Approximation Properties of Discrete Boundary Value Problems for Elliptic Pseudo-Differential Equations

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Abstract We study some discrete boundary value problems which are treated as digital approximation for starting boundary value problem for elliptic pseudo-differential equation. Starting from existence and uniqueness theorem we give a comparison between discrete and continuous solutions for certain boundary value problems.

1 Introduction

We study a discrete variant of the following boundary value problem

$$\begin{cases} (Au)(x) = f(x), & x \in D, \\ (Bu)|_{\partial D} = g \end{cases} \tag{1}$$

where A, B are simplest elliptic pseudo-differential operators [1–3] with symbols $A(\xi), B(\xi)$, acting in Sobolev–Slobodetskii spaces $H^s(D)$, $D \subset \mathbf{R}^m$ is a certain bounded domain, f, g are given functions.

Discrete variants of similar problems for differential operators were studied earlier (see, for example [4] with difference schemes, or [5] with difference potentials), but we would like to develop an approach for more general pseudo-differential operators and related equations. This approach is based on a concept of periodic factorization for an elliptic symbols and it is a discrete analogue of corresponding continuous methods [1].

Some first studies in this direction were done in [6–15].

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F. J. Vermolen, C. Vuik (eds.), *Numerical Mathematics and Advanced Applications ENUMATH 2019*, Lecture Notes in Computational Science and Engineering 139, https://doi.org/10.1007/978-3-030-55874-1_108

2 Digital Operators and Discrete Boundary Value Problems

Here we will describe our approach to studying discrete equations and boundary value problems.

Given function u_d of a discrete variable $\tilde{x} \in h\mathbf{Z}^m$, $h > 0$, we define its discrete Fourier transform by the series

$$(F_d u_d)(\xi) \equiv \widetilde{u}_d(\xi) = \sum_{\tilde{x} \in \mathbf{Z}^m} e^{i\tilde{x}\cdot\xi} u_d(\tilde{x}), \quad \xi \in \hbar\mathbf{T}^m,$$

where $\mathbf{T}^m = [-\pi, \pi]^m$, $\hbar = h^{-1}$, and partial sums are taken over cubes

$$Q_N = \{\tilde{x} \in h\mathbf{Z}^m : \tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_m), \max_{1 \le k \le m} |\tilde{x}_k| \le N\}.$$

We will remind here some definitions of functional spaces [12] and will consider discrete analogue $S(h\mathbf{Z}^m)$ of the Schwartz space $S(\mathbf{R}^m)$. Let us denote $\zeta^2 = h^{-2} \sum_{k=1}^{m} (e^{-ih\cdot\xi_k} - 1)^2$.

The space $H^s(h\mathbf{Z}^m)$ is a closure of the space $S(h\mathbf{Z}^m)$ with respect to the norm

$$||u_d||_s = \left(\int_{\hbar\mathbf{T}^m} (1 + |\zeta^2|)^s |\tilde{u}_d(\xi)|^2 d\xi \right)^{1/2}. \tag{2}$$

Fourier image of the space $H^s(h\mathbf{Z}^m)$ will be denoted by $\widetilde{H}^s(\hbar\mathbf{T}^m)$.

One can define some discrete operators for such functions u_d.

If $\widetilde{A}_d(\xi)$ is a periodic function in $\mathbf{R}^m$ with the basic cube of periods $\hbar\mathbf{T}^m$ then we consider it as a symbol. We will introduce a digital pseudo-differential operator in the following way.

Definition 1 A digital pseudo-differential operator A_d in a discrete domain D_d is called the operator [12]

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbf{Z}^m} \int_{\hbar\mathbf{T}^m} \widetilde{A}_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d,$$

We consider a class of symbols [12] satisfying the following condition

$$c_1(1 + |\zeta^2|)^{\alpha/2} \le |A_d(\xi)| \le c_2(1 + |\zeta^2|)^{\alpha/2}, \quad \alpha \in \mathbf{R}, \tag{3}$$

and universal positive constants c_1, c_2.

Let $D \subset \mathbf{R}^m$ be a domain. We will study the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d, \tag{4}$$

in the discrete domain $D_d \equiv D \cap h\mathbf{Z}^m$ and will seek a solution $u_d \in H^s(D_d)$, $v_d \in H_0^{s-\alpha}(D_d)$ [12, 15].

In this paper we will discuss the case $D \equiv \mathbf{R}^m_+$.

