THE $SU(3)$ SYMMETRY AND MACROSCOPIC DYNAMICS OF MAGNETS WITH SPIN $s = 1$

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In the Hamiltonian approach, we derive nonlinear dynamic equations for magnetic media with spin $s = 1$. We introduce two types of magnetic exchange Hamiltonians corresponding to the Casimir invariants of the $SU(3)$ group. We find the spectra of spin and quadrupole waves corresponding to the states with different symmetries under the time reversal transformation. We consider the effect of dissipative processes and find relaxation fluxes caused by the exchange symmetry of the magnetic Hamiltonian.

Keywords: spin, quadrupole matrix, dynamics, Poisson bracket, Hamiltonian approach, magnon momentum, collective excitation spectrum, relaxation

1. Introduction

The Landau–Lifshitz equation [1] defines the evolution of a magnetic medium in terms of the spin vector. This equation is well justified for the spin $s = 1/2$ and is used to study the dynamic and static properties of a number of magnetic insulators [2], [3]. Investigating quadrupole states and synthesizing high-spin molecules have required clarifying the ideology of a macroscopic description of magnets with a spin $s > 1/2$ [4]–[7]. An additional stimulus came from the discovery of Bose–Einstein condensates of neutral atoms with a nonzero spin [8]–[11]. Two points must be kept in mind regarding the development of notions pertaining to a compact description of nonequilibrium magnetic states. The first is the need to extend the magnetic degrees of freedom in systems with a spin $s > 1/2$. For pure (coherent) quantum states, these degrees of freedom are associated with the number of parameters characterizing the one-particle spin states. The normalization condition and the freedom to choose the wave-function phase lead to $N_{\rm pure}(s)=4s$ such independent parameters for the spin s [12], [13]. In [7], [14]–[16], spin-(s=1) magnets were investigated in agreement with this assumption. The equilibriums were analyzed, and the dynamics in such magnets were studied. In [17], [18], non-one-dimensional solitonic states near the $SU(3)$ point were constructed, and the perturbation spectra were then shown to exhibit a Goldstone behavior, with their decay satisfying the Adler principle. Because of the Hermiticity and the normalization condition for the density matrix in the case of mixed (noncoherent) states, the number of such parameters is $N_{\rm mix}(s) = 4s(s + 1)$. These, in particular, can be the means of the $SU(2s+1)$ Lie group generators. The spin-(s=1) magnets with this number of parameters were studied in [19]–[21]. In [21], new integrable models were obtained that describe the dynamics and soliton states of magnets for the spin value $s = 1$.

The other important point in generalizing the macroscopic description of magnets is related to the notion of normal and degenerate equilibriums of quantum objects. Normal states correspond to a paramagnetic state. The other states of magnets are states with spontaneously broken symmetry [22], [23]. Depending on the pattern of symmetry breaking due to the nature of the order parameter, adequately

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treating the system also requires extending the set of macroscopic parameters. Such a generalization in the investigation of magnets was done in [24]–[26] both in the Lagrangian and Hamiltonian formalisms and in the microscopic approach.

Several options of dynamical behavior with different full sets of parameters of the compact description can be realized in spin-1 magnets. The set of these parameters essentially depends on the symmetry of the Hamiltonian and the symmetry of the equilibrium, which may not coincide in general. In the simplest case, the symmetry of the $SU(3)$ equilibrium and the symmetry of the Hamiltonian coincide (normal states). The spin and the quadrupole matrix are integrals of motion. In another possible case, the symmetry of the equilibrium is lower than the $SO(3)$ symmetry of the Hamiltonian. Quadrupole degrees of freedom are order parameters, and only the spin vector components are integrals of motion (degenerate states). The problem of classifying magnetic equilibriums in that case was studied in [27]. Finally, the case of the total spontaneous violation of the $SU(3)$ symmetry of the equilibrium is possible, with the Hamiltonian having the $SU(3)$ symmetry (degenerate states). In this last case, the total number of dynamical variables is 16 [20]. In this paper, we consider the first of these cases in detail. Using the variational principle of mechanics, we obtain Poisson brackets for a set of macroscopic parameters and find nonlinear equations for the dynamics of magnetic media. We introduce two types of magnetic exchange Hamiltonians and suggest model expressions for homogeneous and inhomogeneous energy densities. We investigate magnetic equilibriums and reveal their stability conditions. We evaluate the spectra of spin and quadrupole waves, which differ in the symmetry of the equilibrium under time reversal. We write dissipative Poisson brackets for densities of additive integrals of motion and reveal the character of relaxation processes in a magnetic medium.

2. Poisson brackets of spin-1 magnetic degrees of freedom

According to the general approach in the mechanics of continuous media, the Lagrangian of an arbitrary physical system is represented as $L = L_{\rm k} - H$, where $L_{\rm k}(\phi, \dot{\phi}) = \int d^3x\, F_a(\mathbf{x}, \phi(\mathbf{x}'))\dot{\phi}_a(\mathbf{x})$ is the kinematic part of the Lagrangian and $H(\phi) = \int d^3x\, \varepsilon(\mathbf{x}, \phi)$ is the system Hamiltonian. The energy density $\varepsilon(\mathbf{x}, \phi(\mathbf{x}'))$ of the medium and the quantities $F_a(\mathbf{x}, \phi(\mathbf{x}'))$ are certain functionals of the dynamical variables $\phi_a(\mathbf{x})$. The stationary action principle implies the dynamical equations for these quantities:

$$\dot{\phi}_a(\mathbf{x}) = \int d^3\mathbf{x}'\, J^{-1}_{ab}(\mathbf{x}, \mathbf{x}'; \phi)\frac{\delta H(\phi)}{\delta\phi_b(\mathbf{x}')} = \{\phi_a(\mathbf{x}), H(\phi)\}, \tag{2.1}$$

where the matrix $J_{ab}(\mathbf{x}, \mathbf{x}'; \phi)$ is defined by

$$J_{ab}(\mathbf{x}, \mathbf{x}'; \phi) = \frac{\delta F_b(\mathbf{x}', \phi)}{\delta\phi_a(\mathbf{x})} - \frac{\delta F_a(\mathbf{x}, \phi)}{\delta\phi_b(\mathbf{x}')}. \tag{2.2}$$

The transformations $\phi_a(\mathbf{x}) \to \phi'_a(\mathbf{x}) = \phi_a(\mathbf{x}, \phi(\mathbf{x}'))$ that leave the kinematic part of the action invariant, $L_{\rm k}(\phi, \dot{\phi}) = L_{\rm k}(\phi', \dot{\phi}')$, are canonical if the relation $F_b(\mathbf{x}'; \phi) = \int d^3x\, F_a(\mathbf{x}; \phi')\delta\phi'_a(\mathbf{x})/\delta\phi_b(\mathbf{x}')$ is satisfied. In the case of infinitesimal transformations $\phi_a(\mathbf{x}) \to \phi'_a(\mathbf{x}) = \phi_a(\mathbf{x}) + \delta\phi_a(\mathbf{x}; \phi(\mathbf{x}'))$, the last formula implies the equality

$$\delta\phi_a(\mathbf{x}) = \{\phi_a(\mathbf{x}), G\}, \tag{2.3}$$

where $G(\phi) \equiv \int d^3x\, F_a(\mathbf{x}, \phi)\delta\phi_a(\mathbf{x})$ is a generator of infinitesimal canonical transformations. We use relation (2.3) to find Poisson brackets of the full set of macroscopic variables. For this, we introduce an expression for the kinematic part of the Lagrangian of the magnetic medium under study:

$$L_{\rm k}(\mathbf{x}) = b_{\alpha\beta}(\mathbf{x})\dot{a}_{\beta\alpha}(\mathbf{x}) \equiv \operatorname{Sp}\hat{b}(\mathbf{x})\dot{\hat{a}}(\mathbf{x}). \tag{2.4}$$

The Hermitian 3×3 matrices $b_{\alpha\beta}$ and $a_{\alpha\beta}$ ($\hat{a} = \hat{a}^{+}$ and $\hat{b} = \hat{b}^{+}$) each contain nine independent variables. For convenience and to shorten the formulas, we omit the spin indices of the matrix elements wherever possible.

