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Special Functions as Solutions to the Euler–Poisson–Darboux Equation with a Fractional Power of the Bessel Operator

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Abstract: In this paper, we consider fractional ordinary differential equations and the fractional Euler–Poisson–Darboux equation with fractional derivatives in the form of a power of the Bessel differential operator. Using the technique of the Meijer integral transform and its modification, fundamental solutions to these equations are derived in terms of the Fox–Wright function, the Fox H-function, and their particular cases. We also provide some explicit formulas for the solutions to the corresponding initial-value problems in terms of the generalized convolutions introduced in this paper.

Keywords: Fox–Wright function; H-function; fractional powers of the Bessel operator; fractional Euler–Poisson–Darboux equation; fractional ODE; Meijer integral transform

MSC: 26A33; 33E20; 34B27; 35C10

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1. Introduction

The role of special functions in applied mathematics and especially in differential equations and mathematical physics can hardly be overestimated. In classical theory, mainly the hypergeometric functions ${}_pF_q$, the Meijer G-function, and their numerous particular cases have been employed until recently ([1,2]). The situation changed dramatically with the development of Fractional Calculus or FC (theory of the integrals and derivatives of the non-integer order) and its applications ([3–7]). It turned out that the solutions to fractional differential equations cannot, in general, be expressed in terms of hypergeometric functions or even in terms of the Meijer G-function, and thus, more general types of special functions came into operation.

In contrast to conventional ODEs and PDEs, different classes of special functions proved to be useful for fractional ODEs and for fractional PDEs, namely Mittag–Leffler-type functions for fractional ODEs ([8–10]) and Wright-type functions for fractional PDEs ([11–13]). Even though the properties and applications of the conventional Mittag–Leffler function and the Wright function are very different, they both are particular cases of the Fox–Wright function, which is a generalization of the hypergeometric function. In more complicated cases, an even more general function, the Fox H-function [4,14], appeared to be useful while dealing with fractional differential equations. The Fox H-function is probably one of the most general special functions that are nowadays in use in mathematics and its applications. It can be interpreted as a generalization of the Meijer G-function.

In this paper, we discuss some new applications of the Fox–Wright function ${}_p\Psi_q(z)$ and the Fox H-function $\mathbf{H}^{m,n}_{p,q}(z)$ in the theory of fractional differential equations with the fractional Bessel operator, which is defined as a power of the conventional Bessel differential operator $(B_\gamma)_t = \frac{\partial^2}{\partial t^2} + \frac{\gamma}{t}\frac{\partial}{\partial t}$. The methods and techniques that we employ in this paper are close to those suggested in [10], where a general schema for the development of the operational calculi for fractional derivatives was presented and applied for the derivation of analytical solutions to some classes of fractional ODEs.

The rest of the paper is organized as follows. In Section 2, we provide the definitions of the special functions that are employed for the derivation of our main results. In Section 3, we discuss some integral transforms with the special functions in the kernels. They are our main tools for solving fractional differential equations with the fractional powers of the Bessel operator. Special attention is given to a modification of the Meijer integral transform, which acts on the fractional Bessel operator in the same way as the Laplace transform does with respect to the derivatives of the integer and fractional order. Moreover, we present a convenient formula for recovering a function from its known Meijer transform (inverse Meijer integral transform). In Section 4, some explicit formulas for the fractional powers of the Bessel operator acting on the functions defined on the positive real semi-axes are presented in terms of the Gauss hypergeometric function. Analytical treatment of fractional ODEs with the fractional Bessel derivative is the subject of Section 5. In particular, in Section 5, a fundamental system of solutions to fractional ODEs with the fractional Bessel operator is derived in terms of the Fox–Wright function. It turns out that for these equations, the Fox–Wright function plays the same role as the Mittag–Leffler function for fractional ODEs with conventional fractional derivatives (the Riemann–Liouville or the Djerbashian–Caputo derivatives). In Section 6, we derive an explicit solution formula for the Cauchy problem for a one-dimensional fractional Euler–Poisson–Darboux equation that contains a fractional power of the Bessel operator with respect to the time variable and the conventional Bessel operator with respect to the spatial variable. The main tool for our derivations in Section 6 is an explicit formula for the Hankel transform of the Fox–Wright function that is also presented in this section.

2. Special Functions Connected to the Fractional Bessel Operator

In this section, we remind the readers of the definitions and basic properties of the special functions that play the main role in the derivation of our main results in the subsequent sections.

The modified Bessel functions of the first and the second kind, I_α and K_α, respectively, are defined as follows ([15]):

$$I_\nu(x) = i^{-\nu} J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}, \tag{1}$$

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\alpha(x)}{\sin(\nu\pi)}, \tag{2}$$

where the index ν in (2) is a non-integer number. For integer values of ν, K_ν is interpreted as the limit value of the expression on the right-hand side of (2). The formula $K_\nu(x) = K_{-\nu}(x)$ easily follows from the definition of K_ν. For the small values of the argument x ($0 < |x| \ll \sqrt{\nu+1}$), the asymptotic behavior of K_ν is well known:

$$K_\nu(x) \sim \begin{cases} -\ln\left(\frac{x}{2}\right) - \vartheta & \text{if} \quad \nu = 0, \\ \frac{\Gamma(\nu)}{2^{1-\nu}} x^{-\nu} & \text{if} \quad \nu > 0, \end{cases} \tag{3}$$

where

$$\vartheta = \lim_{n\to\infty}\left(-\ln n + \sum_{k=1}^{n} \frac{1}{k}\right) = \int_1^\infty \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor}\right) dx$$

is the Euler–Mascheroni constant [16].

The asymptotic behavior of K_ν for the large values of the argument z (in the general complex) is given by the formula

$$K_\nu(z) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-z}}{\sqrt{z}}\left(1 + O\left(\frac{1}{z}\right)\right), \qquad |z| \to \infty. \tag{4}$$

Setting $\nu = \frac{1}{2}$ in Formula (2), we get an important particular case of the modified Bessel functions of the second kind:

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}. \tag{5}$$

The normalized Bessel function of the first kind, j_ν, is defined by the formula

$$j_\nu(x) = \frac{2^\nu \Gamma(\nu+1)}{x^\nu}\, J_\nu(x), \tag{6}$$

where J_ν is Bessel function of the first kind [17].

The functions introduced above are particular cases of the hypergeometric Gauss function that is defined as the following series ([16], p. 373, Formula 15.3.1)

$${}_2F_1(a,b;c;z) = F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k}\frac{z^k}{k!} \tag{7}$$

in the case this series converges, i.e., under the condition $|z| < 1$. For $|z| \geq 1$, ${}_2F_1$ is interpreted as an analytic continuation of this series. In (7), both the parameters a, b, c and the variable z are complex numbers ($c \neq 0, -1, -2, \ldots$). By $(a)_k$, the Pohgammer symbol is denoted ($(z)_n = z(z+1)...(z+n-1)$, $n = 1, 2, ...$, $(z)_0 \equiv 1$).

