure that these zeros are absent in the strip $-\alpha \le \operatorname{Re}\zeta < 0$. Set

$$\varkappa_0 = \frac{1}{2\pi i}[\ln\det b(t)]\big|_0^1 - \frac{1}{2\pi i}\left[\ln\frac{\det(V+B)(\zeta)}{\det(V+1)(\zeta)}\right]\bigg|_{-\alpha-i\infty}^{-\alpha+i\infty}, \tag{10}$$

where the expressions in brackets are determined by continuous branches of the logarithm and the vertical bar stands for the increment in the corresponding limits. One can readily show that $\varkappa_0$ is an integer.

Theorem 1. *Suppose that all arcs Γ_j belong to the class $C^{1,\mu+0}$ (that is, to $C^{1,\mu+\varepsilon}$ with some $\varepsilon > 0$), the matrix function $a(t)$ belongs to $C^{\mu+0}[0,1]$, and problem (6) is a problem of normal type. Then this problem is Fredholm in the class C^{μ}_{-0}, and its index $\varkappa$ is equal to $\varkappa_0 + s$, where s is the number of domains D^p. More precisely, the following assertions hold.*

1. *The homogeneous problem (6) has $k < \infty$ linearly independent solutions in the class C^{μ}_{-0}.*

2. *The inhomogeneous problem with right-hand side $f \in C^{\mu}_{-0}$ is solvable in this class if and only if the orthogonality conditions*

$$\int_0^1 f(t)g_i(t)\,\frac{dt}{t(t-1)} = 0, \qquad 1 \le i \le k', \tag{11}$$

are satisfied with certain linearly independent vector functions $g_i \in C^{\mu}_{+0}$, $1 \le i \le k'$.

3. *$k - k' = \varkappa$.*

The product fg_i in the integrand in (11) is understood as the inner product of the m-component real vectors f and g_i. The product of the functions $f \in C^{\mu}_{-0}$ and $g_i \in C^{\mu}_{+0}$ belongs to C^{μ}_{+0}, and hence these integrals are well defined.

Under certain conditions, Theorem 1 permits one to study the solvability of problem (6) also in the class $C^{\mu}_{+0}(D) \subseteq C(\overline{D})$.

Theorem 2. *Suppose that the matrix function $\zeta(V+B)^{-1}(\zeta)$ occurring in the assumptions of Theorem 1 has no poles on the imaginary axis. Then each solution $\phi \in C^{\mu}_{-0}$ of problem (6) with right-hand side $f \in C^{\mu}_{+0}$ belongs to the class $C^{\mu}_{(+0)}$. In particular, $\varkappa$ is also the index of the problem in the class $C^{\mu}_{(+0)}$.*

Let us present an assertion on the additional smoothness of a solution.

Theorem 3. *Under the assumptions of Theorem* 1, *suppose that the function* $a(t)$ *belongs to* $C^{1,\mu+0}[0,1]$. *Then each solution* $\phi \in C^{\mu}_{-0}$ *of problem* (6) *with right-hand side* $f \in C^{\mu}_{-0}$ *such that*

$$\tilde f(t) = t(t-1)f'(t) \in C^{\mu}_{-0}$$

has a similar property; *i.e.*,

$$\tilde\phi^p(z) = \Pi_{\tau\in F^p}(z-\tau)(\phi^p)'(z) \in C^{\mu}_{-0}(D^p), \qquad p=1,2,3. \tag{12}$$

If, *in addition*, *the assumption of Theorem* 2 *holds and the function* $\tilde f$ *belongs to* C^{μ}_{+0}, *then* $\tilde\phi^p$ *belongs to* $C^{\mu}_{+0}(D^p)$ *as well.*

The verification of the condition $\det(V+B) \neq 0$ and the condition for the inverse matrix $(V+B)^{-1}$ in Theorem 2 is simplified if the matrix $V+B$ has a certain block-diagonal structure.

We say that a $2m \times 2m$ matrix $x = (x_{kr})_1^{2m}$ is block-diagonal with respect to some partition $E = (E_i)_1^n$ of the set of terminal arcs if its entries x_{kr} are zero whenever $\Gamma_{(k)} \in E_i$ and $\Gamma_{(r)} \in E_j$, $i \neq j$. The multiplication of such matrices is reduced to the block multiplication of their diagonal blocks $x(E_i) = \{x_{kr}, \Gamma_{(k)}, \Gamma_{(r)} \in E_i\}$.

Accordingly, $\det x = \Pi_i \det x(E_i)$, and for $\det x \neq 0$ the inverse matrix x^{-1} has the same block-diagonal structure with diagonal blocks $x^{-1}(E_i) = [x(E_i)]^{-1}$.

For definitions (8) and (9), one can conclude that the matrix V has block-diagonal form with respect to the partition G of the set of terminal arcs into the pairs $G^p(\tau) = \{\partial^{\pm} S^p(\tau)\}$ with diagonal blocks

$$V[\zeta, G^p(\tau)] = e^{i\theta\zeta}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where θ is the opening angle of the sector $S^p(\tau)$.

In a similar way, the matrix B is block-diagonal with respect to the partition into two sets $I^0 = \{\Gamma_i^{(0)},\ 1 \le i \le 6\}$ and $I^1 = \{\Gamma_i^{(1)},\ 1 \le i \le m\}$:

$$B(I^0) = \overline{b(0)}, \qquad B(I^1) = b(1).$$

Suppose that the $m \times m$ matrix a of problem (6) is block-diagonal with respect to some partition $O_1, \ldots, O_k$ of the arc set $\{\Gamma_1, \ldots, \Gamma_m\}$. In other words, $a_{ij} = 0$ if $\Gamma_i \in O_p$, $\Gamma_j \in O_q$, and $p \neq q$.