We find the infinitesimal canonical transformations $\delta\phi(\mathbf{x};\phi)$ that leave the kinematic part of the Lagrangian invariant. On one hand, knowing their explicit form and, on the other hand, taking relation (2.3) into account, we easily find the Poisson brackets of the dynamical variables. In particular, it is easy to see that the variations $\delta b_{\alpha\beta}(\mathbf{x}) = 0$, $\delta a_{\alpha\beta}(\mathbf{x}) \neq 0$ (the functions $\delta a_{\alpha\beta}(\mathbf{x})$ are independent of $\hat{a}(\mathbf{x})$ and $\hat{b}(\mathbf{x})$) leave the kinematic part of the Lagrangian invariant and, in accordance with relation (2.3), are representable in the forms

$$\delta a_{\alpha\beta}(\mathbf{x}) = \{a_{\alpha\beta}(\mathbf{x}), G\}, \qquad \delta b_{\alpha\beta}(\mathbf{x}) = \{b_{\alpha\beta}(\mathbf{x}), G\}, \tag{2.5}$$

where the generator of the transformations is $G = \int d^3x\, b_{\alpha\beta}(\mathbf{x})\delta a_{\beta\alpha}(\mathbf{x})$. The Poisson brackets of these matrices hence follow:

$$\{b_{\alpha\beta}(\mathbf{x}), b_{\mu\nu}(\mathbf{x}')\} = 0, \qquad \{a_{\alpha\beta}(\mathbf{x}), b_{\mu\nu}(\mathbf{x}')\} = \delta_{\alpha\nu}\delta_{\beta\mu}\delta(\mathbf{x}-\mathbf{x}'). \tag{2.6}$$

Next, using the kinematic part of the Lagrangian in the form $L_{\mathrm{k}}(\mathbf{x}) \equiv -\operatorname{Sp}\dot{\hat{b}}(\mathbf{x})\hat{a}(\mathbf{x})$, which differs from function (2.4) by a total time derivative, and choosing the variations $\delta b_{\alpha\beta}(\mathbf{x}) \neq 0$, $\delta a_{\alpha\beta}(\mathbf{x}) = 0$, we similarly find

$$\{a_{\alpha\beta}(\mathbf{x}), a_{\mu\nu}(\mathbf{x}')\} = 0, \qquad \{b_{\alpha\beta}(\mathbf{x}), a_{\mu\nu}(\mathbf{x}')\} = -\delta_{\alpha\nu}\delta_{\beta\mu}\delta(\mathbf{x}-\mathbf{x}'). \tag{2.7}$$

The obtained expressions for the Poisson brackets agree with the Hermiticity condition for the matrices and satisfy the Jacobi identity. The dynamical equations for the canonically conjugate variables have the forms

$$\dot{a}_{\alpha\beta}(\mathbf{x}) = \frac{\delta H(\hat{a},\hat{b})}{\delta b_{\beta\alpha}(\mathbf{x})}, \qquad \dot{b}_{\alpha\beta}(\mathbf{x}) = -\frac{\delta H(\hat{a},\hat{b})}{\delta a_{\beta\alpha}(\mathbf{x})}. \tag{2.8}$$

We next relate these matrices to physical variables. We introduce the matrix

$$\hat{g}(\mathbf{x}) \equiv i[\hat{b}(\mathbf{x}), \hat{a}(\mathbf{x})] \tag{2.9}$$

(here and hereafter, square brackets denote the commutator of two matrices). The Hermitian matrix $\hat{g}(\mathbf{x})$ contains eight independent variables in view of the equality $\operatorname{Sp}\hat{g}(\mathbf{x}) = 0$. The symmetric and antisymmetric parts of this matrix are used to define physical variables—the densities of the quadrupole matrix $q_{\alpha\beta}(\mathbf{x})$ and the spin $s_\alpha(\mathbf{x})$:

$$g_{\alpha\beta}(\mathbf{x}) \equiv q_{\alpha\beta}(\mathbf{x}) - \frac{i\varepsilon_{\alpha\beta\gamma}s_\gamma(\mathbf{x})}{2}. \tag{2.10}$$

The quadrupole matrix $q_{\alpha\beta}$ is symmetric and traceless: $q_{\alpha\beta} = q_{\beta\alpha}$, $q_{\alpha\alpha} = 0$. Five of its independent components can be parameterized by the relation

$$q_{\alpha\beta} = q\left(e_\alpha e_\beta - \frac{1}{3}\delta_{\alpha\beta}\right) + q'\left(f_\alpha f_\beta - \frac{1}{3}\delta_{\alpha\beta}\right), \tag{2.11}$$

where q and q' are the matrix moduli. The vectors d_α, e_α, and $f_\alpha = (\mathbf{d}\times\mathbf{e})_\alpha$ constitute an orthonormalized frame. Formula (2.9) can be considered a generalization of the classical analogue of the Holstein–Primakoff relation [28] for spin-1 magnets.

Relations (2.6), (2.7), and (2.9) allow finding the Poisson brackets of the matrices $\hat{a}(\mathbf{x})$ and $\hat{g}(\mathbf{x})$:

$$i\big\{g_{\alpha\beta}(\mathbf{x}), g_{\gamma\rho}(\mathbf{x}')\big\} = \big(-g_{\alpha\rho}(\mathbf{x})\delta_{\gamma\beta} + g_{\gamma\beta}(\mathbf{x})\delta_{\alpha\rho}\big)\delta(\mathbf{x}-\mathbf{x}'), \tag{2.12a}$$

$$i\big\{a_{\alpha\beta}(\mathbf{x}), g_{\gamma\rho}(\mathbf{x}')\big\} = \big(-a_{\alpha\rho}(\mathbf{x})\delta_{\gamma\beta} + a_{\gamma\beta}(\mathbf{x})\delta_{\alpha\rho}\big)\delta(\mathbf{x}-\mathbf{x}'). \tag{2.12b}$$

We note that $\{\det \hat{a}(\mathbf{x}), g_{\gamma\rho}(\mathbf{x}')\} = 0$, and because the right-hand sides of Poisson brackets (2.12) are linear, we can therefore set $\det \hat{a} = 1$ without restricting the generality. Hence, the matrix $\hat{a}(\mathbf{x})$ is a function of eight independent variables. The matrices $\hat{g}(\mathbf{x})$ constitute a subalgebra with respect to Poisson brackets (2.12a) and contain two Casimir invariants:

$$\begin{aligned} &g_2(\mathbf{x}) \equiv \operatorname{Sp} \hat{g}^2(\mathbf{x}), \qquad && g_3(\mathbf{x}) \equiv \operatorname{Sp} \hat{g}^3(\mathbf{x}), \\ &\{g_2(\mathbf{x}), g_{\alpha\beta}(\mathbf{x}')\} = 0, \qquad && \{g_3(\mathbf{x}), g_{\alpha\beta}(\mathbf{x}')\} = 0. \end{aligned} \tag{2.13}$$

The momentum density of a magnetic system is expressed in terms of canonical variables as

$$\pi_k(\mathbf{x}) \equiv -\operatorname{Sp} \hat{b}(\mathbf{x}) \nabla_k \hat{a}(\mathbf{x}). \tag{2.14}$$

It is obvious that in view of formulas (2.6), (2.7), and (2.14), the relations

$$\{\mathsf{P}_k, a_{\alpha\beta}(\mathbf{x})\} = \nabla_k a_{\alpha\beta}(\mathbf{x}), \qquad \{\mathsf{P}_k, b_{\alpha\beta}(\mathbf{x})\} = \nabla_k b_{\alpha\beta}(\mathbf{x})$$

hold, where $\mathsf{P}_k = \int d^3x\, \pi_k(\mathbf{x})$ is the momentum of the medium.