In FC, the Mittag–Leffler function $E_{\alpha,\beta}$ plays a very important role. It is an entire function defined by the following convergent series

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \;\; z \in \mathbb{C},\; \alpha,\beta \in \mathbb{C},\; \mathrm{Re}\,\alpha > 0,\; \mathrm{Re}\,\beta > 0. \tag{8}$$

Another important FC function, the Fox–Wright function ${}_p\Psi_q$, is defined by the series (see [18,19])

$${}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{array}{c} (a_l,\alpha_l)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array} \middle| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p}\Gamma(a_l+\alpha_l k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}\frac{z^k}{k!},\; z \in \mathbb{C},\; a_l, b_j \in \mathbb{C},\; \alpha_l, \beta_j \in \mathbb{R},\; l = 1, ..., p,\; j = 1, ..., q \tag{9}$$

in the case this series converges. When the condition

$$\sum_{j=1}^{q} \beta_j - \sum_{l=1}^{p} \alpha_l > -1 \tag{10}$$

is satisfied, the series at the right-hand side of (9) is convergent for any $z \in \mathbb{C}$. In the case

$$\sum_{j=1}^{q} \beta_j - \sum_{l=1}^{p} \alpha_l = -1,$$

the series in (9) is absolutely convergent for $|z| < \delta$ and for $|z| = \delta$ and $\mathrm{Re}\,\mu > \frac{1}{2}$, where

$$\delta = \prod_{l=1}^{p} |\alpha_l|^{-\alpha_l} \prod_{j=1}^{q} |\beta_j|^{\beta_j},$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{l=1}^{p} a_l + \frac{p-q}{2}.$$

For the fractional powers of the Bessel operator, the Fox–Wright function plays the same role as the Mittag–Leffler function for the conventional fractional derivatives.

Moreover, the Mittag–Leffler function is a particular case of the Fox–Wright function (9):

$$E_{\alpha,\beta}(z) = {}_1\Psi_1\left[\begin{array}{c}(1,1)\\(\beta,\alpha)\end{array}\middle| z\right]. \tag{11}$$

Now we introduce the Fox H-function. Let m, n, p, q be the integers such that $0 \le m \le q$ and $0 \le n \le p$. For the parameters $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ $(i = 1, 2, ..., p; j = 1, 2, ..., q)$, the H-function $H^{m,n}_{p,q}$ is defined via a Mellin–Barnes-type integral ([20])

$$\mathbf{H}^{m,n}_{p,q}(z) = \mathbf{H}^{m,n}_{p,q}\left[z\middle| \begin{array}{c}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{H}^{m,n}_{p,q}(s) z^{-s} ds, \tag{12}$$

where

$$\mathcal{H}^{m,n}_{p,q}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}.$$

Let

$$a^* = \sum_{i=1}^{n}\alpha_i - \sum_{i=n+1}^{p}\alpha_i + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j,$$

$$\Delta = \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i,$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}.$$

The integral at the right-hand side of (12) is well defined in particular under the following conditions: $\Delta > 0, z \neq 0$. $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop that starts at the point $-\infty + i\varphi_1$, terminates at the point $-\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$, encircles all poles of the Gamma functions $\Gamma(b_j + \beta_j s)$, $j = 1, \ldots, m$, and is located in a bounded horizontal strip. Other existence conditions for the Mellin–Barnes-type integral from the definition of $\mathbf{H}^{m,n}_{p,q}$ are listed in [20] (p. 4, Theorem 1.1).

3. Integral Transforms, the Poisson Operator, and a Generalized Convolution

In this section, the Meijer integral transform and its modification, as well as their connection to the integral Laplace transform via the transmutation Poisson operator, are presented. We also provide definitions and properties of the Hankel integral transform and of a generalized convolution that is employed for the equations containing the Bessel differential operator. For the known properties of the Fourier and the Laplace transforms, we refer to [21].

To be able to use the operational method presented in [10], an integral transform suitable for dealing with the fractional Bessel differential operator is needed. It turns out that this integral transform is the one with the modified Bessel function of the second kind (2) in the kernel.

For the functions $f:\mathbb{R}_+\to\mathbb{C}$, an integral transform of the Mellin convolution type with the modified Bessel function K_ν, $\nu\geq 0$ in the kernel is called the Meijer integral transform. It is defined by the formula ([21], p. 93)

$$K_\nu[f](\xi)=\int\limits_0^\infty \sqrt{x\xi}K_\nu(x\xi)\,f(x)\,dx. \tag{13}$$

In (13), the condition $\nu\geq 0$ can be assumed without any loss of generality because of the relation $K_\nu(x)=K_{-\nu}(x)$.

For our aims, it is convenient to use the following modification of the Meijer integral transform:

$$\mathcal{K}_\gamma[f](\xi)=\int\limits_0^\infty x^{\frac{\gamma+1}{2}}K_{\frac{\gamma-1}{2}}(x\xi)\,f(x)\,dx. \tag{14}$$

In particular, for $\gamma=0$ and $\gamma=2$, Formula (5) leads to the following well-known particular cases of the modified Meijer integral transform:

$$\mathcal{K}_0[f](\xi)=\sqrt{\frac{\pi}{2\xi}}\int\limits_0^\infty e^{-x\xi}\,f(x)\,dx=\sqrt{\frac{\pi}{2\xi}}\,\mathcal{L}[f(x)](\xi),$$

$$\mathcal{K}_2[f](\xi)=\sqrt{\frac{\pi}{2\xi}}\int\limits_0^\infty xe^{-x\xi}\,f(x)\,dx=\sqrt{\frac{\pi}{2\xi}}\,\mathcal{L}[xf(x)](\xi),$$

where $\mathcal{L}[f(x)](\xi)$ is the conventional Laplace integral transform.

Let $f\in L_1^{loc}(\mathbb{R}_+)$ and $f(t)=o\left(t^{\beta-\frac{\gamma}{2}}\right)$ as $t\to+0$, where $\beta>\frac{\gamma}{2}-2$ if $\gamma>1$ and $\beta>-1$ if $\gamma=1$. Furthermore, let $f(t)=0(e^{at})$ as $t\to+\infty$. Then, the Meijer integral transform of f exists a.e. for $\operatorname{Re}\xi>a$ ([21], p. 94). The class of the functions that satisfy the conditions mentioned above will be denoted by $\mathbf{K}_\gamma$.

If the condition $0<\gamma<2$ holds true, the function $F(\xi)$ is analytic on the half-plane $H_a=\{p\in\mathbb{C}:\operatorname{Re}p\geq a\}$, $a\leq 0$, and $s^{\frac{\gamma}{2}-1}F(\xi)\to 0$, $|\xi|\to+\infty$, uniformly with respect to $\arg s$, then the inverse Meijer integral transform

$$\mathcal{K}_\gamma^{-1}[\widehat{f}](x)=f(x)=\frac{1}{\pi i}\int\limits_{c-i\infty}^{c+i\infty}\widehat{f}(\xi)i_{\frac{\gamma-1}{2}}(x\xi)\xi^\gamma d\xi \tag{15}$$

is well defined for any $c\in\mathbb{R}$, $c>a$ ([21], p. 94).

The inversion formula (15) is difficult to apply and not convenient for our aims. Moreover, it contains a restrictive condition $0<\gamma<2$. Thus, we introduce another inversion formula in terms of the transmutation Poisson operator that is defined as follows ([22]):

$$\mathcal{P}_x^\gamma f(x)=(\mathcal{P}_t^\gamma f(t))(x)=\frac{2C(\gamma)}{x^{\gamma-1}}\int\limits_0^x\left(x^2-t^2\right)^{\frac{\gamma}{2}-1}f(t)\,dt,\quad C(\gamma)=\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)}. \tag{16}$$

For any summable function $H(x)$, the left inverse transform for the Poisson operator (16) with $\gamma>0$ is given by the formula ([7])

$$(\mathcal{P}_x^\gamma)^{-1}H(x)=\frac{2\sqrt{\pi}x}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(n-\frac{\gamma}{2}\right)}\left(\frac{d}{2xdx}\right)^n\int\limits_0^x H(z)(x^2-z^2)^{n-\frac{\gamma}{2}-1}z^\gamma dz, \tag{17}$$

where $n=\left[\frac{\gamma}{2}\right]+1$.

