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ORIGINAL RESEARCH PAPER

A spectral resolution for digital pseudo-differential operators

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Abstract
We consider a special class of operators acting in discrete spaces and discuss certain its properties related to specters and approximations. These properties can be useful for constructing approximate solutions of corresponding operator equations.

Keywords Calderon–Zygmund operator · Digital pseudo-differential operator · Symbol · Multiplier · Spectra

Mathematics Subject Classification Primary 42A45 · Secondary 42B20

1 Introduction

The discrete Calderon–Zygmund operator is constructed on continual ones by standard digitization. One chooses integer lattice in multi-dimensional space, takes the values of continual kernel in lattice points and constructs the corresponding singular convolution. This is a digitization in space of coordinates. Calderon–Zygmund operator's spectra is defined by image of its symbol. If the spectra is well "visible", for example it is a smooth curve, then this curve can be identified by set of its own points which are near each other. This is frequency digitization in the space of impulses. These digitizations have interesting exceptions and a lot of specific properties. We will discuss some of these properties and consider some generalizations on more general class of digital pseudo-differential operators. These considerations are very closed to so-called inverse problems. According to Wikipedia this is an inverse problem in a certain sense. We cite: "An inverse problem in science is the process of calculating from a set of observations the

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causal factors that produced them: for example, calculating an image in X-ray computed tomography, source reconstruction in acoustics, or calculating the density of the Earth from measurements of its gravity field. It is called an inverse problem because it starts with the results and then calculates the causes. This is the inverse of a forward problem, which starts with the causes and then calculates the results."

We consider some properties of special integral operators (singular integrals) which were systematically studied by Calderon and Zygmund as bounded operators in $L_p(\mathbb{R}^m)$-spaces.

More precisely, given kernel $K(x)$ one constructs an integral operator, in which the integral is treated in the principal value sense [1, 2, 11, 12]

$$(Ku)(x) \equiv \text{ v.p. } \int\limits_{\mathbb{R}^m} K(x-y)u(y)dy \equiv \lim_{\varepsilon\to 0} \int\limits_{|x-y|>\varepsilon} K(x-y)u(y)dy. \tag{1}$$

They considered generalizations (1) also, if the kernel is more complicated namely it is the function $K(x, y)$, $x \in \mathbb{R}^m, y \in \mathbb{R}^m \setminus \{0\}$, and under fixed x the kernel $K(\cdot, y)$ defines Calderon–Zygmund operator according to formula (1). The last operators they called singular integrals with variable kernels.

These considerations and developed methods have led to the calculus of pseudo-differential operators [3, 4] and boundary value problems [8, 13, 14]. Calderon–Zygmund operators are very convenient models for studying more general operators, and since such operators appear in a lot of applications one needs studying equations with such operators. Moreover, the Calderon–Zygmund operator is really the convolution only, but the convolution theory gives the "mathematics" by which one describes the interaction "input–output" for a linear system. Finally, Calderon–Zygmund operator is a multidimensional analogue of Hilbert transform which is widely used in digital signal and image processing.

A lot of questions related to Calderon–Zygmund operators and equations and similar difference and discrete equations were considered in author's papers [16–27]. Everywhere we have used Fourier analysis and methods of the theory of boundary value problems for analytic functions.

2 Digital pseudo-differential operators

2.1 Discrete Calderon–Zygmund operators: symbols and specters

Let $\mathbb{Z}^m$ be an integer lattice in $\mathbb{R}^m$. We will use the following notations. Let $\mathbb{T}^m$ be the m-dimensional cube $[-\pi, \pi]^m, h > 0, \hbar = h^{-1}$.

If $u_d(\tilde{x}), \tilde{x} \in h\mathbb{Z}^m$, is a function of a discrete variable then we call it "discrete function". For such discrete functions one can define the discrete Fourier transform

$$(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^m} e^{i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^m, \quad \xi \in \hbar\mathbb{T}^m,$$

if the last series converges, and the function $\tilde{u}_d(\xi)$ is a periodic function on $\mathbb{R}^m$ with the basic cube of periods $\hbar\mathbb{T}^m$. The discrete Fourier transform is a one-to-one correspondence between the spaces $L_2(h\mathbb{Z}^m)$ and $L_2(\hbar\mathbb{T}^m)$ with norms

$$||u_d||_2 = \left(\sum_{\tilde{x} \in h\mathbb{Z}^m} |u_d(\tilde{x})|^2 h^m \right)^{1/2}$$

and

$$||\tilde{u}_d||_2 = \left(\int\limits_{\xi \in \hbar\mathbb{T}^m} |\tilde{u}_d(\xi)|^2 d\xi \right)^{1/2}.$$