Using the obtained relations (2.12), we find and analyze the dynamics of normal spin-1 magnetic media. In accordance with formulas (2.12a), the Poisson-bracket algebra of arbitrary functionals $A = A(\hat{g}(\mathbf{x}))$ and $B = B(\hat{g}(\mathbf{x}))$ for such states becomes

$$\{A, B\} = i \int d^3x \; \operatorname{Sp} \hat{g}(\mathbf{x}) \left[\frac{\delta \widehat{B}}{\delta g(\mathbf{x})}, \frac{\delta \hat{A}}{\delta g(\mathbf{x})} \right]. \tag{2.15}$$

The exchange energy density is then a function of the matrix $\hat{g}(\mathbf{x})$ and its gradient: $\varepsilon(\mathbf{x}) = \varepsilon(\hat{g}(\mathbf{x}), \nabla \hat{g}(\mathbf{x}))$. For degenerate states, in accordance with formulas (2.12), we obtain the Poisson brackets of the functionals $A = A(\hat{g}(\mathbf{x}), \hat{a}(\mathbf{x}))$ and $B = B(\hat{g}(\mathbf{x}), \hat{a}(\mathbf{x}))$ as

$$\begin{aligned} \{A, B\} = i \int d^3x \; \operatorname{Sp} \hat{g}(\mathbf{x}) \left[\frac{\delta \widehat{B}}{\delta g(\mathbf{x})}, \frac{\delta \hat{A}}{\delta g(\mathbf{x})} \right] + i \int d^3x \; \operatorname{Sp} \hat{g}(\mathbf{x}) \left[\frac{\delta \widehat{B}}{\delta a(\mathbf{x})}, \frac{\delta \hat{A}}{\delta g(\mathbf{x})} \right] + \\ + i \int d^3x \; \operatorname{Sp} \hat{g}(\mathbf{x}) \left[\frac{\delta \widehat{B}}{\delta g(\mathbf{x})}, \frac{\delta \hat{A}}{\delta a(\mathbf{x})} \right]. \end{aligned} \tag{2.16}$$

In this case, the energy density $\varepsilon(\mathbf{x}) = \varepsilon\big(\hat{g}(\mathbf{x}), \nabla \hat{g}(\mathbf{x}), \hat{a}(\mathbf{x}), \nabla \hat{a}(\mathbf{x})\big)$ is a function of both matrices. Formulas (2.15) and (2.16) allow obtaining equations for the magnetic dynamics in the framework of the Hamiltonian approach.

3. Symmetry of the exchange Hamiltonian and equilibrium

The magnetic Hamiltonian $H = \mathsf{H} + V \equiv \int d^3x\, e(\mathbf{x})$ involves strong exchange interactions $\mathsf{H} \equiv \int d^3x\, \varepsilon(\mathbf{x})$ and weak, lower-symmetry relativistic interactions V. The symmetry of the exchange Hamiltonian and of the equilibrium of the medium allows finding several thermodynamic parameters describing the macroscopic magnetic states. To formulate these symmetry properties, we use the construction of the Gibbs equilibrium distribution, which is not part of Hamiltonian mechanics. We give the necessary mathematical formulation and physical clarifications regarding the use of the terms "normal" and "degenerate" equilibriums below, using the language of quantum mechanics. In the case of the $SO(3)$ symmetry, the exchange interaction Hamiltonian and normal equilibriums described by the Gibbs statistical operator $\widehat{w}(Y) = e^{\Omega - Y_a \hat{\gamma}_a}$ satisfy the equalities

$$\lfloor \widehat{\mathsf{H}}, \widehat{S}_\alpha \rfloor = 0, \qquad \lfloor \widehat{w}, \widehat{\Sigma}_\alpha(\mathbf{Y}) \rfloor = 0. \tag{3.1}$$

The generalized operator of spin moment introduced here as

$$\widehat{\Sigma}_\alpha(\mathbf{Y}) \equiv \widehat{S}_\alpha + S^{\mathbf{Y}}_\alpha, \qquad S^{\mathbf{Y}}_\alpha \equiv -i\varepsilon_{\alpha\beta\gamma} Y_\beta \frac{\partial}{\partial \mathbf{Y}_\gamma} \tag{3.2}$$

acts in both the Hilbert space and the space of thermodynamic forces $Y_a = (Y_0, Y_\alpha)$. The thermodynamic forces determine the temperature $Y_0^{-1} \equiv T$ and the internal magnetic field $-Y_\alpha / Y_0 \equiv h_\alpha$. By definition (3.2), the relation $\lfloor Y_\beta \widehat{S}_\beta, \widehat{\Sigma}_\alpha(\mathbf{Y}) \rfloor = 0$ holds, which implies (3.1). Relations (3.1) mean that the Hamiltonian and the equilibrium are invariant under unitary transformations of homogeneous spin rotations $\widehat{U} = \exp\{i\theta_\alpha \widehat{\Sigma}_\alpha(\mathbf{Y})\}$, whose generator is the operator in (3.2). Degenerate equilibriums have a symmetry lower than that of the Hamiltonian, with $\lfloor \widehat{w}, \widehat{\Sigma}_\alpha(\mathbf{Y}) \rfloor \neq 0$. It hence follows that the equilibrium depends on the unitary transformation parameters: $\widehat{w} = \widehat{w}(Y, \theta_\alpha)$.

The Gibbs statistical operator of normal equilibriums of magnets with the $SU(3)$ symmetry has a similar form. In addition to the Hamiltonian, the set of additive integrals of motion $\hat{\gamma}_a \equiv (\widehat{\mathsf{H}}, \widehat{G}_{\alpha\beta})$ then contains the matrix operator $\widehat{G}_{\alpha\beta} = \int d^3x \, \hat{g}_{\alpha\beta}(x)$. Here, following [29], we introduce the tensor density operator $\hat{g}_{\alpha\beta}(\mathbf{x}) \equiv \widehat{\psi}^+_\alpha(\mathbf{x}) \widehat{\psi}_\beta(\mathbf{x}) - \delta_{\alpha\beta} \widehat{\psi}^+_\gamma(\mathbf{x}) \widehat{\psi}_\gamma(\mathbf{x})/3$ in terms of the Bose field creation and annihilation operators $\widehat{\psi}^+_\alpha(\mathbf{x})$ and $\widehat{\psi}_\alpha(\mathbf{x})$. The $SU(3)$ symmetry of normal equilibriums is formulated similarly to relations (3.1) and (3.2). For this, we introduce the operators

$$\widehat{G}_{\alpha\beta}(\mathbf{Y}) \equiv \widehat{G}_{\alpha\beta} + \widehat{G}^{\mathbf{Y}}_{\alpha\beta}, \qquad G^{\mathbf{Y}}_{\alpha\beta} \equiv Y_{\alpha\lambda} \frac{\partial}{\partial Y_{\beta\lambda}} - Y_{\lambda\beta} \frac{\partial}{\partial Y_{\lambda\alpha}}, \tag{3.3}$$

which satisfy the relations $\lfloor \widehat{G}_{\alpha\beta}(\mathbf{Y}), \widehat{G}_{\mu\nu}(\mathbf{Y}) \rfloor = \widehat{G}_{\alpha\nu}(\mathbf{Y}) \delta_{\beta\mu} - \widehat{G}_{\mu\beta}(\mathbf{Y}) \delta_{\alpha\nu}$. Hence, $\lfloor \widehat{G}_{\alpha\beta}(\mathbf{Y}), \mathrm{Sp}\, \widehat{\mathbf{Y}} \widehat{\mathbf{G}} \rfloor = 0$. The $SU(3)$-symmetry conditions for the Hamiltonian and normal equilibriums then become

$$\lfloor \widehat{\mathsf{H}}, \widehat{G}_{\alpha\beta} \rfloor = 0, \qquad \lfloor \widehat{w}, \widehat{G}_{\alpha\beta}(\mathbf{Y}) \rfloor = 0. \tag{3.4}$$

These formulas mean that the Hamiltonian and the equilibrium are invariant under homogeneous linear transformations $\widehat{U} = \exp\{i\theta_{\alpha\beta} \widehat{G}_{\beta\alpha}(\mathbf{Y})\}$, whose generator is the operator in (3.3). In the case of spontaneous symmetry breaking (degenerate states), $\lfloor \widehat{w}, \widehat{G}_{\alpha\beta}(\mathbf{Y}) \rfloor \neq 0$, which results in an additional dependence of the equilibrium on the parameters of the unitary transformation $\widehat{w} = \widehat{w}(Y, \theta_{\alpha\beta})$. In investigating degenerate equilibriums, the concept of quasimeans has been effectively used [30], [31].