We start with the known representation of the modified Bessel function of the second kind ([15], p. 190)

$$K_\nu(x\xi) = \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{\xi}{2x}\right)^\nu \int\limits_x^\infty e^{-\xi z}(z^2-x^2)^{\nu-\frac{1}{2}}dz,$$

and arrive at the formula

$$x^{\frac{\gamma+1}{2}} K_{\frac{\gamma-1}{2}}(x\xi) = \frac{\sqrt{\pi}\, x\xi^{\frac{\gamma-1}{2}}}{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma}{2}\right)} \int\limits_x^\infty e^{-\xi z}(z^2-x^2)^{\frac{\gamma}{2}-1}dz$$

for the kernel of the modified Meijer integral transform. Substituting this formula into (14), we obtain the following convenient factorization of the modified Meijer integral transform:

$$\mathcal{K}_\gamma[f](\xi) = \frac{\pi\,\xi^{\frac{\gamma-1}{2}}}{2^{\frac{\gamma+1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}(\mathcal{L}z^{\gamma-1}\mathcal{P}_z^\gamma z f(z))(\xi), \tag{18}$$

where $\mathcal{L}$ stands for the Laplace integral transform. Representation (18) will be employed in the subsequent section.

Another integral transform that we need for the derivation of our results is the Hankel integral transform. For a function $f \in L_1^\gamma(\mathbb{R}_+^1)$, it is defined as follows ([17]):

$$F_\gamma[f](\xi) = F_\gamma[f(x)](\xi) = f(\xi) = \int\limits_0^\infty f(x)\, j_{\frac{\gamma-1}{2}}(x\xi)x^\gamma dx, \tag{19}$$

where $\gamma > 0$ and j_ν is the normalized Bessel function of the first kind (6).

In the rest of this section, we introduce a generalized translation operator and a generalized convolution that we employ for analytical treatment of the fractional differential equations with the fractional Bessel derivative.

Let $f = f(x)$, $x \in \mathbb{R}$, $\gamma > 0$. The generalized translation is defined by the formula ([22])

$$({}^\gamma T_x^y f)(x) = {}^\gamma T_x^y f = C(\gamma)\int\limits_0^\pi f\left(\sqrt{x^2+y^2-2xy\cos\varphi}\right)\sin^{\gamma-1}\varphi\, d\varphi, \tag{20}$$

where $C(\gamma) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)}$. For $\gamma = 0$, the generalized translation ${}^\gamma T_x^y$ is reduced to the central difference operator:

$${}^0T_x^y = T_x^y f(x) = \frac{f(x+y)-f(x-y)}{2}.$$

The generalized convolution generated by the generalized translation ${}^\gamma T_x^y$ is defined as follows:

$$(f*g)_\gamma(x) = \int\limits_0^\infty f(y)\, {}^\gamma T_x^y g(x) y^\gamma\, dy. \tag{21}$$

It turns out that the generalized convolution (21) is a convolution for Hankel integral transform and the convolution property

$$F_\gamma[(f*g)_\gamma(x)](\xi) = F_\gamma[f(x)](\xi)F_\gamma[g(x)](\xi) \tag{22}$$

holds true. For the proofs of these and other results regarding the generalized translation operators, we refer to [22,23].

4. Fractional Bessel Integral and Derivative

In this section, we shortly present the definitions and the main properties of the fractional Bessel integral and derivative.

A representation of the fractional Bessel integral that was interpreted as a negative power of the Bessel operator $(B_\gamma)_t = \frac{\partial^2}{\partial t^2} + \frac{\gamma}{t}\frac{\partial}{\partial t}$ was first presented in [24] in terms of an integral operator with the Gauss hypergeometric function in the kernel. In [25], the fractional Bessel derivative in the form of the positive powers of the Bessel and the hyper-Bessel differential operators were suggested and studied in detail. The operators we deal with in this paper follow the constructions presented in [25]. In [4,26,27], a more general class of the hyper-Bessel differential operators along with the associated Obrechkoff integral transform was introduced, and the analytical solutions to some linear differential equations with the hyper-Bessel differential operators were deduced. In what follows, we introduce the fractional Bessel integrals and derivatives and provide some of their properties we need in the further discussions.

Let $\alpha > 0$ and $\gamma > 0$. The left-sided Bessel fractional integral $B^{-\alpha}_{\gamma,0+}$ of a function $f \in L[0,\infty)$ defined on the half-axis is given by the formula

$$(B^{-\alpha}_{\gamma,0+}f)(x) = (IB^{\alpha}_{\gamma,0+}f)(x) =$$

$$= \frac{1}{\Gamma(2\alpha)}\int_0^x \left(\frac{y}{x}\right)^\gamma \left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right) f(y)dy. \tag{23}$$

For the properties of the operator (23), we refer to [28].

Let $n = [\alpha]+1$, $f \in L[0,\infty)$, $IB^{n-\alpha}_{\gamma,b-}f, IB^{n-\alpha}_{\gamma,b-}f \in C^{2n}(0,\infty)$. The left-sided Bessel fractional derivative is defined as a composition of the left-sided Bessel fractional integral and the Bessel differential operator as follows:

$$(\mathcal{B}^{\alpha}_{\gamma,0+}f)(x) = (IB^{n-\alpha}_{\gamma,0+}B^n_\gamma f)(x). \tag{24}$$

In [25], the spaces of functions suitable for analysis of the Bessel fractional operators $B^{\alpha}_{\gamma,0+}$, $\alpha \in \mathbb{R}$ were introduced:

$$F_p = \left\{\varphi \in C^\infty(0,\infty) : x^k\frac{d^k\varphi}{dx^k} \in L^p(0,\infty) \text{ for } k = 0,1,2,\ldots\right\}, \qquad 1 \le p < \infty,$$

$$F_\infty = \left\{\varphi \in C^\infty(0,\infty) : x^k\frac{d^k\varphi}{dx^k} \to 0 \text{ as } x \to 0+ \text{ and as } x \to \infty \text{ for } k = 0,1,2,\ldots\right\}$$

and

$$F_{p,\mu} = \{\varphi : x^{-\mu}\varphi(x) \in F_p\}, \qquad 1 \le p \le \infty, \qquad \mu \in \mathbb{C}.$$

For our aims, we need the following result that is a special case of the results presented in [25].

Theorem 1. *Let $\alpha \in \mathbb{R}$. For all p, μ and $\gamma > 0$ such that $\mu \ne \frac{1}{p}-2m$, $\gamma \ne \frac{1}{p}-\mu-2m+1$, $m=1,2\ldots$, the operator $B^{\alpha}_{\gamma,0+}$ is a continuous linear mapping from F_p, μ into $F_{p,\mu-2\alpha}$. If the conditions $2\alpha \ne \mu - \frac{1}{p} + 2m$ and $\gamma - 2\alpha \ne \frac{1}{p} - \mu - 2m + 1$, $m = 1,2\ldots$ also hold true, then $B^{\alpha}_{\gamma,0+}$ is a homeomorphism from F_p, μ onto $F_{p,\mu-2\alpha}$ with the inverse operator $B^{-\alpha}_{\gamma,0+}$.*

Even though the operators (23) and (24) were intensively studied in the literature, until recently, no convenient methods for solving the fractional differential equations with the fractional powers of the Bessel operator were suggested. The situation changed with the publication of the paper [29], where the modified Meijer integral transform (14) was employed for analytical treatment of some fractional differential equations with the fractional Bessel derivatives. In the rest of this section, we present some results from [29] that are needed for our analysis in the next section. To shorten the formulations of the results, each time the Meijer integral transform is applied to a function, we suppose that this function is from the space $\mathbf{K}_\gamma$ introduced in the previous section.