Let $\tilde{A}_d(\xi)$ be a periodic symbol. Let us denote $\Pi_\pm$ half-strips in the complex plane $\mathbf{C}$

$$\Pi_\pm = \{z \in \mathbf{C} : z = s + i\tau, s \in [-\pi, \pi], \pm\tau > 0\}.$$

Definition 2 Periodic factorization of an elliptic symbol $A_d(\xi) \in E_\alpha$ is called its representation in the form

$$A_d(\xi) = A_{d,+}(\xi) A_{d,-}(\xi),$$

where the factors $A_{d,\pm}(\xi)$ admit an analytical continuation into half-strips $\hbar\Pi_\pm$ on the last variable ξ_m for almost all fixed $\xi' \in \hbar\mathbf{T}^{m-1}$ and satisfy the estimates

$$|A^{\pm 1}_{d,+}(\xi)| \leq c_1(1 + |\hat{\zeta}^2|)^{\pm\frac{æ}{2}}, \quad |A^{\pm 1}_{d,-}(\xi)| \leq c_2(1 + |\hat{\zeta}^2|)^{\pm\frac{\alpha-æ}{2}},$$

with constants c_1, c_2 non-depending on h,

$$\hat{\zeta}^2 \equiv \hbar^2 \left(\sum_{k=1}^{m-1} (e^{-ih\xi_k} - 1)^2 + (e^{-ih(\xi_m + i\tau)} - 1)^2 \right), \quad \xi_m + i\tau \in \hbar\Pi_\pm.$$

The number $æ \in \mathbf{R}$ is called an index of periodic factorization.

We consider the following discrete boundary value problem

$$\begin{cases} (A_d u_d)(\tilde{x}) = v_d(\tilde{x}), & \tilde{x} \in \mathbf{R}^m_+ \\ (B_d u_d)_{|\tilde{x}_m = 0} = g_d(\tilde{x}'), & \tilde{x}' \in \mathbf{R}^{m-1}, \end{cases} \tag{5}$$

such that the discrete boundary value problem (5) will have good approximation properties for initial boundary value problem.

3 Solvability and Comparison

This section is devoted to the following questions:

1. to establish solvability for our discrete boundary value problem;
2. to give a comparison between discrete and continuous solutions.

3.1 Solvability

To describe solvability for the boundary value problem (5) we introduce the following notations.

$$(H^{per}_{\xi'}\tilde{u}_d)(\xi',\xi_m) = \frac{h}{2\pi i}\, p.v. \int\limits_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\xi_m-\eta_m)}{2}\tilde{u}_d(\xi',\eta_m)d\eta_m,$$

where

$$p.v. \int\limits_{-\hbar\pi}^{\hbar\pi} \cot\frac{h(\xi_m-\eta_m)}{2}\tilde{u}_d(\xi',\eta_m)d\eta_m$$

$$= \lim_{\varepsilon\to 0+}\left(\int\limits_{-\hbar\pi}^{\xi_m-\varepsilon} + \int\limits_{\xi_m+\varepsilon}^{\hbar\pi}\right)\cot\frac{h(\xi_m-\eta_m)}{2}\tilde{u}_d(\xi',\eta_m)d\eta_m$$

This operator generates two projectors

$$P^{per}_{\xi'} = \frac{1}{2}(I + H^{per}_{\xi'}), \quad Q^{per}_{\xi'} = \frac{1}{2}(I - H^{per}_{\xi'}),$$

which permit to formulate and solve the following problem.

The following theorem was proved in the paper [7].

Theorem 1 *Let* $\text{æ} - s = n+\delta, n \in \mathbf{N}, |\delta| < 1/2$. *Then a general solution of Eq. (4) in Fourier images has the following form*

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi)X_n(\xi)P^{per}_{\xi'}(X^{-1}_n(\xi)\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi)) + \tilde{A}^{-1}_{d,+}(\xi)\sum_{k=0}^{n-1}\tilde{c}_k(\xi')\hat{\zeta}^k_m,$$

where $X_n(\xi)$ *is an arbitrary polynomial of order* n *of variables* $\hat{\zeta}_k = \hbar(e^{-ih\xi_k} - 1), k = 1,\cdots,m$, *satisfying the condition (2),* $c_k(\xi'), j = 0,1,\cdots,n-1$, *are arbitrary functions from* $H^{s_k}(h\mathbf{T}^{m-1}), s_k = s - \text{æ} + k - 1/2$, ℓv_d *is an arbitrary continuation of* v_d. *from* $H^{s-\alpha}(D_d)$ *into* $H^{s-\alpha}(h\mathbf{Z}^m)$

The a priori estimate

$$||u_d||_s \le a(||f||^+_{s-\alpha} + \sum_{k=0}^{n-1}[c_k]_{s_k})$$

holds, where $[\cdot]_{s_k}$ *denotes a norm in the space* $H^{s_k}(h\mathbf{T}^{m-1})$*, and the constant* a *does not depend on* h*.*