Consider the partition $\widehat O$ of the set of terminal arcs into $2k$ elements $O_p^r = \{\Gamma_j^{(r)}, \Gamma_j \in O_p\}$, $r = 0,1$, $1 \le p \le s$. Obviously, the matrix B has block-diagonal form with respect to this partition. Consequently, if each element of the partition E consists of whole elements of both the partition G and the partition $\hat O$, then the matrix $V+B$ has block-diagonal form with respect to the partition E. In this case, for the increment of the argument in the index formula (10), we have the relation

$$\frac{1}{2\pi i}\left[\ln\frac{\det(V+B)(\zeta)}{\det(V+1)(\zeta)}\right]\Bigg|_{-\alpha-i\infty}^{-\alpha+i\infty} = \sum_{k=1}^{n}\frac{1}{2\pi i}\left[\ln\frac{\det(V+B)(\zeta,E_k)}{\det(v+1)(\zeta,E_k)}\right]\Bigg|_{-\alpha-i\infty}^{-\alpha+i\infty}. \tag{13}$$

3. PROBLEM D

Let the arcs $\Gamma_0 = \Gamma_0^p$ and Γ_1^p, $1 \le p \le 3$, comprising the boundary contour ∂D^p of the domain D^p belong to the class $C^{1,\mu+0}$ and have positive sense with respect to D^p. We number these arcs in a unified manner,

$$(\Gamma_0^1, \Gamma_0^2, \Gamma_0^3, \Gamma_1^1, \Gamma_1^2, \Gamma_1^3) = (\Gamma_1, \ldots, \Gamma_6).$$

Then for the lateral sides of the three sectors $S^p(\tau_1)$ and $S^p(\tau_2)$, we have the expressions

$$\partial^+ S^p(\tau_1) = \Gamma_p^{(0)}, \qquad 1 \le p \le 3,$$
$$\partial^- S^1(\tau_1) = \Gamma_4^{(1)}, \qquad \partial^- S^2(\tau_1) = \Gamma_5^{(1)}, \qquad \partial^- S^3(\tau_1) = \Gamma_6^{(1)}$$

and

$$\partial^- S^p(\tau_2) = \Gamma_p^{(1)}, \qquad 1 \le p \le 3,$$
$$\partial^+ S^1(\tau_2) = \Gamma_4^{(0)}, \qquad \partial^+ S^2(\tau_2) = \Gamma_5^{(0)}, \qquad \partial^+ S^3(\tau_2) = \Gamma_6^{(0)},$$

respectively.

In the chosen numbering, we write out Problem D with the boundary conditions (3) and (4_{D}) in the form (6) with 6×6 matrix

$$a = \operatorname{diag}(a_1, a_2), \tag{14}$$

where

$$a_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -i & -i & -i \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The six-component vector f of the right-hand side of (6) in the considered case is related to the right-hand side of the boundary conditions (3) and (4_{D}) by the formulas

$$f_1 = f_2 = 0, \qquad f_3 = C, \qquad f_4 = g^1 \circ \gamma_4, \qquad f_5 = g^2 \circ \gamma_5, \qquad f_6 = g^3 \circ \gamma_6.$$

Accordingly, the matrix b in (7) has the form

$$b = a^{-1}\overline{a} = \operatorname{diag}(c_1, c_2), \tag{15}$$

where

$$c_1 = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}, \qquad c_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In the considered case, the matrix b has block-diagonal form with respect to the partition $\widehat{O}$ of the set $\{\Gamma_1, \dots, \Gamma_6\}$ with elements $O_1 = \{\Gamma_1, \Gamma_2, \Gamma_3\}$, $O_2 = \{\Gamma_4\}$, $O_3 = \{\Gamma_5\}$, and $O_4 = \{\Gamma_6\}$. Consequently, each of the partitions G and $\widehat{O}$ is a refinement of the partition E into two elements

$$E_j = G^1(\tau_j) \cup G^2(\tau_j) \cup G^3(\tau_j), \qquad j = 1, 2,$$

corresponding to the endpoints τ_j of the arc Γ_0. Recall that the lateral sides of the sector $S^p(\tau_j)$ form an element $G^p(\tau_j)$ of the partition G.

We choose the common numbering of terminal arcs composing the set E_j for which the terminal arcs on Γ_0^p are first numbered in the ascending order of p and then those on Γ_1^p are numbered. Specifically, we have the ordered sets

$$\begin{aligned} E_1 &= \{\Gamma_1^{(0)}, \Gamma_2^{(0)}, \Gamma_3^{(0)}, \Gamma_4^{(1)}, \Gamma_5^{(1)}, \Gamma_6^{(1)}\} = \{\Gamma_{(1)}, \dots, \Gamma_{(6)}\}, \\ E_2 &= \{\Gamma_1^{(1)}, \Gamma_2^{(1)}, \Gamma_3^{(1)}, \Gamma_4^{(0)}, \Gamma_5^{(0)}, \Gamma_6^{(0)}\} = \{\Gamma_{(7)}, \dots, \Gamma_{(12)}\}. \end{aligned} \tag{16}$$

In this notation, for the diagonal blocks $(V + B)(\zeta, E_j)$, $j = 1, 2$, of the terminal symbol $V + B$ of Problem D, we have the expression

$$(V + B)(\zeta, E_j) = \begin{pmatrix} c_1 & v_j(\zeta) \\ v_j(\zeta) & 1 \end{pmatrix}, \qquad j = 1, 2. \tag{17}$$

Here the matrix c_1 is given by (15), and the $v_j(\zeta)$, $j = 1, 2$, are the diagonal matrices

$$v_j(\zeta) = \operatorname{diag}[e^{i\theta_1(\tau_j)\zeta}, e^{i\theta_2(\tau_j)\zeta}, e^{i\theta_3(\tau_j)\zeta}], \tag{18}$$

where $\theta_p(\tau_j)$ is the opening angle of the sector $S^p(\tau_j)$.