Theorem 2. *Let $\alpha > 0$, $f \in \mathbf{K}_\gamma$. The modified Meijer integral transform (14) of the Bessel fractional integral $B^{-\alpha}_{\gamma,0+}$ is given by the formula*

$$\mathcal{K}_\gamma[(IB^{\alpha}_{\gamma,0+}f)(x)](\xi) = \xi^{-2\alpha}\mathcal{K}_\gamma f(\xi). \tag{25}$$

Theorem 3. *Let $n \in \mathbb{N}$, $f \in \mathbf{K}_\gamma$, $\frac{d}{dx}[B^{n-k}_\gamma f(x)]$ be bounded, the modified Meijer integral transform of $B^n_\gamma f$ exist, and $\gamma \neq 1$. Then, the formula*

$$\mathcal{K}_\gamma[B^n_\gamma f](\xi) = \xi^{2n}\mathcal{K}_\gamma[f](\xi) - \left(\frac{2}{\xi}\right)^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)\sum_{k=1}^{n}\xi^{2k-2}B^{n-k}_\gamma f(0+) \tag{26}$$

holds true. Moreover, when $\frac{d}{dx}[B^{n-k}_\gamma f(x)] \sim x^\beta$, $\beta > 0$ as $x \to 0+$, then Formula (26) remains valid for $\gamma = 1$.

To formulate the next result, we introduce the space $C^m_{ev} = C^m_{ev}(\mathbb{R})$ that consists of all functions from $C^m(\mathbb{R})$ such that $\left.\frac{\partial^{2k+1}f}{\partial x_i^{2k+1}}\right|_{x=0} = 0$ for all non-negative integers $k \leq \frac{m-1}{2}$ ([17], p. 21).

Remark 1. *If $\gamma = 0$ and $f \in C^{2n}_{ev}$, we get the formulas*

$$\mathcal{K}_0[f(x)](\xi) = \sqrt{\frac{\pi}{2\xi}}\int_0^\infty e^{-x\xi}f(x)\,dx = \sqrt{\frac{\pi}{2\xi}}\mathcal{L}[f(x)](\xi)$$

and

$$\mathcal{L}\left[\frac{d^{2n}}{dx^{2n}}f(x)\right](\xi) = \xi^{2n}\mathcal{L}[f](\xi) - \sum_{k=0}^{2n-1}\xi^k f^{(2n-k-1)}(0+) =$$

$$= \xi^{2n}\mathcal{L}[f](\xi) - f^{(2n-1)}(0+) - sf^{(2n-2)}(0+) - s^2f^{(2n-3)}(0+) - \ldots$$

$$\ldots - s^3f^{(2n-4)}(0+) - s^4f^{(2n-5)}(0+) - s^5f^{(2n-6)}(0+) - \ldots - \xi^{2n-2}f'(0+) - \xi^{2n-1}f(0+).$$

Because $f \in C^{2n}_{ev}$, the conditions $f'(0+) = f'''(0+) = \ldots = f^{(2n-5)}(0+) = f^{(2n-3)}(0+) = f^{(2n-1)}(0+) = 0$ are fulfilled, and we arrive at the formula

$$\mathcal{L}\left[\frac{d^{2n}}{dx^{2n}}f(x)\right](\xi) = \xi^{2n}\mathcal{L}[f](\xi) - sf^{(2n-2)}(0+) - \ldots$$

$$\ldots - s^3f^{(2n-4)}(0+) - s^5f^{(2n-6)}(0+) - \ldots - \xi^{2n-1}f(0+) =$$

$$= \xi^{2n}\mathcal{L}[f](\xi) - \sum_{k=1}^{n}s^{2k-1}f^{(2n-2k)}(0+) = \{m = n-k\} =$$

$$= \xi^{2n}\mathcal{L}[f](\xi) - \sum_{m=0}^{n-1}s^{2(n-m)-1}f^{(2m)}(0+).$$

Thus, the relation

$$\mathcal{L}\left[\frac{d^{2n}}{dx^{2n}}f(x)\right](\xi) = \xi^{2n}\mathcal{L}[f](\xi) - \sum_{m=0}^{n-1}s^{2(n-m)-1}f^{(2m)}(0+)$$

holds true.

From the other side, the formula

$$\sqrt{\frac{2\xi}{\pi}}\mathcal{K}_0[B_0^n f](\xi) = \sqrt{\frac{2\xi}{\pi}}\left(\xi^{2n}\mathcal{K}_0[f](\xi) - \sqrt{\frac{\pi\xi}{2}}\sum_{m=0}^{n-1}\xi^{2(n-m)-2}B_0^m f(0+)\right) =$$

$$= \sqrt{\frac{2\xi}{\pi}}\left(\xi^{2n}\sqrt{\frac{\pi}{2\xi}}\,\mathcal{L}[f(x)](\xi) - \sqrt{\frac{\pi\xi}{2}}\sum_{m=0}^{n-1}\xi^{2(n-m)-2}B_0^m f(0+)\right) =$$

$$= \xi^{2n}\,\mathcal{L}[f(x)](\xi) - \sum_{m=0}^{n-1}\xi^{2(n-m)-1}f^{(2m)}(0+).$$

confirms that the modified Meijer integral transform generalizes the Laplace integral transform.

Theorem 4. *Let $n = [\alpha] + 1$ for non-integer values of α and $n = \alpha$ for $\alpha \in \mathbb{N}$, $k \in \mathbb{N}$, $f \in \mathbf{K}_\gamma$, $\frac{d}{dx}[B_\gamma^k f(x)]$ be bounded, the modified Meijer integral transform of $\mathcal{B}^\alpha_{\gamma,0+}f$ exist, and $\gamma \neq 1$. Then, the formula*

$$\mathcal{K}_\gamma[\mathcal{B}^\alpha_{\gamma,0+}f](\xi) = \xi^{2\alpha}\mathcal{K}_\gamma[f](\xi) - \left(\frac{2}{\xi}\right)^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)\sum_{m=0}^{n-1}\xi^{2(\alpha-m)-2}B_\gamma^m f(0+) \tag{27}$$

holds true. Moreover, if $\frac{d}{dx}[B_\gamma^k f(x)] \sim x^\beta$, $\beta > 0$ as $x \to 0+$, then Formula (27) remains valid for $\gamma = 1$.

The results formulated in this section allow an analytical treatment of the fractional ODEs with the fractional Bessel derivative and of the fractional Euler–Poisson–Darboux equation. These results are presented in the next two sections.

5. Fractional ODEs with the Fractional Bessel Derivative

In this section, we consider a fractional ODE with the fractional Bessel derivative in the form

$$(\mathcal{B}^\alpha_{\gamma,0+}f)(x) = \lambda f(x), \qquad \alpha > 0, \qquad \lambda \in \mathbb{R}. \tag{28}$$

Let $n = [\alpha] + 1$ for non-integer values of α and $n = \alpha$ for $\alpha \in \mathbb{N}$. We look for solutions of Equation (28) that belong to the space C^{2n}_{ev}. Thus, the conditions

$$\left.\frac{d^{2m+1}f}{dx^{2m+1}}\right|_{x=0} = 0, \qquad m = 0, 1, ..., n-1 \tag{29}$$

have to be satisfied. Adding the conditions

$$B_\gamma^m f(0+) = a_m, \qquad m = 0, 1, ..., n-1, \tag{30}$$

we arrive at an initial-value problem for Equation (28) with a total of $2n$ initial conditions ((29) and (30)).

