Direct and Inverse Problems for an Abstract Differential Equation Containing Hadamard Fractional Derivatives

A. V. Glushak and T. A. Manaenkova
Belgorod State University, Belgorod, Russia

1. THE CAUCHY PROBLEM WITH A FRACTIONAL HADAMARD DERIVATIVE

Let X be a Banach space, and let B be a linear closed densely defined operator in X with domain $D(B)$ and with nonempty resolvent set. For $0 < \alpha < 1$, consider the Cauchy type problem

$$ {}^{A}D^{\alpha}_{1+}u(t) = Bu(t), \qquad t > 1, \tag{1} $$

$$ \lim_{t\to 1+} {}^{A}I^{1-\alpha}_{1+}u(t) = u_0, \tag{2} $$

where

$$ {}^{A}I^{1-\alpha}_{a+}u(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_a^t \left(\ln \frac{t}{s}\right)^{-\alpha} u(s)\, \frac{ds}{s} $$

is the left Hadamard fractional integral of order $1-\alpha$, $a > 0$ (see [1, p. 250; 2, p. 110]),

$$ {}^{A}D^{\alpha}_{a+}u(t) = t\frac{d}{dt}\,{}^{A}I^{1-\alpha}_{a+}u(t) $$

is the left Hadamard fractional derivative of order $\alpha \in (0, 1)$, $\Gamma(\cdot)$ is the gamma function, and $u_0 \in X$.

Examples of solutions of some particular differential equations with Hadamard fractional derivatives can be found in [2, p. 212; 3]. Note also that the results proved below were announced in [4].

Definition 1. A *solution* of problem (1), (2) is defined as a continuous function $u(t)$, $t > 1$, such that ${}^{A}I^{1-\alpha}_{1+}u(t)$ is a continuously differentiable function for $t > 1$ and $u(t)$ ranges in $D(B)$ and satisfies problem (1), (2).

Definition 2. Problem (1), (2) is said to be *uniformly well posed* if there exist an operator function ${}^{A}T_{\alpha}(t)$ (defined on X and commuting with B) and numbers $M > 0$ and $\omega \in R$ such that, for any $u_0 \in D(B)$, the function ${}^{A}T_{\alpha}(t)u_0$ is the unique solution of this problem and, in addition,

$$ \|{}^{A}T_{\alpha}(t)\| \le M(\ln t)^{\alpha-1}t^{\omega}, \qquad t > 1. $$

Along with Eq. (1), consider the inhomogeneous equation

$$ {}^{A}D^{\alpha}_{1+}u(t) = Bu(t) + h(t), \qquad t > 1. \tag{3} $$

The change of the independent variable t and the unknown function $u(t)$ by the formulas

$$ t = e^{\tau}, \qquad \tau > 0, \qquad u(t) = u(e^{\tau}) = v(\tau) \tag{4} $$

reduces problems (1), (2) and (3), (2) to the already studied problems for differential equations with Riemann–Liouville fractional derivative $D^{\alpha}_{0+}v(\tau) = \dfrac{d}{d\tau} I^{1-\alpha}_{0+}v(\tau)$, where

$$ I^{1-\alpha}_{0+}v(\tau) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\tau} (\tau - s)^{-\alpha} v(s)\, ds $$

is the left Riemann–Liouville fractional integral of order $1-\alpha$. The indicated changes of variables reduce Eq. (1), the initial condition (2), and Eq. (3) to the relations

$$ D^{\alpha}_{0+}v(\tau) = Bv(\tau), \qquad \tau > 0, \tag{5} $$

$$ \lim_{\tau \to 0+} I^{1-\alpha}_{0+}v(\tau) = u_0, \tag{6} $$

$$ D^{\alpha}_{0+}v(\tau) = Bv(\tau) + h(e^{\tau}), \qquad \tau > 0. \tag{7} $$

The solvability of problems (5), (6) and (7), (6) was established in [5]. Therefore, below in Theorems 1–3, we restate the required results on the solvability of problems (1), (2) and (3), (2) with Hadamard fractional derivatives.