Given Calderon–Zygmund kernel they construct discrete singular convolution

$$(K_d u)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbb{Z}^m, \tilde{y} \neq \tilde{x}} K(\tilde{x} - \tilde{y}) u(\tilde{y}) h^m, \quad \tilde{x} \in h\mathbb{Z}^m, \tag{2}$$

and convergence of the series may be treated as

$$\lim_{N \to \infty} \sum_{\tilde{y} \in C_N \cap (h\mathbb{Z}^m), \tilde{y} \neq \tilde{x}} K(\tilde{x} - \tilde{y}) u(\tilde{y}),$$

where C_N is a cube in $\mathbb{R}^m$ of size $N \in \mathbb{N}$:

$$C_N = \left\{ x \in \mathbb{R}^m : \max_{1 \leq k \leq n} |x_k| \leq N \right\}.$$

In Definition (1) the truncation at infinity is not essential because first such operators were defined on infinitely differentiable functions with compact support, and after obtaining L_p-estimate one can consider limit case taking into account that these functions are dense in $L_p(\mathbb{R}^m)$.

We have considered the operator (1) and (2) in the space $L_2(\mathbb{R}^m)$ and its discrete analogue ℓ_2, because we will seriously apply the Fourier transform. Comparing a symbol of the operator K

$$\sigma(\xi) = v.p. \int\limits_{\mathbb{R}^m} e^{ix\cdot\xi} K(x) dx$$

and a symbol of the operator K_d

$$\sigma_d(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^m \setminus \{0\}} e^{i\tilde{x}\cdot\xi} K(\tilde{x}) h^m$$

we have obtained the following result [17, 18, 20].

Theorem 1 *Let $K(x)$ be a function satisfying the following conditions*

1. $K(tx) = t^{-m}K(x), \forall t > 0$;
2. $K(x) \in C^{\infty}(S^{m-1})$;
3. $\int\limits_{\mathbb{S}^{m-1}} K(\theta)d\theta = 0$

Then operators $K : L_2(\mathbb{R}^m) \to L_2(\mathbb{R}^m)$ and $K_d : L_2(h\mathbb{Z}^m) \to L_2(h\mathbb{Z}^m)$ have the same spectra $\forall h > 0$.

Conserving to introduced in n° 2 discrete Calderon–Zygmund operator we note that as for as the lattice $\mathbf{Z}^m$ is infinite set we need finite approximation in this case too; here the important fact is that *specters of discrete and continual operators are the same.*

2.2 Finite approximations for discrete Calderon–Zygmund operators

Here we will introduce a special discrete periodic kernel $K_{d,N}(\tilde{x})$ which is defined as follows. We take a restriction of the discrete kernel $K_d(\tilde{x})$ on the set $Q_N \cap h\mathbb{Z}^m \equiv Q_N^d$ and periodically continue it to a whole $h\mathbb{Z}^m$. Further we consider discrete periodic functions $u_{d,N}$ with discrete cube of periods Q_N^d. We can define a cyclic convolution for a pair of such functions $u_{d,N}, v_{d,N}$ by the formula

$$(u_{d,N} * v_{d,N})(\tilde{x}) = \sum_{\tilde{y} \in Q_N^d} u_{d,N}(\tilde{x} - \tilde{y}) v_{d,N}(\tilde{y}) h^m. \tag{3}$$

(We would like to remind that such convolutions are used in digital signal processing [5, 6]). Further we introduce finite discrete Fourier transform by the formula

$$(F_{d,N} u_{d,N})(\tilde{\xi}) = \sum_{\tilde{x} \in Q_N^d} u_{d,N}(\tilde{x}) e^{i\tilde{x}\cdot\tilde{\xi}} h^m, \quad \tilde{\xi} \in R_N^d,$$

where $R_N^d = \hbar\mathbb{T}^m \cap \hbar\mathbb{Z}^m$. Let us note that here $\tilde{\xi}$ is a discrete variable.

According to the formula (3) one can introduce the operator

$$K_{d,N} u_{d,N}(\tilde{x}) = \sum_{\tilde{y} \in Q_N^d} K_{d,N}(\tilde{x} - \tilde{y}) u_{d,N}(\tilde{y}) h^m$$

on periodic discrete functions $u_{d,N}$ and a finite discrete Fourier transform for its kernel

$$\sigma_{d,N}(\tilde{\xi}) = \sum_{\tilde{x} \in Q_N^d} K_{d,N}(\tilde{x}) e^{i\tilde{x}\cdot\tilde{\xi}} h^m, \quad \tilde{\xi} \in R_N^d.$$

Definition 1 A function $\sigma_{d,N}(\tilde{\xi}), \tilde{\xi} \in R_N^d$, is called s symbol of the operator $K_{d,N}$. This symbol is called an elliptic symbol if $\sigma_{d,N}(\tilde{\xi}) \neq 0, \forall \tilde{\xi} \in R_N^d$.