We will apply Theorem 1 for the simple case $n = 1$, because we consider only one boundary condition. Then we have

$$\tilde{u}_d(\xi) = \tilde{h}_d(\xi) + \tilde{A}_{d,+}^{-1}(\xi)\tilde{c}_0(\xi'), \tag{6}$$

where we denote

$$\tilde{h}_d(\xi) = \tilde{A}_{d,+}^{-1}(\xi)X_1(\xi)P_{\xi'}^{per}(X_1^{-1}(\xi)\tilde{A}_{d,-}^{-1}(\xi)\widetilde{\ell v_d}(\xi)) \tag{7}$$

The construction of a general solution for starting boundary value problem is very similar and exact, it was obtained in [1]. For our case it has the following form

$$\tilde{u}(\xi) = \tilde{h}(\xi) + \tilde{A}_{+}^{-1}(\xi)\tilde{C}_0(\xi'), \tag{8}$$

$$\tilde{h}(\xi) = \tilde{A}_{+}^{-1}(\xi)Y_1(\xi)P_{\xi'}(Y_1^{-1}(\xi)\tilde{A}_{-}^{-1}(\xi)\widetilde{\ell f}(\xi)), \tag{9}$$

where $P_{\xi'} = 1/2(I + H_{\xi'})$, and $H_{\xi'}$ is the classical Hilbert transform on the last variable ξ_m

$$(H_{\xi'}u(\xi', \xi_m) = \frac{1}{\pi i}\ p.v. \int_{-\infty}^{+\infty} \frac{u(\xi', \tau)d\tau}{\xi_m - \tau},$$

$Y_1(\xi)$ is an arbitrary polynomial of variables $\xi_1, \cdots, \xi_m$ satisfying the condition $|Y_1(\xi)| \sim 1 + |\xi|$, $\tilde{A}_{\pm}(\xi)$ are factors of factorization for the symbol $\tilde{A}(\xi)$.

The formulas (8), (9) are valid under assumptions that the symbols $A(\xi)$ satisfies the condition

$$c_1(1 + |\xi|)^{\alpha} \le |\tilde{A}(\xi)| \le c_2(1 + |\xi|)^{\alpha}, \tag{10}$$

and index factorization of the symbol $A(\xi)$ equals æ.

There are arbitrary functions $\tilde{c}_0$, $\tilde{C}_0$ in the formulas (6), (8). To determine, for example, the function $\tilde{c}_0$ we use the boundary condition from (5). We act by the operator B on the solution u_d and then we take the restriction on the discrete half-plane $\tilde{\xi}_m = 0$. According to properties of the discrete Fourier transform we have

$$\int_{-\hbar\pi}^{+\hbar\pi} \tilde{B}_d(\xi', \xi_m)\tilde{u}_d(\xi', \xi_m)d\xi_m = \int_{-\hbar\pi}^{+\hbar\pi} \tilde{B}_d(\xi', \xi_m)\tilde{h}_d(\xi', \xi_m)d\xi_m + \tilde{c}_0(\xi')b_d(\xi'),$$

where

$$b_d(\xi') = \int\limits_{-\hbar\pi}^{+\hbar\pi} \tilde{B}_d(\xi', \xi_m) A_{d,+}^{-1}(\xi', \xi_m) d\xi_m$$

Here we use the condition $\inf\limits_{\xi' \in \hbar\mathbf{T}^{m-1}} |b_d(\xi'| > 0$; it is a discrete analogue of Shapiro–Lopatinskii condition [1]. Since the left hand side is $\tilde{g}_d(\xi')$ we have the following relation

$$\tilde{c}_0(\xi') = b_d^{-1}(\xi') \left(\tilde{g}_d(\xi') - \tilde{t}_d(\xi') \right), \tag{11}$$

where

$$\tilde{t}_d(\xi') = \int\limits_{-\hbar\pi}^{+\hbar\pi} \tilde{B}_d(\xi', \xi_m) \tilde{h}_d(\xi', \xi_m) d\xi_m.$$

By substitution of (11) into (6), we obtain a unique solution for the discrete boundary value problem (5):

$$\tilde{u}_d(\xi) = \tilde{h}_d(\xi) + \tilde{A}_{d,+}^{-1}(\xi) b_d^{-1}(\xi') \left(\tilde{g}_d(\xi') - \tilde{t}_d(\xi') \right), \tag{12}$$

3.2 A Comparison

According to Vishik–Eskin theory [1] we have a continuous analogue of the formula (12), namely

$$\tilde{u}(\xi) = \tilde{h}(\xi) + \tilde{A}_{+}^{-1}(\xi) b^{-1}(\xi') \left(\tilde{g}(\xi') - \tilde{t}(\xi') \right) \tag{13}$$

under the condition $\inf\limits_{\xi' \in \mathbf{R}^{m-1}} |b(\xi'| > 0$. Now we would like to compare two formulas (12) and (13). To simplify our considerations we put $f \equiv 0$. Then the functions h, h_d, t, t_d are zero.