It follows from (17) that

$$\det(V+B)(\zeta,E_j)=\det[c_1-v_j^2(\zeta)], \tag{19}$$

$$(V+B)^{-1}(\zeta,E_j)=\begin{pmatrix}(c_1-v_j^2)^{-1} & -(c_1-v_j^2)^{-1}v_j\\ -v_j(c_1-v_j^2)^{-1} & v_j(c_1-v_j^2)^{-1}v_j+1\end{pmatrix}.$$

By carrying out similar computations, we obtain

$$\det(V+1)(\zeta,E_j)=\det[1-v_j^2(\zeta)]=(1-e^{2i\theta_1(\tau_j)\zeta})(1-e^{2i\theta_2(\tau_j)\zeta})(1-e^{2i\theta_3(\tau_j)\zeta}), \tag{20}$$

$$(V+1)^{-1}(\zeta,E_j)=\begin{pmatrix}(1-v_j^2)^{-1} & -(1-v_j^2)^{-1}v_j\\ -v_j(1-v_j^2)^{-1} & v_j(1-v_j^2)^{-1}v_j+1\end{pmatrix}.$$

Consider the behavior of the matrix function $(V+B)(\zeta,E_j)$ on the imaginary axis.

Lemma 1. (a) *On the imaginary axis, the function* $\det(V+B)(\zeta,E_j)$ *has a unique zero at the point* $\zeta=0$, *and its order is equal to* 2.

(b) *The matrix function* $(V+B)^{-1}(\zeta,E_j)$ *has a first-order zero at the point* $\zeta=0$.

(c) *One has*

$$\frac{1}{2\pi i}\left[\ln\frac{\det(V+B)(\zeta,E_j)}{\det(V+1)(\zeta,E_j)}\right]\Bigg|_{-\alpha-i\infty}^{-\alpha+i\infty}=\frac{1}{2}. \tag{21}$$

Proof. It suffices to consider the case of $j=1$; for $j=2$, the considerations can be carried out in a similar way.

(a) It follows from (19) that the order of the pole of the function $\det(V+B)(\zeta,E_1)$ coincides with the order of the pole of the matrix $(c_1-v_1^2)(\zeta)$. By setting $t_k=e^{2i\theta_k\zeta}$, $k=1,2,3$, and by using (15), we write out the matrix $c_1-v_1^2$ in the form

$$c_1-v_1^2=\frac{1}{3}\begin{pmatrix}1-3t_1 & -2 & -2\\ -2 & 1-3t_2 & -2\\ -2 & -2 & 1-3t_3\end{pmatrix}. \tag{22}$$

Note that if $\operatorname{Re}\zeta=0$, then all t_k are positive; moreover, $\operatorname{sgn}(t_k-1)$ is independent of k. The determinant $f(t)=\det(c_1-v_1^2)$ of this matrix satisfies the relation

$$3f(t)=-3(t_1t_2t_3+1)+(t_1t_2+t_1t_3+t_2t_3)+(t_1+t_2+t_3). \tag{23}$$

Obviously, $f(t)=0$ for $t_1=t_2=t_3=1$, and

$$f\left(\frac{1}{t}\right)=\frac{f(t)}{t_1t_2t_3},\qquad \frac{1}{t}=\left(\frac{1}{t_1},\frac{1}{t_2},\frac{1}{t_3}\right). \tag{24}$$

To prove the first assertion in (a), it suffices to show that the function $f(t)$ vanishes nowhere in the cube $K=\{0<t_k<1,\ k=1,2,3\}$. On the boundary of this cube, the function $f(t)$ is nonpositive and is zero only at the point $t=(1,1,1)$. One can readily see that $\operatorname{grad}f(t)\neq 0$, $t\in K$; therefore, $f(t)<0$ in the entire cube K.

Let us compute the order of the zero of the function $\det(V+B)(\zeta,E_1)$ at the point $\zeta=0$. By (23), we have

$$3[\det(V+B)]'(0,E_1)=i[-3(2\theta_1+2\theta_2+2\theta_3)+4(\theta_1+\theta_2+\theta_3)+2(\theta_1+\theta_2+\theta_3)]=0,$$
$$3[\det(V+B)]''(0,E_1)=4[(\theta_1^2+\theta_2^2+\theta_3^2)+4(\theta_1\theta_2+\theta_2\theta_3+\theta_1\theta_3)-(\theta_1+\theta_2+\theta_3)]\neq 0.$$

Therefore, the function $\det(V+B)(\zeta,E_j)$, $j=1,2$, has a second-order zero at the point $\zeta=0$.

(b) It follows from (19) that the orders of the poles of the matrices $(V+B)^{-1}(\zeta, E_1)$ and $(c_1 - v_1^2)^{-1}(\zeta)$ coincide. We rewrite relation (22) in the form

$$3(c_1 - v_1^2) = \begin{pmatrix} d_1 & -2 & -2 \\ -2 & d_2 & -2 \\ -2 & -2 & d_3 \end{pmatrix}, \qquad d_k = 1 - 3t_k.$$

Then

$$(c_1 - v_1^2)^{-1} = \frac{3}{\det(c - v_1^2)} \begin{pmatrix} d_2 d_3 - 4 & 2(d_3+2) & 2(d_2+2) \\ 2(d_3+2) & d_1 d_3 - 4 & 2(d_1+2) \\ 2(d_2+2) & 2(d_1+2) & d_1 d_2 - 4 \end{pmatrix}.$$

If $\zeta = 0$, i.e., $d_1 = d_2 = d_3 = -2$, then all entries of the matrix on the right-hand side in the last relation are zero. Since the matrix $\det(V+B)(\zeta, E_1)$ has a second-order zero at the point $\zeta = 0$, it follows that the matrix $(V+B)^{-1}(\zeta, E_1)$ and hence the matrix $(V+B)^{-1}(\zeta, E_2)$ have a first-order pole at this point.

(c) By (20), in the above-introduced notation, we have

$$\det(V+1)(\zeta, E_1) = (1-t_1)(1-t_2)(1-t_3).$$

Therefore, the function

$$x(\zeta) = \frac{\det(V+B)(\zeta, E_1)}{\det(V+1)(\zeta, E_1)}$$

can be represented as the ratio of the functions $f(t)$ and $g(t) = (1-t_1)(1-t_2)(1-t_3)$, where $t = (t_1, t_2, t_3)$ and $t_k = e^{2i\theta_k \zeta}$.