The main result of this section is formulated in the following theorem:

Theorem 5. *Let $n = [\alpha] + 1$ for the non-integer values of α and $n = \alpha$ for $\alpha \in \mathbb{N}$, $\lambda \in \mathbb{R}$.*

The fundamental system of solutions to Equation (28) is built by the following n functions

$$y_m = x^{2m}\,{}_2\Psi_2\left[\begin{matrix}(m+\frac{\gamma}{2}+1,\alpha),(1,1)\\(m+1,\alpha),(2m+\gamma+1,2\alpha)\end{matrix}\,\middle|\,\lambda x^{2\alpha}\right], \qquad m = 0, ..., n-1.$$

These functions satisfy the initial conditions in the form

$$\left.\frac{d^{2m+1}y_m}{dx^{2m+1}}\right|_{x=0}=0,\qquad B_\gamma^m y_m(0+)=1,\qquad m=0,1,...,n-1.$$

The unique solution to the initial-value problem (28)–(30) *is given by the formula*

$$f(x)=\frac{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}}\sum_{m=0}^{n-1}a_m x^{2m}\,{}_2\Psi_2\left[\begin{array}{c}(m+\frac{\gamma}{2}+1,\alpha),(1,1)\\(m+1,\alpha),(2m+\gamma+1,2\alpha)\end{array}\middle|\lambda x^{2\alpha}\right].$$

Proof. Applying the modified Meijer integral transform $\mathcal{K}_\gamma$ to Equation (28) and taking into account the initial conditions (29) and (30), we get the following chain of relations:

$$\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi)-\left(\frac{2}{\xi}\right)^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)\sum_{m=0}^{n-1}a_m\xi^{2(\alpha-m)-2}=\lambda\mathcal{K}_\gamma[f](\xi),$$

$$(\xi^{2\alpha}-\lambda)\mathcal{K}_\gamma[f](\xi)=\left(\frac{2}{\xi}\right)^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)\sum_{m=0}^{n-1}a_m\xi^{2(\alpha-m)-2}$$

$$\mathcal{K}_\gamma[f](\xi)=2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)\sum_{m=0}^{n-1}a_m\frac{\xi^{2(\alpha-m)-2+\frac{1-\gamma}{2}}}{\xi^{2\alpha}-\lambda}.$$

Taking into account the factorization formula (18), we first obtain the formula

$$\frac{\pi\,\xi^{\frac{\gamma-1}{2}}}{2^{\frac{\gamma+1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}(\mathcal{L}z^{\gamma-1}\mathcal{P}_z^\gamma zf(z))(\xi)=2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)\sum_{m=0}^{n-1}a_m\frac{\xi^{2(\alpha-m)-2+\frac{1-\gamma}{2}}}{\xi^{2\alpha}-\lambda}$$

and then arrive at the relation

$$(\mathcal{L}x^{\gamma-1}\mathcal{P}_x^\gamma xf(x))(\xi)=\frac{2^\gamma}{\pi}\Gamma^2\left(\frac{\gamma+1}{2}\right)\sum_{m=0}^{n-1}a_m\frac{\xi^{2(\alpha-m)-\gamma-1}}{\xi^{2\alpha}-\lambda}.$$

To get a solution formula, we first apply the inverse Laplace integral transform to both sides of the last formula. Taking into account the formula ([3], p. 50, Formula 1.10.9)

$$\mathcal{L}^{-1}\left[\frac{\xi^{2(\alpha-m)-1-\gamma}}{\xi^{2\alpha}-\lambda}\right](x)=x^{2m+\gamma}E_{2\alpha,2m+\gamma+1}(\lambda x^{2\alpha}),$$

we immediately obtain the representation

$$\mathcal{P}_x^\gamma xf(x)=\frac{2^\gamma}{\pi}\Gamma^2\left(\frac{\gamma+1}{2}\right)\sum_{m=0}^{n-1}a_m x^{2m+1}E_{2\alpha,2m+\gamma+1}(\lambda x^{2\alpha}).$$

Now we apply the inverse Poisson integral transform to the last equation and obtain the solution in the form

$$xf(x)=\frac{2^\gamma}{\pi}\Gamma^2\left(\frac{\gamma+1}{2}\right)\sum_{m=0}^{n-1}a_m(\mathcal{P}_x^\gamma)^{-1}x^{2m+1}E_{2\alpha,2m+\gamma+1}(\lambda x^{2\alpha}).$$

Finally, taking into account Representation (17), we can represent the solution as follows:

$$f(x)=\frac{2^{\gamma+1}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma(p-\frac{\gamma}{2})}\left(\frac{d}{2xdx}\right)^p\sum_{m=0}^{n-1}a_m\int_0^x z^{2m+\gamma+1}E_{2\alpha,2m+\gamma+1}(\lambda z^{2\alpha})(x^2-z^2)^{p-\frac{\gamma}{2}-1}dz,$$

where $p = \left[\frac{\gamma}{2}\right] + 1$.

Let us now derive an explicit formula for the integral

$$I = \int_0^x z^{2m+\gamma+1} E_{2\alpha,2m+\gamma+1}(\lambda z^{2\alpha})(x^2 - z^2)^{p-\frac{\gamma}{2}-1} dz.$$

First of all, we mention that this integral converges for all values of its parameters because of the known estimate $|E_{\alpha,\beta}(z)| \le C\, e^{\sigma |z|^{1/\alpha}}$ for the Mittag–Leffler function that is valid for any $\sigma > 1$ ([8], p. 71). By using Definition (8) of the Mittag–Leffler function, we have the formula

$$E_{2\alpha,\beta}(\lambda z^{2\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{2k\alpha}}{\Gamma(2\alpha k + \beta)}.$$

Substituting this representation into the integral I, we get the following chain of relations:

$$I = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(2\alpha k + 2m + \gamma + 1)} \int_0^x z^{2m+2k\alpha+\gamma+1}(x^2 - z^2)^{p-\frac{\gamma}{2}-1} dz =$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(2\alpha k + 2m + \gamma + 1)} \frac{\Gamma\left(p - \frac{\gamma}{2}\right)\Gamma\left(m + \frac{\gamma}{2} + 1 + k\alpha\right)}{2\Gamma(m + p + 1 + k\alpha)}\, x^{2\alpha k + 2m + 2p} =$$

$$= \frac{1}{2} x^{2m+2p} \Gamma\left(p - \frac{\gamma}{2}\right) \sum_{k=0}^{\infty} \frac{\Gamma(1+k)}{\Gamma(2\alpha k + 2m + \gamma + 1)} \frac{\Gamma\left(m + \frac{\gamma}{2} + 1 + k\alpha\right)}{\Gamma(m + p + 1 + k\alpha)} \frac{(\lambda x^{2\alpha})^k}{k!} =$$

$$= \frac{1}{2} x^{2m+2p} \Gamma\left(p - \frac{\gamma}{2}\right) {}_2\Psi_2\left[\begin{array}{c} \left(m + \frac{\gamma}{2} + 1, \alpha\right), (1,1) \\ (m + p + 1, \alpha), (2m + \gamma + 1, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right],$$

where ${}_p\Psi_q(z)$ stands for the Fox–Wright function (9).