Returning to Hamiltonian mechanics, we note that in this approach, the difference in the description of nonequilibrium normal and degenerate states is manifested as a functional dependence of energy on macroscopic quantities. For normal spin-1 magnetic states, the Hamiltonian is a functional of only the matrix $\hat{g}(\mathbf{x})$: $H = H(\hat{g}(\mathbf{x}))$. Hence, using formulas (2.1) and (2.15), we obtain the dynamical equation for this variable,

$$\dot{\hat{g}}(\mathbf{x}) = i \left[\hat{g}(\mathbf{x}), \frac{\delta \widehat{H}(\hat{g})}{\delta g(\mathbf{x})} \right], \tag{3.5}$$

which generalizes the Landau–Lifshitz equation to the spin-(s=1) magnetic medium under consideration. In terms of the matrices $q_{\alpha\beta}$ and $\varepsilon_{\alpha\beta} \equiv \varepsilon_{\alpha\beta\gamma} s_\gamma$, the last equation can be rewritten in the form

$$\begin{aligned} \dot{\hat{q}}(\mathbf{x}) &= \frac{1}{2} \left[\hat{\varepsilon}(\mathbf{x}), \frac{\delta \widehat{H}(\hat{q}, \hat{\varepsilon})}{\delta q(\mathbf{x})} \right] - 2 \left[\hat{q}(\mathbf{x}), \frac{\delta \widehat{H}(\hat{q}, \hat{\varepsilon})}{\delta \varepsilon(\mathbf{x})} \right], \\ \dot{\hat{\varepsilon}}(\mathbf{x}) &= 2 \left[\frac{\delta \widehat{H}(\hat{q}, \hat{\varepsilon})}{\delta q(\mathbf{x})}, \hat{q}(\mathbf{x}) \right] + 2 \left[\frac{\delta \widehat{H}(\hat{q}, \hat{\varepsilon})}{\delta \varepsilon(\mathbf{x})}, \hat{\varepsilon} \right]. \end{aligned} \tag{3.6}$$

The right-hand sides of these equations demonstrate the mutual effect of spin and the quadrupole matrix on their dynamical behaviors. If we neglect the dependence of the Hamiltonian on the quadrupole matrix in Eqs. (3.6), then the second of these equations passes into an equation in [1]. The structure of Eq. (3.5) is such that the Casimir invariants are unchanged: $g_2 = \text{const}$ and $g_3 = \text{const}$. The existence of these invariants reduces the number of independent magnetic parameters to six in the two-axis case and to four in the one-axis case of the quadrupole matrix. In what follows, we refine the forms of Eqs. (3.5) and (3.6) by taking specific expressions for the exchange energy and lower-symmetry relativistic interactions into account in them.

For degenerate nonequilibriums, the Hamiltonian depends on both the matrices $\hat{g}(\mathbf{x})$ and $\hat{a}(\mathbf{x})$: $H = H(\hat{g}(\mathbf{x}), \hat{a}(\mathbf{x}))$. By virtue of formulas (2.1) and (2.16), the equations of motion for them become

$$\begin{aligned}\dot{\hat{g}}(\mathbf{x}) &= i\left[\hat{g}(\mathbf{x}), \frac{\delta\widehat{H}(\hat{g},\hat{a})}{\delta g(\mathbf{x})}\right] + i\left[\hat{a}(\mathbf{x}), \frac{\delta\widehat{H}(\hat{g},\hat{a})}{\delta a(\mathbf{x})}\right],\\ \dot{\hat{a}}(\mathbf{x}) &= i\left[\hat{a}(\mathbf{x}), \frac{\delta\widehat{H}(\hat{g},\hat{a})}{\delta g(\mathbf{x})}\right].\end{aligned}$$

4. Differential conservation laws and models of the Hamiltonian

The basic interactions in magnetic systems are exchange interactions. Considering dynamical processes in such media requires refining the formulation of conservation laws in differential form by taking the symmetry of the exchange Hamiltonian into account. In the case of the $SU(3)$ symmetry of Hamiltonian (3.1), the set of the integrals of motion consists of the exchange Hamiltonian and the matrix $G_{\alpha\beta}$:

$$\gamma_a \equiv (\mathsf{H}, G_{\alpha\beta}) = \int d^3x\,\zeta_a(\mathbf{x}), \qquad \{\mathsf{H}, \gamma_a\} = 0. \tag{4.1}$$

Here, $\zeta_a(\mathbf{x}) = \varepsilon(\mathbf{x})$ and $g_{\alpha\beta}(\mathbf{x})$ are densities of the additive integrals of motion, and $a = 0, \alpha\beta$. Using the representation of flux densities of additive integrals of motion in [32], we obtain dynamical equations reflecting the conservation laws in differential form:

$$\begin{aligned}\dot{\varepsilon}(\mathbf{x}) &= -\nabla_k q_k(\mathbf{x}), \qquad q_k(\mathbf{x}) = \frac{1}{2}\int d^3x'\,x'_k \int_0^1 d\lambda\,\{\varepsilon(\mathbf{x}+\lambda\mathbf{x}'), \varepsilon(\mathbf{x}-(1-\lambda)\mathbf{x}')\},\\ \dot{\hat{g}}(\mathbf{x}) &= -\nabla_k \hat{j}_k(\mathbf{x}), \qquad \hat{j}_k(\mathbf{x}) = \int d^3x'\,x'_k \int_0^1 d\lambda\,\{\hat{g}(\mathbf{x}+\lambda\mathbf{x}'), \varepsilon(\mathbf{x}-(1-\lambda)\mathbf{x}')\},\end{aligned} \tag{4.2}$$

where $q_k(\mathbf{x})$ is the energy flux density and $\hat{j}_k(\mathbf{x})$ is the flux density corresponding to the conserved quantity $\widehat{G}$. In obtaining the last equality, we take the $SU(3)$ symmetry of the exchange energy density into account, which is related to the invariance under homogeneous linear transformations:

$$\{\widehat{G}, \varepsilon(\mathbf{x})\} = 0. \tag{4.3}$$

Taking the relativistic interactions into account modifies the second equation in (4.2),

$$\dot{\hat{g}}(\mathbf{x}) = -\nabla_k \hat{j}_k(\mathbf{x}) + \hat{\eta}(\mathbf{x}), \tag{4.4}$$

where the source $\hat{\eta}(\mathbf{x})$ has the form

$$\hat{\eta}(\mathbf{x}) \equiv \{\hat{g}(\mathbf{x}), V\}. \tag{4.5}$$

In addition to symmetry properties (4.1), the exchange Hamiltonian H is translation invariant ($\{\mathsf{P}_k, \mathsf{H}\} = 0$) and is also invariant under rotations in the configuration space ($\{\mathsf{L}_k, \mathsf{H}\} = 0$). Here, $\mathsf{L}_k = \varepsilon_{kij} \int d^3x\, x_i \pi_j(\mathbf{x})$ is the orbital moment of the system. Symmetry relations for the exchange energy density hold in the form

$$\{\mathsf{P}_k, \varepsilon(\mathbf{x})\} = \nabla_k \varepsilon(\mathbf{x}), \qquad \{\mathsf{L}_i, \varepsilon(\mathbf{x})\} = \varepsilon_{ikl} x_k \nabla_l \varepsilon(\mathbf{x}). \tag{4.6}$$