To obtain a good approximation we choose certain elements for the discrete solution in a particular way.

First, let us denote by q_h the following operator of restriction and periodization; this operator acts in Fourier images. Given function $\tilde{u}$ the notation $q_h \tilde{u}$ means that we take a restriction of $\tilde{u}$ on $\hbar\mathbf{T}^m$ and periodically continue it into whole $\mathbf{R}^m$. The symbol $\tilde{A}_d(\xi)$ of the discrete operator A_d is the following. We take the factorization

$$\tilde{A}(\xi) = \tilde{A}_+(\xi) \cdot \tilde{A}_-(\xi)$$

and introduce the periodic symbol by the formula

$$\tilde{A}_d(\xi) \equiv (q_h\tilde{A}_+)(\xi)\cdot(q_h\widetilde{A}_-)(\xi),$$

so we have immediately the needed periodic factorization.

Secondly, we define the symbol $\tilde{B}_d(\xi)$ of the boundary operator B_d by

$$\tilde{B}_d(\xi) \equiv (q_h\tilde{B})(\xi).$$

Third, we choose $g_d = F_d^{-1}(q_h\tilde{g})$, where

$$(F_d^{-1}\tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^m}\int\limits_{\hbar\mathbf{T}^m} e^{i\tilde{x}\cdot\xi}\tilde{u}_d(\xi)d\xi, \quad \tilde{x}\in h\mathbf{Z}^m.$$

Lemma 1 *Let the boundary symbol $\tilde{B}(\xi)$ satisfy the condition (10) with order β. Then the following estimate*

$$|\tilde{b}_d(\xi') - \tilde{b}(\xi')| \le ch^{æ-1-\beta}$$

holds.

Proof We give corresponding estimates:

$$|b(\xi') - b_d(\xi')| = \left|\int\limits_{-\infty}^{+\infty}\tilde{B}(\xi',\xi_m)\tilde{A}_+^{-1}(\xi',\xi_m)d\xi_m - \int\limits_{-\hbar\pi}^{\hbar\pi}\tilde{B}(\xi',\xi_m)\tilde{A}_{d,+}^{-1}(\xi',\xi_m)d\xi_m\right| =$$

$$\left|\left(\int\limits_{-\infty}^{-\hbar\pi} + \int\limits_{\hbar\pi}^{+\infty}\right)\tilde{B}(\xi',\xi_m)\tilde{A}_+^{-1}(\xi',\xi_m)d\xi_m\right|.$$

Two integrals have the same estimate and we consider the second one.

$$\int\limits_{\hbar\pi}^{+\infty}|\tilde{B}(\xi',\xi_m)A_+^{-1}(\xi',\xi_m)|d\xi_m \le c_5\int\limits_{\hbar\pi}^{+\infty}(1+|\xi'|+|\xi_m|)^{\beta-æ}d\xi_m =$$

$$\frac{c_5}{æ-1-\beta}(1+|\xi'|+\hbar\pi)^{1-æ} \le ch^{æ-1-\beta}.$$

Theorem 2 *Let* $f \equiv 0$, $v_d \equiv 0$, $g \in H^{s-\beta-1/2}(\mathbf{R}^{m-1})$, $g_d \in H^{s-\beta-1/2}(h\mathbf{Z}^{m-1})$, $s-\beta > 1/2$, $æ > 1+\beta$, *and*

$$\inf_{\xi' \in \mathbf{R}^{m-1}} |b(\xi')| > 0, \qquad \inf_{\xi' \in \mathbf{T}^{m-1}, h>0} |b_d(\xi')| > 0.$$

Then boundary value problems (1) and (5) have unique solutions in spaces $H^s(\mathbf{R}^m_+)$ *and* $H^s(h\mathbf{Z}^m_+)$ *respectively.*

If $g \in L_1(\mathbf{R}^{m-1})$ *then we have the estimate*

$$|\tilde{u}_d(\xi) - \tilde{u}(\xi)| \le c h^{æ-1-\beta}, \quad \xi \in \hbar\mathbf{T}^m.$$

Proof The existence and uniqueness for the problems was proved in [1] for continuous case and in [8] for discrete case, and here we have described the construction for solving discrete boundary value problem. Therefore we need to prove the estimate. We have

$$\tilde{u}(\xi) - \tilde{u}_d(\xi) = b^{-1}(\xi')\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m) - b_d^{-1}(\xi')\tilde{g}_d(\xi')A_{d,+}^{-1}(\xi', \xi_m) =$$

$$(b^{-1}(\xi') - b_d^{-1}(\xi'))\tilde{g}_d(\xi')A_{d,+}^{-1}(\xi', \xi_m), \quad \xi \in \hbar\mathbf{T}^m,$$

and using Lemma 1 and boundedness of $\tilde{g}$ we complete the estimate.

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