Next, let $L(X)$ be the space of linear bounded operators in X, and let $E_{\alpha,\beta}(\cdot)$ be the Mittag-Leffler function given by the relation

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \qquad \alpha > 0, \qquad \beta > 0. $$

Theorem 1. *Let $B \in L(X)$ and $u_0 \in X$, and let $h(t) \in C(1,\infty)$ be an absolutely integrable function in a neighborhood of the point $t = 1$. Then problem* (1), (2) *is uniformly well posed,*

$$ {}^{A}T_{\alpha}(t)u_0 = (\ln t)^{\alpha-1} E_{\alpha,\alpha}((\ln t)^{\alpha} B)u_0, $$

and the inhomogeneous problem (3), (2) *has the unique solution given by the formula*

$$ u(t) = (\ln t)^{\alpha-1} E_{\alpha,\alpha}((\ln t)^{\alpha} B)u_0 + \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\left(\ln \frac{t}{s}\right)^{\alpha} B\right) h(s) \frac{ds}{s}. $$

Throughout the following, we use the nonnegative function

$$ f_{\tau,\alpha}(t) = t^{-1} e^{1,0}_{1,\alpha}(-\tau t^{-\alpha}), $$

$$ e^{\mu,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \mu)\Gamma(\delta - \beta n)}, \qquad \alpha > \max\{0;\beta\}, \qquad \mu, z \in C, $$

where $e^{\mu,\delta}_{\alpha,\beta}(z)$ is a Wright type function (see [6, Chap. 1]).

Condition 1. The initial element u_0 belongs to $D(B)$, and B is the generator of an exponentially bounded semigroup $T(t)$ of the class C_0; moreover, $\|T(t)\| \le Me^{\omega t}$.

Theorem 2. *Let Condition* 1 *be satisfied. Then the homogeneous problem* (1), (2) *is uniformly well posed*, *and in addition*,

$$ {}^{A}T_{\alpha}(t)u_0 = \int\limits_0^{\infty} f_{\tau,\alpha}(\ln t)T(\tau)u_0\,d\tau, \tag{8}$$

$$ \|{}^{A}T_{\alpha}(t)\| \le M(\ln t)^{\alpha-1}t^{\omega_1}, \qquad \omega_1 > \omega^{1/\alpha}. \tag{9}$$

Condition 2. One of the following requirements is satisfied: (a) the function $h(t) \in C(1,\infty)$ is absolutely integrable in a neighborhood of the point $t = 1$ and ranges in $D(B)$, and the function $Bh(t) \in C(1,\infty)$ is absolutely integrable in a neighborhood of the point $t = 1$ as well; (b) the function ${}^{A}I_{1+}^{1-\alpha}h(t)$ is continuous for $t \ge 1$ and is continuously differentiable for $t > 1$, and ${}^{A}D^{\alpha}h(t)$ is absolutely integrable in a neighborhood of the point $t = 1$.

Theorem 3. *Let problem* (1), (2) *be uniformly well posed*, *let* $u_0 \in D(B)$, *and let Condition* 2 *be satisfied. Then problem* (3), (2) *has the unique solution given by the formula*

$$ u(t) = {}^{A}T_{\alpha}(t)u_0 + \int\limits_1^{t} {}^{A}T_{\alpha}\left(\frac{t}{s}\right) h(s)\,\frac{ds}{s}. \tag{10}$$