Using relations (2.14), (4.3), and (4.6) and the flux density representation in the form proposed in [32] allows expressing the magnon momentum density conservation law in differential form:

$$\begin{aligned} &\dot{\pi}_i(\mathbf{x}) = -\nabla_k t_{ik}(\mathbf{x}), \\ &t_{ik}(\mathbf{x}) = -e(\mathbf{x})\delta_{ik} + \int d^3x'\, x'_k \int_0^1 d\lambda\, \{\pi_i(\mathbf{x} + \lambda \mathbf{x}'), \varepsilon(\mathbf{x} - (1-\lambda)\mathbf{x}')\}. \end{aligned} \tag{4.7}$$

Here, $t_{ik}(\mathbf{x})$ is the momentum flux density.

The magnetic Heisenberg Hamiltonian is given by

$$\mathsf{H} = \int d^3x\, \varepsilon(\mathbf{x}) = -\int d^3x\, d^3x'\, J(|\mathbf{x} - \mathbf{x}'|) s_\alpha(\mathbf{x}) s_\alpha(\mathbf{x}'),$$

where $J(|\mathbf{x}-\mathbf{x}'|)$ is the exchange integral of the two-particle interaction. Up to the terms quadratic in spatial gradients of spin density, we give an expression for the magnetic energy density and for this Hamiltonian:

$$\varepsilon(\mathbf{x}) = -J s_\alpha(\mathbf{x}) s_\alpha(\mathbf{x}) + \frac{1}{2} \bar{J} \nabla_k s_\alpha(\mathbf{x}) \nabla_k s_\alpha(\mathbf{x}). \tag{4.8}$$

Here, $J \equiv \int d^3x\, J(|x|)$ and $\bar{J} \equiv \int d^3x\, x^2 J(|x|)/3 > 0$ are the effective exchange integrals of the two-particle interaction. The first and second terms in (4.8) describe the respective homogeneous and inhomogeneous exchange interactions, and the functional form of the homogeneous part of this energy is determined by the Casimir invariant.

We construct an analytic form of the $SU(3)$-symmetric exchange Hamiltonian by analogy with the Heisenberg Hamiltonian. We write the magnetic Hamiltonian such that the homogeneous part of the energy density is expressed in terms of the invariants g_2 and g_3:

$$\begin{aligned} &\mathsf{H}(g_2, g_3) = \mathsf{H}(g_2) + \mathsf{H}(g_3), \\ &\mathsf{H}(g_2) = -2 \int d^3x\, d^3x'\, J(|\mathbf{x} - \mathbf{x}'|)\, \mathrm{Sp}\, \hat{g}(\mathbf{x}) \hat{g}(\mathbf{x}'), \\ &\mathsf{H}(g_3) = -\int d^3x\, d^3x'\, d^3x'' I(|\mathbf{x} - \mathbf{x}'|, |\mathbf{x} - \mathbf{x}''|, |\mathbf{x}' - \mathbf{x}''|)\, \mathrm{Sp}\, \hat{g}(\mathbf{x}) \hat{g}(\mathbf{x}') \hat{g}(\mathbf{x}''). \end{aligned} \tag{4.9}$$

Here, $J(|\mathbf{x} - \mathbf{x}'|)$ and $I(|\mathbf{x} - \mathbf{x}'|, |\mathbf{x} - \mathbf{x}''|, |\mathbf{x}' - \mathbf{x}''|)$ are the exchange integrals of the two-particle and three-particle magnetic interactions. The energy density corresponding to the Hamiltonian $\mathsf{H}(g_2)$ has the form

$$\varepsilon(\mathbf{x}, g_2(\mathbf{x})) = -2J g_2(\mathbf{x}) + \bar{J}\, \mathrm{Sp}\, \nabla_k \hat{g}(\mathbf{x}) \nabla_k \hat{g}(\mathbf{x}). \tag{4.10}$$

The signs and the coefficient in energy density (4.10) are chosen such that this expression passes into formula (4.8) in the absence of quadrupole degrees of freedom. Similarly, the Hamiltonian energy density $\mathsf{H}(g_3)$ has the form

$$\varepsilon(\mathbf{x}, g_3(\mathbf{x})) = -I g_3(\mathbf{x}) + 2\bar{I}\, \mathrm{Sp}\, \hat{g}(\mathbf{x}) \nabla_k \hat{g}(\mathbf{x}) \nabla_k \hat{g}(\mathbf{x}) \tag{4.11}$$

and satisfies $SU(3)$-symmetry relation (4.3) with respect to transformations whose generators are the $\widehat{G}$ matrices. Here,

$$I \equiv \int d^3x\, d^3x' I(|\mathbf{x}|, |\mathbf{x}'|, |\mathbf{x}''|),$$

$$\bar{I} \equiv \int d^3x\, d^3x' (x^2 + x'^2 - \mathbf{x}\mathbf{x}') \frac{I(|\mathbf{x}|, |\mathbf{x}'|, |\mathbf{x} - \mathbf{x}'|)}{3}$$

are the effective exchange integrals of the homogeneous and inhomogeneous three-particle magnetic interaction.

Next, to find stationary values of the magnetic variables and analyze the linearized dynamical equations and the spectra of collective excitations, we use the homogeneous part of the magnetic energy in the form

$$e(s, q) \equiv -2J\left(\frac{2q^2}{3} + \frac{s^2}{2}\right) + B\left(\frac{2q^2}{3} + \frac{s^2}{2}\right)^2 + Aq^2, \tag{4.12}$$

where we take into account that in the one-axis case of the quadrupole matrix, $\mathrm{Sp}\, \hat{g}^2 = 2q^2/3 + s^2/2$, where s is the spin modulus and q is modulus of this matrix. The first two terms in (4.12) correspond to the exchange isotropic interaction with the $SU(3)$ symmetry, and the last term is anisotropic. These terms are necessary for the definition of the ground state of the magnet. It is easy to see that three equilibriums are possible for this model Hamiltonian. The solution

$$s_0 = q_0 = 0 \tag{4.13}$$

corresponds to a stable paramagnetic state if $J < 0$ and $J < 3A/4$. The solution

$$s_0^2 = \frac{2J}{B}, \qquad q_0 = 0 \tag{4.14}$$

is the ferromagnetic equilibrium. Its stability requires that the inequalities $B > 0$ and $8J > 3A > 0$ be satisfied. Finally, the solution

$$s_0 = 0, \qquad q_0^2 = \frac{3(4J - 3A)}{8B} \tag{4.15}$$

corresponds to the quadrupole state (spin nematic), whose stability is ensured by the inequalities $B > 0$, $A < 0$, and $8J > 3A$.