Note that the function $x(\zeta)$ is odd; thus,

$$\frac{1}{2\pi i} \ln x(\zeta)\Big|_{-\alpha - i\infty}^{-\alpha + i\infty} = -\frac{1}{2\pi i} \ln x(\zeta)\Big|_{\alpha - i\infty}^{\alpha + i\infty}.$$

On the other hand, by the argument principle for analytic functions, we have

$$\frac{1}{2\pi i} \ln x(\zeta)\Big|_{\alpha - i\infty}^{\alpha + i\infty} - \frac{1}{2\pi i} \ln x(\zeta)\Big|_{-\alpha - i\infty}^{-\alpha + i\infty} = m,$$

where m stands for the difference between the number of zeros and the number of poles of the function $x(\zeta)$ with regard of multiplicities in the strip $-\alpha < \operatorname{Re}\zeta < \alpha$. It follows from these two relations that the left-hand side of relation (21) is equal to $-m/2$.

Obviously, the function $\det(V+1)(\zeta, E_1)$ has a unique zero of multiplicity 3 at the point $\zeta = 0$ in this strip. This, together with assertion (a), implies that $m = -1$, which completes the proof of relation (21) and the lemma.

Theorem 4. *Let all arcs Γ_j belong to the class $C^{1,\mu+0}$. Then the solutions of the homogeneous problem* (6), (14) *consist of constant functions, and the inhomogeneous problem is unconditionally solvable in the class C^μ_{-0}. In this case, the assertions of Theorems* 2 *and* 3 *hold.*

Proof. It follows from Theorem 1 and Lemma 1 (c) that the index of the considered problem is equal to

$$\varkappa = -\frac{1}{2} - \frac{1}{2} + 3 = 2. \tag{25}$$

It follows from Lemma 1 (b) that the matrix function $\zeta(V+B)^{-1}(\zeta, E_j)$ has no poles on the imaginary axis. Therefore, the assumptions of Theorems 2 and 3 hold for the considered problem. In particular, any solution $\phi \in C^\mu_{-0}$ of the homogeneous problem belongs to the class $C^\mu_{(+0)}$.

Let us show that the solution space of the homogeneous problem consists of constants and hence has dimension 2. This, together with relation (25), implies the unconditional solvability of the inhomogeneous problem.

Let $\phi = (\phi^p)_1^3$ be a solution of the homogeneous problem (6), (14) in the class C^{μ}_{-0}. Then, as was mentioned above, the function ϕ^p is continuous in the closed domain $\overline{D}^p$. Consequently, the harmonic functions $u^p \in C(\overline{D}^p)$ are a solution of the homogeneous problem (1), (2_{D}), i.e., satisfy the homogeneous boundary conditions

$$u^1 = u^2 = u^3, \quad \frac{\partial u^1}{\partial n} + \frac{\partial u^2}{\partial n} + \frac{\partial u^3}{\partial n} = 0 \quad \text{on} \quad \Gamma_0, \tag{26}$$

$$u^p|_{\Gamma_1^p} = 0, \qquad 1 \le p \le 3. \tag{27_{D}}$$

Suppose that one of these harmonic functions is nonzero. By virtue of (26) and (27_{D}), the maximum value of each of the functions $|u^p|$ can be attained only at some common interior point c of the arc Γ_0. But then, by the Zaremba–Giraud theorem, the normal derivatives $\partial u^p/\partial n$ have the same sign at this point, and at least one of them is nonzero. But this contradicts the second relation in (26).

Thus, all functions $u^p = \operatorname{Re}\phi^p$ are zero, and all functions ϕ^p are imaginary constants.

4. PROBLEM N

Problem N can be considered by analogy with the previous one. In this case, in (14) and (15), one should set

$$a_2 = \operatorname{diag}(-i, -i, i) \tag{28}$$

and

$$c_2 = \operatorname{diag}(-1, -1, -1).$$

Then relations (17) and (19) acquire the form

$$\begin{gathered}
(V+B)(\zeta, E_j) = \begin{pmatrix} c_1 & v_j(\zeta) \\ v_j(\zeta) & -1 \end{pmatrix}, \qquad j = 1, 2, \\
\det(V+B)(\zeta, E_j) = -\det[c_1 + v_j^2(\zeta)], \\
(V+B)^{-1}(\zeta, E_j) = \begin{pmatrix} (c_1 + v_j^2)^{-1} & (c_1 + v_j^2)^{-1} v_j \\ v_j (c_1 + v_j^2)^{-1} & v_j (c_1 + v_j^2)^{-1} v_j - 1 \end{pmatrix}.
\end{gathered} \tag{29}$$

Accordingly, the following assertions provide an analog of Lemma 1.

Lemma 2. (a) *On the imaginary axis, the function* $\det(V+B)(\zeta, E_j)$ *has a unique first-order zero at the point* $\zeta = 0$.

(b) *The matrix function* $(V+B)^{-1}(\zeta, E_j)$ *has a first-order pole at the point* $\zeta = 0$.

(c) *One has*

$$\frac{1}{2\pi i} \left[\ln \frac{\det(V+B)(\zeta, E_j)}{\det(V+1)(\zeta, E_j)} \right] \Bigg|_{-\alpha - i\infty}^{-\alpha + i\infty} = 1. \tag{30}$$

Proof. The proof can be carried out by analogy with Lemma 1 with some modifications.