Theorem 2 *Let $\sigma_d(\xi)$ be an elliptic symbol. Then for enough large N the symbol $\sigma_{d,N}(\tilde{\xi})$ is elliptic symbol also.*

Proof The function

$$\sum_{\tilde{x} \in Q_N^d} K_{d,N}(\tilde{x}) e^{i\tilde{x}\cdot\xi} h^m, \quad \xi \in \hbar\mathbb{T}^m,$$

is a segment of the Fourier series

$$\sum_{\tilde{x} \in h\mathbb{Z}^m} K_d(\tilde{x}) e^{i\tilde{x}\cdot\xi} h^m, \quad \xi \in \hbar\mathbb{T}^m,$$

and according our assumptions this is continuous function on $\hbar\mathbb{T}^m$. Therefore values of the partial sum coincide with values of $\sigma_{d,N}$ in points $\tilde{\xi} \in R_N^d$. Besides these partial sums are continuous functions on $\hbar\mathbb{T}^m$. □

As before an elliptic symbol $\sigma_{d,N}(\tilde{\xi})$ corresponds to the invertible operator $K_{d,N}$ in the space $L_2(Q_N^d)$.

2.3 General concept

We will use the discrete Fourier transform to introduce special discrete Sobolev–Slobodetskii spaces which are very convenient for studying discrete pseudo-differential operators and related equations.

For the divided difference of second order we have

$$\begin{aligned}(\Delta_k^{(2)} u_d)(\tilde{x}) = & h^{-2}(u_d(x_1, \ldots, x_k + 2h, \cdots, x_m) \\ & - 2u_d(x_1, \ldots, x_k + h, \ldots, x_m) + u_d(x_1, \ldots, x_k + h, \cdots, x_m))\end{aligned}$$

and its discrete Fourier transform

$$(\widetilde{\Delta_k^{(2)} u_d})(\xi) = h^{-2}(e^{-ih\cdot\xi_k} - 1)^2 \tilde{u}_d(\xi).$$

Thus, for the discrete Laplacian we have

$$(\Delta_d u_d)(\tilde{x}) = \sum_{k=1}^m (\Delta_k^{(2)} u_d)(\tilde{x}),$$

so that

$$(\widetilde{\Delta_d u_d})(\xi) = h^{-2} \sum_{k=1}^m (e^{-ih\cdot\xi_k} - 1)^2 \tilde{u}_d(\xi).$$

Let us denote $\zeta^2 = h^{-2} \sum\limits_{k=1}^m (e^{-ih\cdot\xi_k} - 1)^2$ and introduce the following

Definition 2 The space $H^s(h\mathbb{Z}^m)$ consists of discrete functions $u_d(\tilde{x})$ for which the norm

$$||u_d||_s = \left(\int\limits_{\hbar\mathbb{T}^m} (1 + |\zeta_h^2|)^s |\tilde{u}_d(\xi)|^2 d\xi \right)^{1/2}$$

is finite.

We will consider all functions defined on the cube $\hbar\mathbb{T}^m$ as periodic functions in $\mathbb{R}^m$ with the same cube of periods.

Let $\widetilde{A}_d(\xi)$ be a periodic function in $\mathbb{R}^m$ with the basic cube of periods $\hbar\mathbb{T}^m$. Such functions are called symbols. As usual we will define a digital pseudo-differential operator by its symbol.

Definition 3 A digital pseudo-differential operator A_d in a discrete domain D_d is called an operator of the following kind

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbb{Z}^m} \int\limits_{\hbar\mathbb{T}^m} \widetilde{A}_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \ \ \tilde{x} \in D_d,$$

An operator A_d is called an elliptic operator if

$$ess \inf_{\xi \in \hbar\mathbb{T}^m} |\widetilde{A}_d(\xi)| > 0.$$

Remark 1 One can introduce the symbol $\widetilde{A}_d(\tilde{x}, \xi)$ depending on a spatial variable $\tilde{x}$ and define a general pseudo-differential operator by the formula

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbb{Z}^m} \int\limits_{\hbar\mathbb{T}^m} \widetilde{A}_d(\tilde{x}, \xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \ \ \tilde{x} \in D_d,$$

For studying such operators and related equations one needs to use more fine and complicated technique.