Thus, we obtain the solution formula

$$f(x) = \frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}} \left(\frac{d}{2x dx}\right)^p \sum_{m=0}^{n-1} a_m x^{2m+2p}\, {}_2\Psi_2\left[\begin{array}{c} \left(m + \frac{\gamma}{2} + 1, \alpha\right), (1,1) \\ (m + p + 1, \alpha), (2m + \gamma + 1, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right].$$

Applying the formula

$$\left(\frac{d}{2x dx}\right)^p x^{2\mu+2p} = \frac{\Gamma(\mu + p + 1)}{\Gamma(\mu + 1)} x^{2\mu},$$

we can rewrite the solution in the final form:

$$f(x) = \frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}} \sum_{m=0}^{n-1} a_m x^{2m}\, {}_2\Psi_2\left[\begin{array}{c} \left(m + \frac{\gamma}{2} + 1, \alpha\right), (1,1) \\ (m + 1, \alpha), (2m + \gamma + 1, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right].$$

To determine the convergence conditions for the Fox–Wright functions from the last formula, we substitute their parameters values

$$q = p = 2, \qquad \alpha_1 = \alpha, \qquad \alpha_2 = 1, \qquad \beta_1 = \alpha, \qquad \beta_2 = 2\alpha$$

into Condition (10) that in our case takes the form

$$(\alpha + 2\alpha) - (\alpha + 1) = 2\alpha - 1 > -1 \Rightarrow \alpha > 0.$$

This means that the series for the Fox–Wright functions in the solution formula are convergent for any $z \in \mathbb{C}$.

Moreover, the series representation (9) of the Fox–Wright functions in the solution formula ensures that the functions

$$y_m = x^{2m}\,{}_2\Psi_2\left[\begin{array}{c}(m+\frac{\gamma}{2}+1,\alpha),(1,1)\\(m+1,\alpha),(2m+\gamma+1,2\alpha)\end{array}\bigg|\lambda x^{2\alpha}\right],\qquad m=0,\ldots,n-1$$

satisfy the initial conditions (29) and (30). $\square$

As an example, consider the initial-value problem (28)–(30) in the integer-order case $\alpha = 1$:

$$B_\gamma f(x) = \lambda f(x),\qquad \lambda\in\mathbb{R},\tag{31}$$

$$f(0) = 1.\tag{32}$$

According to Theorem 5, the unique solution to the initial-value problem (31) and (32) is given by the formula

$$f(x) = \frac{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}}\,{}_2\Psi_2\left[\begin{array}{c}(\frac{\gamma}{2}+1,1),(1,1)\\(1,1),(\gamma+1,2)\end{array}\bigg|\lambda x^2\right] = \frac{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}}\,{}_1\Psi_1\left[\begin{array}{c}(1+\frac{\gamma}{2},1)\\(\gamma+1,2)\end{array}\bigg|\lambda x^2\right] =$$

$$= \frac{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}}\sum_{m=0}^\infty\frac{\lambda^m\Gamma(1+\frac{\gamma}{2}+m)}{\Gamma(\gamma+1+2m)}\frac{x^{2m}}{m!}.$$

Let us represent this solution in terms of some simpler special functions. Using the Legendre duplication formula ([16]) for the Euler gamma function

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

we obtain the following chain of equations

$$f(x) = 2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)\sum_{m=0}^\infty\frac{\lambda^m\Gamma(1+\frac{\gamma}{2}+m)}{2^{\gamma+2m}\Gamma(1+\frac{\gamma}{2}+m)\Gamma\left(\frac{\gamma+1}{2}+m\right)}\frac{x^{2m}}{m!} =$$

$$= \frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}{x^{\frac{\gamma-1}{2}}}\sum_{m=0}^\infty\frac{\lambda^m}{\Gamma\left(\frac{\gamma+1}{2}+m\right)}\frac{1}{m!}\left(\frac{x}{2}\right)^{2m+\frac{\gamma-1}{2}}.$$

For $\lambda = -\tau^2$, the last formula can be rewritten in terms of the normalized Bessel function $j_{\frac{\gamma-1}{2}}$:

$$\frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}{x^{\frac{\gamma-1}{2}}}\sum_{m=0}^\infty\frac{(-1)^m}{\Gamma\left(\frac{\gamma+1}{2}+m\right)}\frac{1}{m!}\left(\frac{\tau x}{2}\right)^{2m+\frac{\gamma-1}{2}} = j_{\frac{\gamma-1}{2}}(\tau x).$$

Thus, we obtain the solution to the initial-value problem (31) and (32) in the classical form. Because of the relation

$$B_\gamma j_{\frac{\gamma-1}{2}}(\tau x) = -\tau^2 j_{\frac{\gamma-1}{2}}(\tau x)$$

and taking into account the result formulated in Theorem 5, the function

$${}_2\Psi_2\left[\begin{array}{c}(1+\frac{\gamma}{2},\alpha),(1,1)\\(1,\alpha),(\gamma+1,2\alpha)\end{array}\bigg|\lambda x^{2\alpha}\right]$$

can be considered as a generalization of the normalized Bessel function $j_{\frac{\gamma-1}{2}}$.

Let us consider another example of the initial-value problem (28)–(30). It is well-known ([3,30]) that the functions

$$y_j(x) = x^j E_{\alpha,j+1}(\lambda x^\alpha), \qquad j = 0, ..., l-1$$

build the fundamental system of solutions to the fractional differential equation

$$(D^\alpha_{0+}y)(x) = \lambda y(x), \qquad x > 0, \qquad l-1 < \alpha \le l, \qquad l \in \mathbb{N}, \qquad \lambda \in \mathbb{R}$$

with the Djerbashian–Caputo fractional derivative

$$(D^\alpha_{0+}y)(x) = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^x \frac{y^{(n)}(t)dt}{(x-t)^{\alpha-n+1}}, \qquad \alpha \notin \mathbb{N}, \qquad n = [\alpha] + 1.$$

Theorem 5 provides the fundamental system of solutions to the equation

$$(\mathcal{B}^\alpha_{\gamma,0+}f)(x) = \lambda f(x), \qquad \alpha > 0, \qquad \lambda \in \mathbb{R}$$

in the form

$$y_m(x) = x^{2m}\, {}_2\Psi_2\left[\begin{array}{c} (m+\frac{\gamma}{2}+1,\alpha), (1,1) \\ (m+1,\alpha), (2m+\gamma+1, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right], \qquad m = 0, ..., n-1,$$

where $n = [\alpha] + 1$ for the non-integer values of α and $n = \alpha$ for $\alpha \in \mathbb{N}$. Because of the relation $(\mathcal{B}^\alpha_{0,0+}f)(x) = (D^{2\alpha}_{0+}f)(x)$, the Fox–Wright function ${}_2\Psi_2$ can be considered as a generalization of the Mittag–Leffler function from the viewpoint of the eigenfunctions of the fractional differential operators.

Our last example in this section is the initial-value problem (28)–(30) with $0 < \alpha \le 1$ ($n = 1$):

$$(\mathcal{B}^\alpha_{\gamma,0+}f)(x) = \lambda f(x), \qquad \alpha > 0, \qquad \lambda \in \mathbb{R},$$

$$f(0) = a, \qquad f'(0) = 0.$$

According to Theorem 5, its unique solution has the form

$$f(x) = \frac{a 2^\gamma \Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}}\, {}_2\Psi_2\left[\begin{array}{c} (\frac{\gamma}{2}+1,\alpha), (1,1) \\ (1,\alpha), (\gamma+1, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right].$$

6. The Euler–Poisson–Darboux Equation with the Fractional Bessel Derivative

In this section, we apply a method similar to the one employed in the previous section for analytical treatment of a one-dimensional fractional PDE.

Let $u = u(x,t)$, $t \ge 0$, $x \ge 0$ be a function of two variables. In this section, we deal with the fractional PDE in the form

$$(\mathcal{B}^\alpha_{\gamma,0+})_t u(x,t) = (B_\gamma)_x u(x,t) \tag{33}$$

equipped with the Cauchy initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = 0. \tag{34}$$

Equation (33) is a fractional generalization of the well-known Euler–Poisson–Darboux equation (see, for example, [31] and the references therein).