2. CAUCHY PROBLEM WITH A REGULARIZED HADAMARD FRACTIONAL DERIVATIVE

On continuous functions $u(t)$, $t \ge a > 0$, that have left Hadamard fractional derivative of order $\alpha \in (0,1)$, we define the regularized Hadamard fractional derivative

$$ {}^{A}\partial_{a+}^{\alpha}u(t) = {}^{A}D_{a+}^{\alpha}(u(t)-u(a)) = {}^{A}D_{a+}^{\alpha}u(t) - \left(\ln\frac{t}{a}\right)^{-\alpha}\frac{u(a)}{\Gamma(1-\alpha)}$$

and consider the Cauchy problem

$$ {}^{A}\partial_{1+}^{\alpha}u(t) = Bu(t) + h(t), \qquad t \ge 1, \tag{11}$$

$$ u(1) = u_0. \tag{12}$$

Definition 3. A *solution* of problem (11), (12) is a continuous function $u(t)$, $t \ge 1$, such that ${}^{A}I_{1+}^{1-\alpha}u(t)$ is a continuously differentiable function for $t \ge 1$ and $u(t)$ ranges in $D(B)$ and satisfies problem (11), (12).

Definition 4. The homogeneous problem (11), (12) $[h(t) \equiv 0]$ is said to be *uniformly well posed* if there exist an operator function ${}^{A}S_{\alpha}(t)$ (defined on X and commuting with B) and numbers $M > 0$ and $\omega \in R$ such that the function ${}^{A}S_{\alpha}(t)u_0$ is the unique solution of this problem for each $u_0 \in D(B)$ and, in addition,

$$ \|{}^{A}S_{\alpha}(t)\| \le Mt^{\omega}, \qquad t \ge 1.$$

The change of variables (4) reduces problem (11), (12) to the problem

$$ \partial_{0+}^{\alpha}v(\tau) = Bv(\tau) + h(e^{\tau}), \qquad \tau \ge 0, \tag{13}$$

$$ v(0) = u_0 \tag{14}$$

with the Caputo fractional derivative $\partial_{0+}^{\alpha}v(\tau) = D_{0+}^{\alpha}(v(\tau) - v(0))$.

The homogeneous $[h(t) \equiv 0]$ problem (13), (14) was studied in [7, Chap. 2]. Let $g_{\tau,\alpha}(t) = t^{-\alpha}e_{1,\alpha}^{1,1-\alpha}(-\tau t^{-\alpha})$; by taking into account Theorem 3.1 in [7], we obtain the following assertion.

Theorem 4. *Let Condition* 1 *be satisfied. Then the homogeneous problem* (11), (12) *is uniformly well posed, and in addition,*

$$ {}^{A}S_\alpha(t)u_0 = \int\limits_0^\infty g_{\tau,\alpha}(\ln t)T(\tau)u_0\,d\tau, \tag{15}$$

$$ \|{}^{A}S_\alpha(t)\| \le Mt^{\omega_1}, \qquad \omega_1 > \omega^{1/\alpha}. \tag{16}$$

Corollary. *Let problem* (1), (2) *be uniformly well posed, and let* $\ln t = s$. *Then the solution of problem* (1), (2) *and the solution of the homogeneous problem* (11), (12) *are related by the formula*

$$ {}^{A}S_\alpha(\exp s)u_0 = I^{1-\alpha}_{0+,s}{}^{A}T_\alpha(\exp s)u_0.$$

This assertion is a straightforward consequence of formula 1.2.12 in [6].

Condition 3. One of the following requirements is satisfied: (a) the function $h(t) \in C[1,\infty)$ ranges in $D(B)$, and $Bh(t) \in C[1,\infty)$; (b) the function $h(t)$ is continuously differentiable for $t \ge 1$.

Theorem 5. *Let* $u_0 \in D(B)$, *let problem* (1), (2) *be uniformly well posed, and let Condition* 3 *be satisfied. Then problem* (11), (12) *has the unique solution given by the formula*

$$ u(t) = {}^{A}S_\alpha(t)u_0 + \int\limits_1^t {}^{A}T_\alpha\left(\frac{t}{s}\right)h(s)\,\frac{ds}{s}. \tag{17}$$