(a) By setting $t_k = e^{2i\theta_k \zeta}$ and by using (14), (15), and (28), we obtain

$$c_1 + \operatorname{diag}(t_1, t_2, t_3) = \frac{1}{3} \begin{pmatrix} 1 + 3t_1 & -2 & -2 \\ -2 & 1 + 3t_2 & -2 \\ -2 & -2 & 1 + 3t_3 \end{pmatrix}. \tag{31}$$

For the determinant $f(t)$, $t = (t_1, t_2, t_3)$, of the last matrix, we have

$$27 f(t) = (1 + 3t_1)(1 + 3t_2)(1 + 3t_3) - 16 - 4[3 + 3(t_1 + t_2 + t_3)],$$

or

$$3f(t) = 3(t_1t_2t_3 - 1) + t_1t_2 + t_1t_3 + t_2t_3 - (t_1 + t_2 + t_3). \tag{32}$$

Therefore,

$$\det(V + B)(\zeta, E_1) = -f(t), \qquad t_k = e^{2i\theta_k\zeta}. \tag{33}$$

On the line $\operatorname{Re}\zeta = 0$, all quantities $t_k = e^{2i\theta_k\zeta}$ are real and positive; moreover, they are either simultaneously less than unity, or simultaneously larger than unity, or simultaneously equal to unity. It follows from (32) that if $t_1 = t_2 = t_3 = 1$, then the function $f(t)$ is zero and that

$$f(1/t) = -(t_1t_2t_3)^{-1}f(t), \qquad 1/t = (1/t_1, 1/t_2, 1/t_3). \tag{34}$$

By arguing as in Lemma 1, we find that the function $f(t)$ is negative in the entire cube $K = \{0 < t_k < 1,\ k = 1, 2, 3\}$ and is zero only at the point $t = (1, 1, 1)$, i.e., for $\zeta = 0$.

It remains to evaluate the order of the zero of the function $\det(V + B)(\zeta, E_1)$ at the point $\zeta = 0$. By (32), we have

$$3[\det(V + B)]'(0, E_1) = 2i[3(\theta_1 + \theta_2 + \theta_3) + 2(\theta_1 + \theta_2 + \theta_3) - (\theta_1 + \theta_2 + \theta_3)] \neq 0.$$

Consequently, the function $\det(V + B)(\zeta, E_1)$ and hence the function $\det(V + B)(\zeta, E_2)$ have a first-order zero at the point $\zeta = 0$.

(b) It follows from (29) that the order of the pole of the matrix $(V + B)^{-1}(\zeta, E_1)$ is equal to the order of the pole of the matrix $(c_1 + v_1^2)(\zeta)$. Arguing as in item (c) in Lemma 1, we set $d_k = 1 + 3t_k$ and obtain

$$(c_1 + v_1^2)^{-1} = \frac{3}{\det(c_1 + v_1^2)} \begin{pmatrix} d_2d_3 - 4 & 2(d_3 + 2) & 2(d_2 + 2) \\ 2(d_3 + 2) & d_1d_3 - 4 & 2(d_1 + 2) \\ 2(d_2 + 2) & 2(d_1 + 2) & d_1d_2 - 4 \end{pmatrix}.$$

Obviously, the entries of the matrix on the right-hand side in this relation are nonzero for any ζ. Since the matrix $\det(V + B)(\zeta, E_j)$, $j = 1, 2$, has a first-order zero at the point $\zeta = 0$, it follows that the matrix $(V + B)^{-1}(\zeta, E_j)$ has a first-order pole at this point.

(c) The proof of this assertion with regard of (a) can be performed by analogy with Lemma 1 (c).

Theorem 5. *Let all arcs Γ_j belong to the class $C^{1,\mu+0}$. Then all solutions of the homogeneous problem* (6), (28) *are constant functions, and the inhomogeneous problem is unconditionally solvable in the class C^{μ}_{-0}. Moreover, the assertions of Theorems* 2 *and* 3 *remain valid.*

Proof. It follows from Theorem 1 and Lemma 2 (c) that the index of the considered problem is equal to

$$\varkappa = -1 - 1 + 3 = 1. \tag{35}$$

Lemma 2 (b) implies that the matrix function $\zeta(V + B)^{-1}(\zeta, E_j)$ has no poles on the imaginary axis. Therefore, the assumptions of Theorems 2 and 3 hold for the considered problem. In particular, each solution $\phi \in C^{\mu}_{-0}$ of the homogeneous problem belongs to the class $C^{\mu}_{(+0)}$.

Let us show that the solution space of the homogeneous problem consists of constants and hence is one-dimensional. This, together with relation (35), implies the unconditional solvability of the inhomogeneous problem.

Let $\phi = (\phi^p)_1^3$ be a solution of the homogeneous problem (6), (28) in the class C^{μ}_{-0}. Then, by Theorem 3,

$$\tilde{\phi}^p(z) = (z - \tau_1)(z - \tau_2)(\phi^p)'(z) \in C^{\mu}_{+0}(D^p), \qquad p = 1, 2, 3.$$

Therefore, the function $|(\phi^p)'(z)|$ is square integrable in the domain D^p, and for the harmonic function $u^p = \operatorname{Re}\phi^p$, one can use the Green formula

$$\int\limits_{D^p} \left[\left(\frac{\partial u^p}{\partial x}\right)^2 + \left(\frac{\partial u^p}{\partial y}\right)^2\right] dx\, dy = \int\limits_{\partial D^p} u^p \frac{\partial u^p}{\partial n}\, ds, \qquad 1 \le p \le 3.$$

By virtue of the homogeneous conditions on the arc Γ_1^p, the last integral is equal to

$$\int\limits_{\partial D^p} u^p \frac{\partial u^p}{\partial n}\, ds = \int\limits_{\Gamma_0} u^p \frac{\partial u^p}{\partial n}\, ds.$$

It follows from the homogeneous conditions on the arc Γ_0 that the sum of these integrals over $1 \le p \le 3$ is zero. Consequently,

$$\sum_{p=1}^{3} \int\limits_{D^p} \left[\left(\frac{\partial u^p}{\partial x} \right)^2 + \left(\frac{\partial u^p}{\partial y} \right)^2 \right] dx\, dy = 0;$$

thus, each function u^p is constant in the respective domain D^p.

Since, by virtue of the boundary condition on Γ_1^p, the imaginary parts of the functions ϕ^p are zero, it follows that the space of solutions of the homogeneous problem indeed consists of constants and is one-dimensional.