5. Spectra of collective excitations

The basic thermodynamic relation for normal states of a magnetic medium is written as

$$d\varepsilon = \mathrm{Sp}\, \frac{\partial \hat{\varepsilon}}{\partial g} d\hat{g} + \frac{\partial \varepsilon}{\partial s} ds + \mathrm{Sp}\, \frac{\partial \hat{\varepsilon}}{\partial \nabla_k g} d\nabla_k \hat{g}, \tag{5.1}$$

where s is the entropy density. Its Poisson brackets with the other variables are equal to zero. Using the symmetry property of the exchange interactions for energy density (4.3), we obtain the relation

$$\left[\frac{\partial \hat{\varepsilon}}{\partial g}, \hat{g}\right] + \left[\frac{\partial \hat{\varepsilon}}{\partial \nabla_k g}, \nabla_k \hat{g}\right] = 0.$$

Taking this equality into account, we use formulas (4.1)–(4.3) to obtain expressions for the flux densities of the additive integrals of motion:

$$\hat{j}_k = i\left[\hat{g}, \frac{\partial \hat{\varepsilon}}{\partial \nabla_k g}\right], \qquad q_k = \mathrm{Sp}\, \frac{\delta \widehat{\mathsf{H}}}{\delta g} \hat{j}_k. \tag{5.2}$$

Equations (4.2) and the flux densities in form (5.2) describe the adiabatic dynamics of normal nonequilibrium spin-1 states of spin-1 magnets.

We formulate the dynamical equation for the magnon momentum density. Because $\mathsf{H}(\hat{b},\hat{a}) = \mathsf{H}(\hat{g}(\hat{b},\hat{a}))$, neglecting the anisotropic interaction allows rewriting Eqs. (2.8) as

$$\dot{\hat{a}}(\mathbf{x}) = i\left[\hat{a}(\mathbf{x}), \frac{\delta\widehat{\mathsf{H}}}{\delta g(\mathbf{x})}\right], \qquad \dot{\hat{b}}(\mathbf{x}) = i\left[\hat{b}(\mathbf{x}), \frac{\delta\widehat{\mathsf{H}}}{\delta g(\mathbf{x})}\right].$$

Using definition (2.14), we obtain the dynamical equation for magnon momentum density (4.7), where the magnon momentum flux density is

$$t_{ik} = \delta_{ik}\left(-\varepsilon + \operatorname{Sp}\frac{\delta\widehat{\mathsf{H}}}{\delta g}\hat{g}\right) + \operatorname{Sp}\frac{\partial\hat{\varepsilon}}{\partial\nabla_k g}\nabla_i\hat{g}.$$

Using (5.2), it is easy to transform nonlinear equation (4.2) into equations for the quadrupole matrix and the antisymmetric matrix $\varepsilon_{\alpha\beta}$:

$$\dot{\hat{q}} = -\nabla_k\hat{j}_k^{(q)}, \qquad \dot{\hat{\varepsilon}} = -\nabla_k\hat{j}_k^{(\varepsilon)}. \tag{5.3}$$

The flux densities involved here are given by

$$\begin{aligned} \hat{j}_k^{(q)} &= 2\left[\hat{q}, \frac{\partial\hat{\varepsilon}}{\partial\nabla_k\varepsilon}\right] + \frac{1}{2}\left[\hat{\varepsilon}, \frac{\partial\hat{\varepsilon}}{\partial\nabla_k q}\right], \\ \hat{j}_k^{(\varepsilon)} &= 2\left[\hat{q}, \frac{\partial\hat{\varepsilon}}{\partial\nabla_k q}\right] - 2\left[\hat{\varepsilon}, \frac{\partial\hat{\varepsilon}}{\partial\nabla_k\varepsilon}\right]. \end{aligned} \tag{5.4}$$

Our subsequent analysis of Eqs. (5.3) and (5.4) involves model expressions for the energy density in (4.10) and (4.12). Using (4.12), we obtain

$$\dot{\hat{q}} = \bar{J}[\Delta\hat{\varepsilon}, \hat{q}] + \bar{J}[\Delta\hat{q}, \hat{\varepsilon}], \qquad \dot{\hat{\varepsilon}} = 4\bar{J}[\hat{q}, \Delta\hat{q}] + \bar{J}[\Delta\hat{\varepsilon}, \hat{\varepsilon}]. \tag{5.5}$$

Linearizing these equations near the equilibrium $(\hat{\varepsilon}_0)_{\alpha\beta} \equiv 0$, $(\hat{q}_0)_{\alpha\beta} \neq 0$ (T-even states, a spin nematic) and passing to the Fourier representation, we obtain the dispersion equation

$$\det\widehat{D}(\mathbf{k},\omega) = 0, \qquad D_{\beta\alpha}(\mathbf{k},\omega) = \delta_{\beta\alpha}(\omega^2 - 2\bar{J}^2k^4\operatorname{Sp}(\hat{q}_0^2)) + 3\bar{J}^2k^4(\hat{q}_0^2)_{\beta\alpha}.$$

In the equilibrium, if the quadrupole matrix is one-axis, we obtain the solutions $\omega = 0$ and $\omega = \pm\bar{J}k^2q_0$, where q_0 is defined by formula (4.15). For a two-axis quadrupole matrix in the equilibrium, the solution of linearized Eqs. (5.5) leads to three spectra of quadrupole waves:

$$\omega_\pm^{(1)} = \pm 2\bar{J}k^2q_0, \qquad \omega_\pm^{(2)} = \pm 2\bar{J}k^2q_0', \qquad \omega_\pm^{(3)} = \pm 2\bar{J}k^2|q_0 - q_0'|.$$

Here, q_0 and q_0' are moduli of this quadrupole matrix in the equilibrium.

Linearizing equations (5.3) and (5.4) near the equilibrium $(\hat{\varepsilon}_0)_{\alpha\beta} \neq 0$, $(\hat{q}_0)_{\alpha\beta} \equiv 0$ (T-odd states, a ferromagnet), we obtain the spectra of spin waves $\omega_\pm^{(1)} = 0$, $\omega_\pm^{(2)} = \pm\bar{J}k^2s_0$, and $\omega_\pm^{(3)} = \pm\bar{J}k^2s_0/2$, where s_0 is defined by formula (4.14). The parabolic dependence of the frequency on the wave vector in these spectra agrees with the results in [33].

We similarly consider dynamical processes associated with the magnetic Hamiltonian $\mathsf{H}(g_3)$. As a result, we obtain the dispersion equation

$$7\delta s_\beta(\mathbf{k},\omega)D_{\beta\alpha}(\mathbf{k},\omega) = 0,$$

$$D_{\beta\alpha}(\mathbf{k},\omega) \equiv \delta_{\beta\alpha}\big(\omega^2 - 4\bar{I}^2k^4(2\operatorname{Sp}\hat{q}_0^4 - (\operatorname{Sp}q_0^2)^2)\big) + 4\bar{I}^2k^4\big[3(\hat{q}_0^4) - 2(\hat{q}_0^2)\operatorname{Sp}\hat{q}_0^2\big]_{\beta\alpha}.$$

If the state of a spin nematic is one-axis, then we obtain two spectra of quadrupole waves: $\omega = 0$ and $\omega = \pm 2\bar{I}q_0^2k^2/3$. In the two-axis case of the quadrupole matrix, we obtain the excitation spectra

$$\omega_\pm^{(1)} = \pm\frac{2\bar{I}k^2}{3}\sqrt{(q_0^2 - q_0'^2)^2 + 4q_0q_0'(q_0^2 + q_0'^2)},$$

$$\omega_\pm^{(2)} = \pm\frac{2\bar{I}k^2}{3}\sqrt{q_0'(q_0'^3 + 4q_0^3 + 4q_0^2q_0')},$$

$$\omega_\pm^{(3)} = \pm\frac{2\bar{I}k^2}{3}\sqrt{q_0(q_0^3 + 4q_0'^3 + 4q_0'^2q_0)}.$$

They are qualitatively similar to the case of the Hamiltonian $\mathsf{H}(g_2)$, but the spectrum dependence on the quadrupole matrix moduli is more complicated.