Definition 4 By definition the class E_α includes symbols satisfying the following condition

$$c_1(1 + |\zeta^2|)^{\alpha/2} \leq |A_d(\xi)| \leq c_2(1 + |\zeta^2|)^{\alpha/2} \tag{4}$$

with positive constants c_1, c_2 non-depending on h.

The number $\alpha \in \mathbb{R}$ is called an order of a digital pseudo-differential operator A_d.

Roughly speaking the order of a digital pseudo-differential operator is the power of h with the sign "minus".

Using the last definition one can easily get the following property.

Lemma 1 *A digital pseudo-differential operator $A_d \in E_\alpha$ is a linear bounded operator $H^s(h\mathbb{Z}^m) \to H^{s-\alpha}(h\mathbb{Z}^m)$.*

2.4 Reduction of order

According to our previous considerations a discrete Calderon–Zygmund operator is an operator of order zero. For an arbitrary pseudo-differential operator of order α with symbol $A_d(\xi)$ we can write

$$A_d(\xi) = \left(\frac{A_d(\xi)}{k^2 + \zeta^2}\right)^\alpha (k^2 + \zeta^2)^\alpha,$$

where $(k^2 + \zeta^2)^\alpha$ satisfies the condition (4) for some k. Thus, the first factor is symbol of Calderon–Zygmund operator, and the second one is a fractional power of the discrete Laplacian. Hence, we need studying such operators, but here we have considered only the first operator.

3 Spectral projectors

For simplicity we consider the plane case $m = 2$. Then $\mathbb{S}^1$ is the unit circumference. Let S_a be a part of $\mathbb{S}^1$ intersected by the cone (see also [7, 15])

$$C_+^a = \left\{x \in \mathbb{R}^2 : x_2 > a|x_1|,\, a > 0\right\},$$

$m_a(x)$ is a function (multiplier) equals to 1 on S_a and 0 on other piece of $\mathbb{S}^1$. We formulate the problem as follows. What kind of an operator corresponds to such multiplier in Fourier image ?

Theorem 3 *We have the following property $\forall u \in L_2(\mathbb{R}^2)$*

$$F(m_a \cdot u) = \lim_{\tau \to 0+} \int\limits_{\mathbf{R}^2} \frac{2a\tilde{u}(\eta_1, \eta_2)\, d\eta_1 d\eta_2}{(\xi_1 - \eta_1)^2 - a^2(\xi_2 - \eta_2 + i\tau)^2}. \tag{5}$$

Proof Let $\Theta(x, y)$ be the indicator of C_+^a. Let us consider the integral ($\tau > 0$)

$$\int\int\limits_{\mathbb{R}^2} e^{i(x\xi + y\eta)} \Theta(x, y) u(x, y) e^{-\tau y} dy;$$

it is a Fourier transform of the product of two functions $u(x, y)$ and $\Theta(x, y)e^{-\tau y}$ which are absolutely integrable (the last property allows us to apply a convolution theorem). Let us find a Fourier transform of function $\Theta(x, y)e^{-\tau y}$.

$$\begin{aligned}
&\int\int\limits_{\mathbb{R}^2} e^{i(x\xi+y\eta)}\Theta(x,y)e^{-\tau y}dxdy = \int\int\limits_{C_+^a} e^{i(x\xi+y\eta)}e^{-\tau y}dxdy \\
&= \int\int\limits_{C_+^a} e^{ix\xi}e^{iy(\eta+i\tau)}dxdy = \int\limits_{-\infty}^{+\infty}\left(\int\limits_{a|x|}^{+\infty} e^{iy(\eta+i\tau)}dy\right)e^{ix\xi}dx \\
&= -\frac{1}{\eta+i\tau}\int\limits_{-\infty}^{+\infty} e^{ia|x|(\eta+i\tau)}e^{ix\xi}dx \\
&= -\frac{1}{i(\eta+i\tau)}\left(\int\limits_{-\infty}^{0} e^{-iax(\eta+i\tau)}e^{ix\xi}dx + \int\limits_{0}^{+\infty} e^{iax(\eta+i\tau)}e^{ix\xi}dx\right) \\
&= -\frac{1}{i(\eta+i\tau)}\left(\int\limits_{-\infty}^{0} e^{ix(\xi-a(\eta+i\tau))}dx + \int\limits_{0}^{+\infty} e^{ix(\xi+a(\eta+i\tau))}dx\right) \\
&= -\frac{1}{i(\eta+i\tau)}\left(\frac{1}{i(\xi-a(\eta+i\tau))} - \frac{1}{i(\xi+a(\eta+i\tau))}\right) \\
&= \frac{1}{\eta+i\tau}\frac{2a(\eta+i\tau)}{\xi^2-a^2(\eta+i\tau)^2} = \frac{2a}{\xi^2-a^2(\eta+i\tau)^2}.
\end{aligned}$$