5. PROBLEM DN

Just as above, we assume that the arcs $\Gamma_0 = \Gamma_0^p$ and the arcs Γ_1^p and Γ_2^p forming the curve Γ_+^p, $1 \le p \le 3$, of the boundary contour ∂D^p of the domain D^p belong to the class $C^{1,\mu+0}$ and have positive sense with respect to D^p. We number these arcs in the following unified way: $(\Gamma_0^1, \Gamma_0^2, \Gamma_0^3, \Gamma_1^1, \Gamma_1^2, \Gamma_1^3, \Gamma_2^1, \Gamma_2^2, \Gamma_2^3) = (\Gamma_1, \dots, \Gamma_9)$. Then, for the lateral sides of the six sectors $S^p(\tau_1)$ and $S^p(\tau_2)$, we have the expressions

$$\begin{aligned}
&\partial^+ S^p(\tau_1) = \Gamma_p^{(0)}, && 1 \le p \le 3, && \\
&\partial^- S^1(\tau_1) = \Gamma_4^{(1)}, && \partial^- S^2(\tau_1) = \Gamma_5^{(1)}, && \partial^- S^3(\tau_1) = \Gamma_6^{(1)}, \\
&\partial^- S^p(\tau_2) = \Gamma_p^{(1)}, && 1 \le p \le 3, && \\
&\partial^+ S^1(\tau_2) = \Gamma_7^{(0)}, && \partial^+ S^2(\tau_2) = \Gamma_8^{(0)}, && \partial^+ S^3(\tau_2) = \Gamma_9^{(0)},
\end{aligned}$$

and the lateral sides of the three sectors $S^p(\tau^p)$ are given by the relations

$$\begin{aligned}
&\partial^+ S^1(\tau^1) = \Gamma_4^{(0)}, && \partial^- S^1(\tau^1) = \Gamma_7^{(1)}, \\
&\partial^+ S^2(\tau^2) = \Gamma_5^{(0)}, && \partial^- S^2(\tau^2) = \Gamma_8^{(1)}, \\
&\partial^+ S^3(\tau^3) = \Gamma_6^{(0)}, && \partial^- S^3(\tau^3) = \Gamma_9^{(1)}.
\end{aligned}$$

In the chosen numbering, we write out Problem DN with the boundary conditions (3) and (4_{DN}) in the form (6) with the 9×9 matrix

$$a = \mathrm{diag}(a_1, a_2, a_3), \tag{36}$$

where a_1 and a_2 are the same matrices as in (14), $a_3 = \mathrm{diag}(-i, -i, -i)$, and the vector f on the right-hand side in relation (6) has the form

$$f_1 = f_2 = 0, \qquad f_3 = C, \qquad f_j = h^p \circ \gamma_j, \qquad j \ge 4.$$

By (7), the matrix b has the form

$$b = a^{-1}\overline{a} = \mathrm{diag}(c, c_1, c_2) \in \mathbb{C}^{9\times 9}; \tag{37}$$

here the matrices c and c_1 coincide with the matrices c_1 and c_2, respectively, in (15), and

$$c_2 = \mathrm{diag}(-1, -1, -1).$$

The matrix b has block-diagonal form with respect to the partition $\widehat{O}$ of the set $\{\Gamma_1, \dots, \Gamma_9\}$ into the elements $O_1 = \{\Gamma_1, \Gamma_2, \Gamma_3\}$, $O_2 = \{\Gamma_4\}$, $O_3 = \{\Gamma_5\}$, $O_4 = \{\Gamma_6\}$, $O_5 = \{\Gamma_7\}$, $O_6 = \{\Gamma_8\}$, and $O_7 = \{\Gamma_9\}$. The matrix V has block-diagonal form with respect to the partition of the set of terminal arcs into the pairs $G^p(\tau) = \{\partial^{\pm} S^p(\tau)\}$, $\tau \in F^p$.

We form a partition E as follows:

$$E_1 = G^1(\tau_1) \cup G^2(\tau_1) \cup G^3(\tau_1), \qquad E_2 = G^1(\tau_2) \cup G^2(\tau_2) \cup G^3(\tau_2),$$
$$E^1 = G^1(\tau^1), \qquad E^2 = G^2(\tau^2), \qquad E^3 = G^3(\tau^3).$$

It consists of entire elements of the partitions G and $\widehat{O}$. Consequently, the matrices V and B have block-diagonal form with respect to this partition, and therefore, the same is true for their sum $(V+B)(\zeta, E_j)$. We choose a common numbering of terminal arcs as follows:

$$\begin{aligned}
E_1 &= \{\Gamma_1^{(0)}, \Gamma_2^{(0)}, \Gamma_3^{(0)}, \Gamma_4^{(1)}, \Gamma_5^{(1)}, \Gamma_6^{(1)}\} = \{\Gamma_{(1)}, \Gamma_{(2)}, \Gamma_{(3)}, \Gamma_{(4)}, \Gamma_{(5)}, \Gamma_{(6)}\},\\
E_2 &= \{\Gamma_1^{(1)}, \Gamma_2^{(1)}, \Gamma_3^{(1)}, \Gamma_7^{(0)}, \Gamma_8^{(0)}, \Gamma_9^{(0)}\} = \{\Gamma_{(7)}, \Gamma_{(8)}, \Gamma_{(9)}, \Gamma_{(10)}, \Gamma_{(11)}, \Gamma_{(12)}\},\\
E^1 &= \{\Gamma_4^{(0)}, \Gamma_7^{(1)}\} = \{\Gamma_{(13)}, \Gamma_{(14)}\},\\
E^2 &= \{\Gamma_5^{(0)}, \Gamma_8^{(1)}\} = \{\Gamma_{(15)}, \Gamma_{(16)}\},\\
E^3 &= \{\Gamma_6^{(0)}, \Gamma_9^{(1)}\} = \{\Gamma_{(17)}, \Gamma_{(18)}\}.
\end{aligned}$$