6. Taking dissipative processes into account

We consider relaxation processes in spin-1 magnets. For this, we use the approach proposed in [34], where dissipative Poisson brackets were introduced in the Hamiltonian framework and relaxation dynamical equations were obtained for a number of normal and degenerate condensed media. The equations of motion for the density of additive integrals of motion are written as

$$\dot{\zeta}_a(\mathbf{x}) \equiv \{\zeta_a(\mathbf{x}), H\} - T_0\{\zeta_a(\mathbf{x}), \Sigma\}_{\mathrm{D}}, \tag{6.1}$$

where $\Sigma = \int d^3x\, s(\mathbf{x})$ is the entropy and T_0 is a constant with the dimension of temperature. The reactive Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ describes the system dynamics in the adiabatic approximation, and the dissipative bracket $\{\,\cdot\,,\,\cdot\,\}_{\mathrm{D}}$ describes relaxation processes. The reactive Poisson brackets are antisymmetric and satisfy the Leibnitz rule and the Jacobi identity:

$$\{A,B\} = -\{B,A\}, \qquad \{A,BC\} = -\{A,B\}C + B\{A,C\},$$

$$\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0.$$

The dissipative brackets are symmetric and satisfy the Leibnitz identity:

$$\{A,B\}_{\mathrm{D}} = \{B,A\}_{\mathrm{D}}, \qquad \{A,BC\}_{\mathrm{D}} = \{A,B\}_{\mathrm{D}}C + B\{A,C\}_{\mathrm{D}}.$$

Because the relaxation equations of motion are formulated in what follows for densities of additive integrals of motion, we use formulas (2.12) and (5.2) to obtain the reactive Poisson brackets for these variables in advance:

$$\begin{aligned}\{\zeta_a(\mathbf{x}), \zeta_b(\mathbf{x}')\} = &-i\delta_{a,\alpha\beta}\delta_{\beta,\gamma\rho}\delta(\mathbf{x}-\mathbf{x}')(g_{\gamma\beta}(\mathbf{x})\delta_{\alpha\rho} - g_{\alpha\rho}(\mathbf{x})\delta_{\gamma\beta}) + \\ &+ \big[\delta_{a0}\zeta_{bk}^{(0)}(\mathbf{x}) + \delta_{b0}\zeta_{ak}^{(0)}(\mathbf{x}')\big]\nabla_k'\delta(\mathbf{x}-\mathbf{x}').\end{aligned} \tag{6.2}$$

The right-hand side of Poisson brackets (6.2) is given in terms of densities and the corresponding fluxes of additive integrals of motion (5.2). The validity of this relation can be proved easily, using the assumption that the Casimir invariants g_2 and g_3 given by (2.13) are independent of the coordinate: $g_2 = \text{const}$ and $g_3 = \text{const}$.

Taking the relaxation processes into account leads to the appearance of dissipative terms in the dynamical equations for the densities of the additive integrals of motion:

$$\dot{\zeta}_a(\mathbf{x}) = -\nabla_k(\zeta_{ak}^{(0)}(\mathbf{x}) + \zeta_{ak}^{(1)}(\mathbf{x})) \equiv L_a^{\mathrm{R}}(\mathbf{x}) + L_a^{\mathrm{D}}(\mathbf{x}), \tag{6.3}$$

where in view of (6.1), we have

$$L_a^{\mathrm{D}}(\mathbf{x}) = -T_0 \int d^3x' \frac{\delta\Sigma}{\delta\zeta_b(\mathbf{x}')}\{\zeta_a(\mathbf{x}), \zeta_b(\mathbf{x}')\}_{\mathrm{D}}.$$

Relation (5.1) and (6.3) allow obtaining the dynamical equation for the entropy density

$$\dot{s}(\mathbf{x}) = -\nabla_k j_{sk}^{(1)}(\mathbf{x}) + I(\mathbf{x}), \tag{6.4}$$

where

$$j_{sk}^{(1)} = Y_a\zeta_{ak}^{(1)}, \qquad I = \zeta_{ak}^{(1)}\nabla_k Y_a \tag{6.5}$$

are the respective dissipative flux density and entropy production. The dissipative Poisson brackets can be expressed explicitly in terms of the dissipation function, which for the magnetic medium under consideration is given by

$$R \equiv \frac{1}{2}\int d^3x\, \nabla_k Y_a(\mathbf{x}) I_{ak,bl}(\mathbf{x}) \nabla_l Y_b(\mathbf{x}) = \int d^3x\, r(\mathbf{x}). \tag{6.6}$$

Here, $Y_a(\mathbf{x}) = \delta\Sigma/\delta\zeta_a(\mathbf{x})$ are the thermodynamic forces conjugate to the additive integrals of motion, and $I_{ak,bl}$ are the generalized kinetic coefficients, which satisfy the Onsager principle of symmetry of kinetic coefficients

$$I_{ak,bl} = I_{bl,ak}. \tag{6.7}$$

Because the matrices $\hat{g}$ are traceless, the additional relations $I_{\alpha\alpha k,bl} = 0$ and $I_{ak,\gamma\gamma l} = 0$ hold. With formulas (6.1), (6.3), and (6.6), we see that the dissipation function is related to the density of dissipative fluxes of additive integrals of motion as

$$L_a^{\mathrm{D}}(\mathbf{x}) = -\nabla_k\zeta_{ak}^{(1)}(\mathbf{x}) = \frac{\delta R}{\delta Y_a(\mathbf{x})}. \tag{6.8}$$

With expression (6.6), the dissipative Poisson brackets of the densities of additive integrals of motion become

$$\{\zeta_a(\mathbf{x}), \zeta_b(\mathbf{x}')\}_{\mathrm{D}} \equiv -\frac{\delta^2 R}{\delta Y_a(\mathbf{x})\delta Y_b(\mathbf{x}')} = -\frac{1}{T_0}\nabla_k\nabla_l'(I_{ak,bl}(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')). \tag{6.9}$$

In the exchange approximation, the tensor structure of the generalized kinetic coefficients is such that the spatial and spin indices do not mix and there are no preferred directions in the configuration space. Therefore, $I_{ak,bl} = \delta_{kl}I_{ab}$. In this case, we obtain the dissipative flux densities of additive integrals of motion in the form

$$\begin{aligned} j_{\alpha\beta}^{(1)k} &= -D_{\alpha\beta}\nabla_k T - \sigma_{\alpha\beta,\gamma\rho}\nabla_k h_{\rho\gamma}, \\ q_k^{(1)} &= -(\kappa + h_{\beta\alpha}D_{\alpha\beta})\nabla_k T - TD_{\alpha\beta}\nabla_k h_{\beta\alpha} - \sigma_{\alpha\beta,\gamma\rho}h_{\beta\alpha}\nabla_k h_{\rho\gamma}. \end{aligned} \tag{6.10}$$

The kinetic heat conduction coefficient κ, the magnetic thermodiffusion coefficient $D_{\alpha\beta}$, and the magnetic diffusion coefficient $\sigma_{\alpha\beta,\gamma\rho}$ are related to the generalized kinetic coefficients as

$$I_{\alpha\beta,0} = T^2 D_{\alpha\beta} + T h_{\gamma\rho}\sigma_{\alpha\beta,\rho\gamma}, \qquad I_{\alpha\beta,\gamma\rho} = T\sigma_{\alpha\beta,\rho\gamma},$$
$$I_{0,0} = T^2\kappa + 2T^2 h_{\gamma\rho}D_{\rho\gamma} + T h_{\beta\alpha}h_{\gamma\rho}\sigma_{\alpha\beta,\rho\gamma}.$$

Using formulas (6.5) and (6.10), we find the dissipative flux and the entropy production:

$$j^{(1)}_{sk} = -\frac{\kappa}{T}\nabla_k T - D_{\alpha\beta}\nabla_k h_{\beta\alpha},$$
$$I = \left(\frac{\sqrt{\kappa}}{T}\nabla_k T + \frac{D_{\alpha\beta}}{\sqrt{\kappa}}\nabla_k h_{\beta\alpha}\right)^2 + \nabla_k h_{\beta\alpha}\left(\frac{1}{T}\sigma_{\alpha\beta,\gamma\rho} - \frac{1}{\kappa}D_{\alpha\beta}D_{\gamma\rho}\right)\nabla_k h_{\rho\gamma}. \qquad (6.11)$$