The convolution of the last function with $\tilde{u}(\xi,\eta)$ and passage to the limit under $\tau \to +0$ gives the formula (5). □

Remark 2 The sector size is

$$\alpha = 2\arctan(1/a)$$

We will denote the operator (5) in the following way

$$\tilde{u} \longmapsto G_a\tilde{u}.$$

4 Calderon–Zygmund operators and spectral resolution

Here we consider Calderon–Zygnund operators with kernels as in Theorem 1. If is it possible to reconstruct the operator knowing its spectra? The answer is positive with some reservations if we know what kind of an operator we would like to construct. More precisely we can prove the following theorem if *the spectra of K is a smooth closed curve in a complex plane*.

Theorem 4 *Two-dimensional Calderon–Zygmund operator can be reconstructed on its spectra up to rotations* $\mathbb{S}^1 \to \mathbb{S}^1$.

Proof In the case $m = 2$ the image $\sigma : S^1 \to$ spec $K \equiv \gamma$ necessarily goes around smooth curve in a complex plane. Let's take a partition of curve by points (vertices) $\lambda_k \in \gamma$, $k = 1, 2, \ldots, n$, and on each arc $\lambda_{k-1}\lambda_k$ on γ we choose an arbitrary point $\tilde{\lambda}_k$. We are doing the same for S^1 taking its partition for n (equal) pieces $s_{k-1}s_k$, and then each piece corresponds to certain $\tilde{\lambda}$ (for near pieces of $\tilde{\lambda}$ we take near pieces of S^1). Each arc $s_{k-1}s_k$ in S^1 will correspond to multiplier $m_{a_k}(x)$ (let's note that every G_{a_k} is Calderon–Zygmund operator), so that

$$F(\tilde{\lambda}_k m_{a_k}) = \tilde{\lambda}_k G_{a_k},$$

and therefore

$$F\left(\sum_{k=1}^{n} \tilde{\lambda}_k m_{a_k}\right) = \sum_{k=1}^{n} \tilde{\lambda}_k G_{a_k}.$$

We will treat the latter formula in the following sense

$$F^{-1}\left(\sum_{k=1}^{n} \tilde{\lambda}_k m_{a_k} \tilde{u}\right) = \sum_{k=1}^{n} \tilde{\lambda}_k G_{a_k} u.$$

Further, we can write

$$F^{-1}\left(\sum_{k=1}^{n} \tilde{\lambda}_k m_{a_k} \tilde{u}\right) \Delta s_k = \mu \sum_{k=1}^{n} \tilde{\lambda}_k G_{a_k} \Delta\lambda_k u, \tag{6}$$

where $\mu = \frac{mes\, \mathbb{S}^1}{mes\, \gamma}$.

Since Δs_k represents a certain sector, it will be denoted by α_k. Then we we rewrite the formula (6) in the following way

$$F^{-1}\left(\sum_{k=1}^{n} \tilde{\lambda}_k m_{a_k} \tilde{u}\right) = \mu \sum_{k=1}^{n} \tilde{\lambda}_k \alpha_k^{-1} G_{a_k} \Delta\lambda_k u,$$

Now we recall that α_k, a_k are angle sizes which should be connected to $\tilde{\lambda}_k$. Using the notation $\alpha_k^{-1} G_{a_k} \equiv \tilde{G}_{\lambda_k}$ and taking into account that the limit in left-hand side of the latter formula exists we obtain (up to a constant) the line integral over γ

$$K = \int_{\gamma} \lambda \tilde{G}_\lambda \, d\lambda. \tag{7}$$

in the right-hand side.

It is obviously that if we will have some small additional information on the spectra, we can reconstruct the operator exactly. □

Remark 3 The formula (7) is very similar to spectral decomposition for a self-adjoint operator (so called spectral theorem) [9].

5 Conclusion

Some observations and connections between discrete and continuous operators are presented in the paper. These author's observations related to a spectra of operators are inspired by Mark Kac question "Can one hear the shape of a drum" [10]. The author hopes obtaining certain results for multidimensional case also.

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Compliance with ethical standards

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Research involving human participants and/or animals This article does not contain any studies with human participants or animals performed by the author.

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