Then the diagonal blocks of the matrices V and B have the form

$$V(\zeta, E_j) = \begin{pmatrix} 0 & v_j(\zeta) \\ v_j(\zeta) & 0 \end{pmatrix} \in \mathbb{C}^{6\times 6}, \qquad v_j(\zeta) = \mathrm{diag}[e^{i\theta_1(\tau_j)\zeta}, e^{i\theta_2(\tau_j)\zeta}, e^{i\theta_3(\tau_j)\zeta}],$$

where $\theta_p(\tau_j)$ is the opening angle of the sector $S^p(\tau_j)$,

$$B(\zeta, E_j) = \mathrm{diag}(c, (-1)^j) \in \mathbb{C}^{6\times 6}, \qquad j = 1, 2,$$

and c is found from (40) (see below). The remaining 2×2 diagonal blocks are given by the relations

$$V(\zeta, E^j) = e^{i\theta_j(\tau^j)\zeta} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B(E^j) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad 1 \le j \le 3.$$

It follows from Theorem 1 that, to prove the solvability of Problem DN in the class C^{μ}_{-0}, it suffices to study the zeros of the determinant $\det(V+B)(\zeta)$ on the line $\mathrm{Re}\,\zeta = 0$ and the order of the pole of the inverse matrix function $(V+B)^{-1}(\zeta)$ at the point $\zeta = 0$. As was mentioned above, it suffices to do this for the diagonal blocks of the corresponding matrices.

Obviously,

$$\begin{aligned}
&\det(V+B)(\zeta, E^j) = -(1 + e^{2i\theta_j(\tau^j)\zeta}), \qquad 1 \le j \le 3,\\
&(V+B)^{-1}(\zeta, E^j) = [\det(V+B)(\zeta, E^j)]^{-1} \begin{pmatrix} -1 & -e^{i\theta_j(\tau^j)\zeta} \\ -e^{i\theta_j(\tau^j)\zeta} & 1 \end{pmatrix},
\end{aligned} \tag{38}$$

$$\det(V+B)(\zeta, E_1) = -\det[c + v_1^2(\zeta)], \qquad \det(V+B)(\zeta, E_2) = \det[c - v_2^2(\zeta)], \tag{39}$$

$$\begin{aligned}
(V+B)^{-1}(\zeta, E_1) &= \begin{pmatrix} (c+v_1^2)^{-1} & (c+v_1^2)^{-1} v_1 \\ v_1 (c+v_1^2)^{-1} & v_1 (c+v_1^2)^{-1} v_1 - 1 \end{pmatrix},\\
(V+B)^{-1}(\zeta, E_2) &= \begin{pmatrix} (c-v_2^2)^{-1} & -(c-v_2^2)^{-1} v_2 \\ -v_2 (c-v_2^2)^{-1} & v_2 (c-v_2^2)^{-1} v_2 + 1 \end{pmatrix}.
\end{aligned} \tag{40}$$

By similar computations, we obtain

$$\det(v+1)(\zeta, E^j) = 1 - e^{2i\theta_j(\tau^j)\zeta}, \qquad 1 \le j \le 3, \tag{41}$$

$$(V+1)^{-1}(\zeta, E^j) = [\det(v+1)(\zeta, E^j)]^{-1} \begin{pmatrix} 1 & -e^{i\theta_j(\tau^j)\zeta} \\ -e^{i\theta_j(\tau^j)\zeta} & 1 \end{pmatrix},$$

$$\det(V+1)(\zeta, E_j) = \det[1 - v_j^2(\zeta)] = (1 - e^{2i\theta_1(\tau_j)\zeta})(1 - e^{2i\theta_2(\tau_j)\zeta})(1 - e^{2i\theta_3(\tau_j)\zeta}), \tag{42}$$

$$(V+1)^{-1}(\zeta, E_j) = \begin{pmatrix} (1-v_j^2)^{-1} & -(1-v_j^2)^{-1}v_j \\ -v_j(1-v_j^2)^{-1} & v_j(1-v_j^2)^{-1}v_j + 1 \end{pmatrix}.$$

By virtue of (38), we have $\det(V+B)(\zeta, E^j) \ne 0$ and $\operatorname{Re}\zeta = 0$. Therefore, it suffices to consider the behavior of the function $(V+B)(\zeta, E_j)(\zeta)$ on the imaginary axis.

Lemma 3. (a) *The function* $\det(V+B)(\zeta, E_j)$, $j = 1, 2$, *has no zeros other than* $\zeta = 0$ *on the imaginary axis and has a zero of order* j *at the point* $\zeta = 0$.

(b) *The matrix function* $(V+B)^{-1}(\zeta, E_j)$ *has a first-order zero at the point* $\zeta = 0$ *for both values of* j.

(c) *One has*

$$\frac{1}{2\pi i}\left[\ln\frac{\det(V+B)(\zeta, E_j)}{\det(V+1)(\zeta, E_j)}\right]\Bigg|_{-\alpha-i\infty}^{-\alpha+i\infty} = \begin{cases} 1 & \text{for} \quad j = 1 \\ 1/2 & \text{for} \quad j = 2, \end{cases} \tag{43}$$

$$\frac{1}{2\pi i}\left[\ln\frac{\det(V+B)(\zeta, E^j)}{\det(V+1)(\zeta, E^j)}\right]\Bigg|_{-\alpha-i\infty}^{-\alpha+i\infty} = \frac{1}{2}. \tag{44}$$

Proof. (a) By (39), it suffices to prove the assertion of the lemma for the matrices $(c+v_1^2)$ and $(c-v_2^2)$. First, we prove the lemma for the first matrix. By setting $t_k = e^{2i\theta_k\zeta}$, $1 \le k \le 3$, and by taking into account (39), we represent the matrix $(c+v_1^2)$ in the form

$$c + \operatorname{diag}(t_1, t_2, t_3) = \frac{1}{3}\begin{pmatrix} 1+3t_1 & -2 & -2 \\ -2 & 1+3t_2 & -2 \\ -2 & -2 & 1+3t_3 \end{pmatrix}.$$

By Lemma 2 (a), the function $\det(V+B)(\zeta, E_1)$ has no zeros other than $\zeta = 0$ on the imaginary axis and has a first-order zero at the point $\zeta = 0$.