We start with some auxiliary statements that are employed for analytical treatment of the initial-value problem (33) and (34). In particular, we will need an explicit formula for the inverse Hankel transform of the Fox–Wright function. It can be derived using the following known result ([20], p. 50, Corollary 2.5.1):

Lemma 1. *Let $a^* > 0$ or $a^* = \Delta = 0$ and $\operatorname{Re}\mu < -1$. Let $\eta, \omega \in \mathbb{C}$, $\tau > 0$ and $\sigma > 0$ be such that*

$$\sigma \operatorname{Re}\eta + \operatorname{Re}\omega + \tau \min_{1\le j\le m}\left(\frac{\operatorname{Re} b_j}{\beta_j}\right) > -1,$$

$$\tau \min_{1\le i\le m}\left(\frac{1-\operatorname{Re} a_i}{\alpha_i}\right) > \operatorname{Re}\omega - \frac{\sigma}{2} + 1$$

and

$$\operatorname{Re}\eta > -\frac{1}{2}.$$

Then, for $a > 0$, $b > 0$, $x > 0$, the formula

$$\int_0^\infty (x\xi)^\omega J_\eta(a(x\xi)^\sigma)\mathbf{H}^{m,n}_{p,q}\left[b\xi^\tau \middle| \begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\right] d\xi =$$

$$= \frac{1}{2\sigma x}\left(\frac{2}{a}\right)^{\frac{\omega+1}{\sigma}} \mathbf{H}^{m,n+1}_{p+2,q}\left[b\left(\frac{2}{a}\right)^{\frac{\tau}{\sigma}}\frac{1}{x^\tau}\middle| \begin{matrix}\left(1-\frac{\omega+1}{2\sigma}-\frac{\eta}{2},\frac{\tau}{2\sigma}\right),(a_i,\alpha_i)_{1,p},\left(1-\frac{\omega+1}{2\sigma}+\frac{\eta}{2},\frac{\tau}{2\sigma}\right)\\ (b_j,\beta_j)_{1,q}\end{matrix}\right].$$

holds valid.

Using Lemma 1, we then derive an explicit formula for the inverse Hankel transform of the Fox–Wright function.

Lemma 2. *For $1 < \gamma < 2$, $\frac{\gamma}{\gamma+1} < \alpha < 1$, the inverse Hankel transform of the Fox–Wright function is given by the formula*

$$F_\gamma^{-1}\left[{}_2\Psi_2\left[\begin{matrix}(\frac{\gamma}{2}+1,\alpha),(1,1)\\ (1,\alpha),(\gamma+1,2\alpha)\end{matrix}\middle| -\xi^2 t^{2\alpha}\right]\right](x) =$$

$$= \frac{2}{x^{\gamma+1}\Gamma\left(\frac{\gamma+1}{2}\right)}\mathbf{H}^{1,2}_{3,2}\left[\frac{4t^{2\alpha}}{x^2}\middle| \begin{matrix}\left(\frac{1-\gamma}{2},1\right),\left(-\frac{\gamma}{2},\alpha\right),(0,1)\\ (0,\alpha),(-\gamma,2\alpha)\end{matrix}\right].$$

Proof. Because of the representation $j_\nu(x) = \frac{2^\nu\Gamma(\nu+1)}{x^\nu} J_\nu(x)$, the formula $j_{\frac{\gamma-1}{2}}(x\xi) = 2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)(x\xi)^{\frac{1-\gamma}{2}} J_{\frac{\gamma-1}{2}}(x\xi)$ holds true. Then, the inverse Hankel transform can be represented as follows:

$$F_\gamma^{-1}[\widehat{f}(\xi)](x) = f(x) = \frac{2^{1-\gamma}}{\Gamma^2\left(\frac{\gamma+1}{2}\right)}\int_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\widehat{f}(\xi)\xi^\gamma\,d\xi = \frac{2^{\frac{1-\gamma}{2}}}{x^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\int_0^\infty (x\xi)^{\frac{\gamma+1}{2}} J_{\frac{\gamma-1}{2}}(x\xi)\widehat{f}(\xi)\,d\xi.$$

Now we apply this formula to the Fox–Wright function represented in the form of a particular case of the Fox H-function:

$$F_\gamma^{-1}\left[{}_2\Psi_2\left[\begin{matrix}(\frac{\gamma}{2}+1,\alpha),(1,1)\\ (1,\alpha),(\gamma+1,2\alpha)\end{matrix}\middle| -\xi^2 t^{2\alpha}\right]\right](x) = F_\gamma^{-1}\left[\mathbf{H}^{1,2}_{2,3}\left[\xi^2 t^{2\alpha}\middle| \begin{matrix}(-\frac{\gamma}{2},\alpha),(0,1)\\ (0,1),(0,\alpha),(-\gamma,2\alpha)\end{matrix}\right]\right](x) =$$

$$= \frac{2^{\frac{1-\gamma}{2}}}{x^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\int_0^\infty (x\xi)^{\frac{\gamma+1}{2}} J_{\frac{\gamma-1}{2}}(x\xi)\mathbf{H}^{1,2}_{2,3}\left[\xi^2 t^{2\alpha}\middle| \begin{matrix}(-\frac{\gamma}{2},\alpha),(0,1)\\ (0,1),(0,\alpha),(-\gamma,2\alpha)\end{matrix}\right] d\xi.$$

To apply Lemma 1 to the last formula, we first check the conditions formulated in Lemma 1. In the case under consideration, we have the following parameter values:

$$\omega = \frac{\gamma+1}{2},\ \eta = \frac{\gamma-1}{2},\ a = 1,\ \sigma = 1,\ m = 1,\ n = 2,\ p = 2,\ q = 3,\ b = t^{2\alpha},\ \tau = 2,$$

$$a_1 = -\frac{\gamma}{2},\ a_2 = 0,\ \alpha_1 = \alpha,\ \alpha_2 = 1,\ b_1 = 0,\ b_2 = 0,\ b_3 = -\gamma,\ \beta_1 = 1,\ \beta_2 = \alpha,\ \beta_3 = 2\alpha.$$

Then, the values mentioned in Lemma 1 are as follows:

$$a^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j = \alpha_1 + \alpha_2 + \beta_1 - \beta_2 - \beta_3 =$$

$$= \alpha + 1 + 1 - \alpha - 2\alpha = 2 - 2\alpha > 0 \Leftrightarrow \alpha < 1,$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2} = b_1 + b_2 + b_3 - a_1 - a_2 + \frac{p-q}{2} =$$

$$= 0 + 0 - \gamma + \frac{\gamma}{2} - 0 - \frac{1}{2} = -\frac{\gamma+1}{2} < -1 \Leftrightarrow \gamma > 1,$$

$$\eta = \frac{\gamma-1}{2} > -\frac{1}{2} \Leftrightarrow \gamma > 0, \qquad \tau = 2 > 0, \qquad \sigma = 1 > 0,$$

$$\sigma \cdot \eta + \omega + \tau \min_{1 \le j \le m} \left(\frac{b_j}{\beta_j} \right) = \frac{\gamma-1}{2} + \frac{\gamma+1}{2} + 2 \min\left\{ \frac{0}{1}, \frac{0}{\alpha}, -\frac{\gamma}{2\alpha} \right\} =$$

$$= \gamma - \frac{\gamma}{\alpha} = \gamma\left(1 - \frac{1}{\alpha}\right) > -1 \Rightarrow \frac{\gamma}{\gamma+1} < \alpha,$$

$$\tau \min_{1 \le i \le m} \left(\frac{1 - a_i}{\alpha_i} \right) - \omega + \frac{\sigma}{2} = 2 \min\left\{ \frac{1 + \frac{\gamma}{2}}{\alpha}, \frac{1-0}{1} \right\} - \frac{\gamma+1}{2} + \frac{1}{2} = 2 - \frac{\gamma}{2} > 1 \Rightarrow \gamma < 2.$$