Further specifying the tensor structure of the kinetic coefficients depends on the equilibrium of the system. If there are no preferred directions in the spin space in the equilibrium, i.e., $g_{\alpha\beta}=0$ and $h_{\alpha\beta}=0$ (paramagnetic states), then the expressions for the tensor kinetic coefficients are simplified and become

$$\sigma_{\alpha\beta,\gamma\rho} = \frac{1}{4}\sigma(\delta_{\alpha\rho}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\rho}) + \frac{1}{2}\sigma'\left(\delta_{\alpha\gamma}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\gamma} - \frac{2}{3}\delta_{\alpha\beta}\delta_{\gamma\rho}\right), \qquad D_{\alpha\beta} = 0. \qquad (6.12)$$

Here, σ and σ' are the respective coefficients of spin diffusion and of the quadrupole matrix diffusion. As a result, we obtain expressions for the flux densities for the matrix $\hat{g}$ and the energy density,

$$j^{(1)k}_{\alpha\beta} = \frac{1}{2}i\sigma\varepsilon_{\alpha\beta\gamma}\nabla_k h_\gamma - \sigma'\nabla_k h^s_{\alpha\beta}, \qquad q^{(1)}_k = -\kappa\nabla_k T, \qquad (6.13)$$

where $h^a_{\alpha\beta} \equiv -i\varepsilon_{\alpha\beta\gamma}h_\gamma$ and $h^s_{\alpha\beta} \equiv (h_{\alpha\beta} + h_{\beta\alpha})/2$. Formulas (6.11)–(6.13) imply expressions for the dissipative flux and entropy production:

$$j^{(1)k}_s = -\frac{\kappa}{T}\nabla_k T, \qquad I = \frac{\kappa}{T^2}(\nabla_k T)^2 + \frac{\sigma}{T}(\nabla_k h_\alpha)^2 + \frac{\sigma'}{T}(\nabla_k h^s_{\alpha\beta})^2 \geq 0.$$

The positivity of entropy production is ensured by the inequalities $\kappa \geq 0$, $\sigma \geq 0$, and $\sigma' \geq 0$.

We consider another case, where the anisotropy of a magnetic equilibrium is characterized by a vector $s_\alpha \neq 0$, $h_\alpha \neq 0$, $q_{\alpha\beta} = 0$, $h^s_{\alpha\beta} = 0$ (ferromagnetic states). Such states are not invariant under time reversal. The tensor structure of the kinetic coefficients is

$$\sigma_{\alpha\beta,\gamma\rho} = \frac{\sigma(\delta_{\alpha\rho}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\rho})}{4} + \frac{1}{2}\sigma'\left(\delta_{\alpha\gamma}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\gamma} - \frac{2\delta_{\alpha\beta}\delta_{\gamma\rho}}{3}\right) -$$
$$- \sigma_1\varepsilon_{\alpha\beta\delta}h_\delta\varepsilon_{\gamma\rho\lambda}h_\lambda + \sigma_2\left(h_\alpha h_\beta - \frac{h^2\delta_{\alpha\beta}}{3}\right)\left(h_\gamma h_\rho - \frac{h^2\delta_{\gamma\rho}}{3}\right),$$
$$D_{\alpha\beta} = \frac{i\varepsilon_{\alpha\beta\gamma}h_\gamma D}{2} + D'\left(h_\alpha h_\beta - \frac{\delta_{\alpha\beta}h^2}{3}\right). \qquad (6.14)$$

Relaxation processes near this magnetic state are characterized by two magnetic thermodiffusion coefficients and four diffusion coefficients. Formulas (6.10) and (6.14) solve the question of the form of dissipative fluxes $j^{(1)k}_{\alpha\beta}$ and $q^{(1)}_k$ in this case. By virtue of formulas (6.5) and (6.14), we obtain the expression for entropy production

$$I = \left(\frac{\sqrt{\kappa}}{T}\nabla_k T - \frac{Dh}{\sqrt{\kappa}}\nabla_k h + \frac{2D'h^2}{3\sqrt{\kappa}}\nabla_k \underline{\underline{h}}\right)^2 + \frac{\sigma h^2}{T}(\nabla_k n_\alpha)^2 +$$
$$+ \left(\sqrt{A}\nabla_k h + \frac{C}{2\sqrt{A}}\nabla_k \underline{\underline{h}}\right)^2 + \left(B - \frac{c^2}{4A}\right)(\nabla_k \underline{\underline{h}})^2 \geq 0,$$

where $\underline{\underline{h}} \equiv 3n_\alpha h^s_{\alpha\beta} n_\beta/2$, $A \equiv (\sigma + 4\sigma_1 h^2)/T - D^2 h^2/\kappa$, $B \equiv 2(3\sigma' + 2\sigma_2 h^4)/(9T) - 4D'^2 h^4/(9\kappa)$, and $C \equiv 4h^3 DD'/(3\kappa)$. The requirement of its positivity leads to the inequalities $\kappa > 0$, $\sigma > 0$, $A > 0$, and $4AB - C^2 > 0$.

We finally consider a T-invariant equilibrium, for which $s_\alpha = 0$, $h_\alpha = 0$, and $q_{\alpha\beta} = q(n_\alpha n_\beta - \delta_{\alpha\beta}/3) \neq 0$, $h^s_{\alpha\beta} \neq 0$ (states of one-axis spin nematics). In this case, the magnetic thermodiffusion tensor $D_{\alpha\beta}$ and the magnetic diffusion tensor $\sigma_{\alpha\beta,\gamma\rho}$ become

$$\begin{aligned}
\sigma_{\alpha\beta,\gamma\rho} = {} & \frac{1}{4}\sigma(\delta_{\alpha\rho}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\rho}) + \frac{1}{2}\sigma'\left(\delta_{\alpha\gamma}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\gamma} - \frac{2\delta_{\alpha\beta}\delta_{\gamma\rho}}{3}\right) + \\
& + \sigma_3 h^s_{\alpha\beta} h^s_{\gamma\rho} + \frac{1}{2}\sigma_4 (h^s_{\alpha\rho} h^s_{\beta\gamma} - h^s_{\alpha\gamma} h^s_{\beta\rho}) + \\
& + \frac{1}{2}\sigma_5 \left(h^s_{\alpha\gamma} h^s_{\beta\rho} + h^s_{\alpha\rho} h^s_{\beta\gamma} - \frac{2}{9}\delta_{\alpha\beta}\underline{\underline{h}} h^s_{\gamma\rho} - \frac{2}{9}\delta_{\gamma\rho}\underline{\underline{h}} h^s_{\alpha\beta}\right), \\
D_{\alpha\beta} = {} & \underline{\underline{D}} h^s_{\alpha\beta}.
\end{aligned}$$

Relaxation processes near this magnetic state are characterized by the magnetic thermodiffusion coefficient and five diffusion coefficients. The requirement of positive definiteness of entropy production leads to the inequalities $\kappa > 0$, $\sigma + 2\sigma_4 \underline{\underline{h}}^2/9 > 0$, $\sigma' - 2\sigma_5 \underline{\underline{h}}^2/9 > 0$, and $(3\sigma' + 2\sigma_3 \underline{\underline{h}}^2 + \sigma_5 \underline{\underline{h}}^2)\kappa > 2T\underline{\underline{h}}^2 \underline{\underline{D}}^2$.

Thus, for spin-1 magnets, relaxation processes are described in terms of three kinetic coefficient in the paramagnetic case and by seven kinetic coefficients in the case of ferromagnetic and quadrupole states. The revealed structure of relaxation terms in dynamical equations in principle allows obtaining damping decrements of the collective excitation spectra in the appropriate magnetic media.

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