By (39), the matrix $(c - v_2^2)$ has the form

$$c - \operatorname{diag}(t_1, t_2, t_3) = \frac{1}{3}\begin{pmatrix} 1-3t_1 & -2 & -2 \\ -2 & 1-3t_2 & -2 \\ -2 & -2 & 1-3t_3 \end{pmatrix},$$

where $t_k = e^{2i\theta_k\zeta}$, $1 \le k \le 3$. By Lemma 1 (a), the function $\det(V+B)(\zeta, E_2)$ has no zeros other than $\zeta = 0$ on the imaginary axis and has a second-order zero at the point $\zeta = 0$. The proof of assertion (a) is complete.

(b) It follows from (40) that the order of the pole of $(V+B)^{-1}(\zeta, E_j)$ coincides with that of the pole of $[c - (-1)^j v_j^2]^{-1}$. The assertion follows from Lemma 2 (b) for $j = 1$ and from Lemma 1 (b) for $j = 2$.

(c) Let us prove the first relation. The determinants in the numerator are given by formulas (39), and the determinants in the denominator can be computed directly by virtue of (42):

$$\det(V+1)(\zeta, E_j) = (1-t_1)(1-t_2)(1-t_3), \qquad t_k = e^{2i\theta_k(\tau_j)\zeta}, \qquad k = 1, 2, 3.$$

By setting

$$g(t) = (1-t_1)(1-t_2)(1-t_3) \quad \text{for} \quad t = (t_1, t_2, t_3),$$

we obtain

$$x_1(\zeta) = \frac{\det(V+B)(\zeta, E_1)}{\det(V+1)(\zeta, E_1)} = -\frac{f(t)}{g(t)}, \qquad x_2(\zeta) = \frac{\det(V+B)(\zeta, E_2)}{\det(V+1)(\zeta, E_2)} = \frac{f(-t)}{g(t)}.$$

These functions satisfy the relation

$$\frac{1}{2\pi i} \ln x_j(\zeta)\Big|_{-\alpha-i\infty}^{-\alpha+i\infty} = -\frac{1}{2\pi i} \ln x_j(\zeta)\Big|_{\alpha-i\infty}^{\alpha+i\infty}.$$

By the argument principle for analytic functions, we have

$$\frac{1}{2\pi i} \ln x_j(\zeta)\Big|_{\alpha-i\infty}^{\alpha+i\infty} - \frac{1}{2\pi i} \ln x_j(\zeta)\Big|_{-\alpha-i\infty}^{-\alpha+i\infty} = m_j,$$

where m_j stands for the difference between the number of zeros and the number of poles of the function $x_j(\zeta)$ with regard of multiplicities in the strip $-\alpha < \operatorname{Re}\zeta < \alpha$.

Obviously, the function $\det(V+1)(\zeta, E_j)$ has a unique zero of multiplicity 3 at the point $\zeta = 0$ in this strip. It follows from Lemma 3 (a) that $m_1 = -2$ and $m_2 = -1$. As a result, we obtain relation (43). By (38) and (41), for the second relation, we have

$$\frac{1}{2\pi i}\left[\ln \frac{\det(V+B)(\zeta, E^j)}{\det(V+1)(\zeta, E^j)}\right]\Bigg|_{-\alpha-i\infty}^{-\alpha+i\infty} = \frac{1}{2\pi i}\left[\ln \frac{1+e^{2i\theta_j(\tau^j)\zeta}}{1-e^{2i\theta_j(\tau^j)\zeta}}\right]\Bigg|_{-\alpha-i\infty}^{-\alpha+i\infty} = -\frac{1}{2\pi i} \ln \frac{1+z}{1-z}\Big|_0^\infty,$$

where the increment in the last term is taken along the ray $\arg z = -\varepsilon$ with small $\varepsilon > 0$. One can readily see that this increment is equal to $-\pi i$; consequently, relation (44) holds. The proof of the lemma is thereby complete.

Theorem 6. *Let all arcs Γ_j belong to the class $C^{1,\mu+0}$. Then Problem* DN *is uniquely solvable in the class C^μ_{-0}. If the right-hand side f belongs to the class C^μ_{+0}, then its solution ϕ belongs to $C^\mu_{(+0)}$.*

Proof. Recall that the index $\varkappa_0$ is computed by formula (10). By substituting the values of both terms found in Lemma 3 (c), we obtain $\varkappa_0 = -1 - 1/2 - 3/2 = -3$. It follows from Theorem 1 that the index of Problem DN is zero.

The analysis of the solvability of Problem DN in the class $C^\mu_{(+0)}$ can be reduced to the verification of the assumptions of Theorem 2. It follows from the second assertion of Lemma 3 that the matrix function $\zeta(V+B)^{-1}(\zeta, E_j)$ has no poles on the imaginary axis. Therefore, the assumptions of Theorem 2 hold for the considered problem.

Let us analyze the uniqueness of the solution of Problem DN. Let $\phi^p \in C^\mu_{-0}$ be a solution of the homogeneous problem. Then, by virtue of Theorem 2 and the last assertion of Theorem 3, the functions ϕ^p have property (12) with respect to the class C^μ_{+0}. In other words,

$$\tilde\phi^p(z) = (z-\tau_1)(z-\tau_2)(z-\tau^p)(\phi^p)'(z) \in C^\mu_{+0}(D^p), \qquad p = 1, 2, 3.$$

Therefore, the function $|(\phi^p)'(z)|$ is square integrable in the domain D^p; by analogy with the proof of Theorem 5, one can show that each of the functions $u^p = \operatorname{Re}\phi^p$ is constant in the domain D^p. By virtue of the homogeneous condition on the arc Γ^p_0, all u^p are actually zero.

Therefore, the numbers k and k' in Theorem 1 are zero for the considered problem, and hence this problem is uniquely solvable in the class C^μ_{-0}. By Theorem 2, this is also true for the class $C^\mu_{(+0)}$.

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