As a result, we are allowed to apply Lemma 1 to get an explicit formula for the inverse Hankel transform of the Fox–Wright function:

$$F_\gamma^{-1}\left[{}_2\Psi_2\left[\begin{array}{c} \left(\frac{\gamma}{2}+1, \alpha\right), (1,1) \\ (1,\alpha), (\gamma+1, 2\alpha) \end{array} \middle| -\xi^2 t^{2\alpha} \right] \right](x) =$$

$$= \frac{2^{\frac{1-\gamma}{2}}}{x^\gamma \Gamma\left(\frac{\gamma+1}{2}\right)} \int_0^\infty (x\xi)^{\frac{\gamma+1}{2}} J_{\frac{\gamma-1}{2}}(x\xi) \mathbf{H}_{2,3}^{1,2}\left[\xi^2 t^{2\alpha} \middle| \begin{array}{c} \left(-\frac{\gamma}{2}, \alpha\right), (0,1) \\ (0,1), (0,\alpha), (-\gamma, 2\alpha) \end{array} \right] d\xi =$$

$$= \frac{2^{\frac{1-\gamma}{2}}}{x^\gamma \Gamma\left(\frac{\gamma+1}{2}\right)} \frac{1}{2x} 2^{\frac{\gamma+3}{2}} \mathbf{H}_{4,3}^{1,3}\left[\frac{4t^{2\alpha}}{x^2} \middle| \begin{array}{c} \left(\frac{1-\gamma}{2}, 1\right), \left(-\frac{\gamma}{2}, \alpha\right), (0,1), (0,1) \\ (0,1), (0,\alpha), (-\gamma, 2\alpha) \end{array} \right] =$$

$$= \frac{2}{x^{\gamma+1} \Gamma\left(\frac{\gamma+1}{2}\right)} \mathbf{H}_{3,2}^{1,2}\left[\frac{4t^{2\alpha}}{x^2} \middle| \begin{array}{c} \left(\frac{1-\gamma}{2}, 1\right), \left(-\frac{\gamma}{2}, \alpha\right), (0,1) \\ (0,\alpha), (-\gamma, 2\alpha) \end{array} \right].$$

□

In the rest of this section, we derive an explicit form of the solution to the initial-value problem (33) and (34) for the fractional Euler–Poisson–Darboux equation. The main result is given in the following theorem.

Theorem 6. *Let $1 < \gamma < 2$ and $\frac{\gamma}{\gamma+1} < \alpha \le 1$ and $f \in C_{ev}^2$ be an exponentially bounded function. Then, the initial-value problem (33) and (34) for the fractional Euler–Poisson–Darboux equation possesses an unique solution $u = u(x,t)$ in the form*

$$u(x,t) = \int_0^\infty \frac{1}{y} \mathbf{H}_{3,2}^{1,2}\left[\frac{4t^{2\alpha}}{y^2} \middle| \begin{array}{c} \left(\frac{1-\gamma}{2}, 1\right), \left(-\frac{\gamma}{2}, \alpha\right), (0,1) \\ (0,\alpha), (-\gamma, 2\alpha) \end{array} \right] {}^\gamma T_x^y f(x)\, dy$$

provided that the integral at the right-hand side of this formula is convergent.

Proof. Applying the Hankel transform (19) to the initial-value problem (33) and (34) for the fractional Euler–Poisson–Darboux equation with respect to the spatial variable x and using the formula (1.95) from [23] (p. 41), we get the following initial-value problem for the fractional Bessel derivative:

$$(\mathcal{B}^{\alpha}_{\gamma,0+})_t \widehat{u}(\xi,t) = -\xi^2 \widehat{u}(\xi,t), \tag{35}$$

$$\widehat{u}(\xi,0) = \widehat{f}(\xi), \qquad \widehat{u}_t(\xi,0) = 0. \tag{36}$$

The unique solution to the initial-value problem (35) and (36) is provided by Theorem 5 in the form

$$\widehat{u}(\xi,t) = \frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}} \widehat{f}(\xi)\, {}_2\Psi_2\left[\begin{array}{c} \left(\frac{\gamma}{2}+1,\alpha\right),(1,1) \\ (1,\alpha),(\gamma+1,2\alpha) \end{array}\middle| -\xi^2 t^{2\alpha}\right].$$

Then, we employ the convolution property (22) for the Hankel integral transform and the generalized convolution (21) and arrive at the following solution representation in terms of the generalized convolution

$$u(x,t) = \frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}} \left(f(x) * (F_{\gamma}^{-1})_{\xi}\left[{}_2\Psi_2\left[\begin{array}{c} \left(\frac{\gamma}{2}+1,\alpha\right),(1,1) \\ (1,\alpha),(\gamma+1,2\alpha) \end{array}\middle| -\xi^2 t^{2\alpha}\right]\right](x)\right)_{\gamma}.$$

Finally, the result formulated in Lemma 2 directly leads to the desired solution formula

$$u(x,t) = \frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}} \left(f(x) * \frac{2}{x^{\gamma+1}\Gamma\left(\frac{\gamma+1}{2}\right)} \mathbf{H}^{1,2}_{3,2}\left[\frac{4t^{2\alpha}}{x^2}\middle| \begin{array}{c} \left(\frac{1-\gamma}{2},1\right),\left(-\frac{\gamma}{2},\alpha\right),(0,1) \\ (0,\alpha),(-\gamma,2\alpha) \end{array}\right]\right)_{\gamma}$$

in terms of the generalized convolution (21). □

7. Conclusions

The role of the higher transcendental functions both in mathematical treatises and in numerous applications permanently increases. Even more general special functions are introduced and employed in the framework of mathematical theories and application domains. One of the prime examples of this sort is the theory of the integrals and derivatives of the non-integer order (Fractional Calculus) and its applications. In the framework of this theory, and especially for analytical treatment of the fractional ODEs and PDEs, some particular types of the higher transcendental functions became extremely important, including the Mittag–Leffler function and its generalizations, the Fox–Wright function, and the Fox H-function.

In this paper, we focused on some new applications of the Fox–Wright function and the Fox H-function in the theory of the fractional differential equations with the fractional Bessel operator. We started with a discussion of the integral transforms with the special functions in the kernels, whicht were then employed for analytical treatment of the fractional differential equations with the fractional powers of the Bessel operator. In particular, a suitable modification of the Meijer integral transform and its inversion formula was introduced and studied in detail.

One of the main results presented in the paper is the derivation of the fundamental system of solutions to the fractional ODEs with the fractional Bessel operator in terms of the Fox–Wright function. It turns out that for these equations, the Fox–Wright function plays the same role that the Mittag–Leffler function does for the fractional ODEs with the conventional fractional derivatives. Another important result is an explicit solution formula for the Cauchy problem for a one-dimensional fractional Euler–Poisson–Darboux equation that contains a fractional power of the Bessel operator with respect to the time variable and the conventional Bessel operator with respect to the spatial variable. The solution is described in terms of the generalized convolutions introduced in this paper, the Fox–Wright function, the Fox H-function, and their particular cases.

Because the Bessel-type ODEs and the Euler–Poisson–Darboux PDEs are extremely important for different applications, their fractional generalizations considered in this paper are not just interesting objects for mathematical treatment, but they are also for sure potentially useful for applications. These matters are worth further investigating. Of course, for applications, not only analytical formulas but mainly numerical results are needed. Thus, numerical schemata for the initial and boundary value problems for the fractional Bessel ODEs and the fractional Euler–Poisson–Darboux PDEs should be developed. It is another direction for further research worth considering by the FC community.

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