Proof. Let us show that if Condition 3 is satisfied, then the function

$$ w(t) = \int\limits_1^t {}^{A}T_\alpha\left(\frac{t}{s}\right)h(s)\,\frac{ds}{s}$$

satisfies the zero initial condition (12). By taking into account inequality (9) and the continuity of the function $h(t)$, we estimate the norm of the function $w(t)$ for $t \in [1, 1+\delta\,]$ and $\delta > 0$ as follows:

$$ \left\| \int\limits_1^t {}^{A}T_\alpha\left(\frac{t}{s}\right)h(s)\,\frac{ds}{s}\right\| \le M_0t^{\omega_1}\int\limits_1^t (\ln t - \zeta)^{\alpha-1}\,d\zeta = \frac{M_0t^{\omega_1}}{\alpha}(\ln t)^\alpha.$$

Consequently, the function $w(t)$ satisfies the zero initial condition (12).

The relation ${}^{A}\partial^\alpha_{1+}w(t) = {}^{A}D^\alpha_{1+}w(t)$ holds for the function $w(t)$, which is zero for $t = 1$; therefore, by Theorem 3, the first term on the right-hand side in (17) satisfies the homogeneous equation (11) and condition (12), and by Theorem 4, the second one satisfies Eq. (11) and the zero initial condition (12). In addition, Condition 3 provides the desired smoothness of the solution. The proof of the theorem is complete.

In the case in which B is a bounded operator, the assertions of Theorems 4 and 5 acquire the following form (cf. formula (4.1.66) in [2]).

Theorem 6. *Let* $B \in L(X)$ *and* $u_0 \in X$, *and let the function* $h(t)$ *belong to* $C[1,\infty)$. *Then the homogeneous problem* (11), (12) *is uniformly well posed,*

$$ {}^{A}S_\alpha(t)u_0 = E_{\alpha,1}((\ln t)^\alpha B)u_0, \tag{18}$$

and the inhomogeneous problem (11), (12) *has the unique solution given by the formula*

$$ u(t) = E_{\alpha,1}((\ln t)^\alpha B)u_0 + \int\limits_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\left(\ln\frac{t}{s}\right)^\alpha B\right)h(s)\frac{ds}{s}. \tag{19}$$

3. INVERSE PROBLEM

Consider the problem of finding a function $u(t)$ that belongs to $D(B)$ for $t \in (1, e]$ and an element $p \in X$ from the conditions

$$ {}^A D^{\alpha}_{1+} u(t) = Bu(t) + (\ln t)^{k-1} p, \tag{20} $$

$$ \lim_{t \to 1} {}^A I^{1-\alpha}_{1+} u(t) = u_0, \tag{21} $$

$$ \lim_{t \to e} {}^A I^{\beta}_{1+} u(t) = u_1, \tag{22} $$

where $k > 0$ and ${}^A I^{\beta}_{1+} u(t)$ is the left Hadamard fractional integral of order $\beta \geq 0$. (Note that ${}^A I^{\beta}_{1+}$ is the identity operator for $\beta = 0$.) The interval $t \in (1, e]$ is chosen so as to have more concise formulas.

Definition 5. A *solution* of problem (20)–(22) is a pair $(u(t), p)$, where $u(t) \in D(B)$, $t \in (1, e]$, is a continuous function such that ${}^A I^{1-\alpha}_{1+} u(t)$ is continuously differentiable for $t \in (1, e]$, $p \in X$, and $u(t)$ and p satisfy relations (20)–(22).

Problem (20)–(22) is called the *inverse problem*, as opposed to the direct Cauchy type problem (20), (21) with known element $p \in X$, and can be treated as problem of reconstructing the nonstationary term $(\ln t)^{k-1} p$ in Eq. (20) from the additional boundary condition (22).

A survey of publications on inverse problems for equations of integer order can be found in [8], while the inverse problem (20)–(22) has not been considered earlier.

Theorem 7. *Let $B \in L(X)$ and $u_0, u_1 \in X$. Problem (20)–(22) has a unique solution if and only if the condition*

$$ E_{\alpha, k+\alpha+\beta}(z) \neq 0, \qquad z \in \sigma(B), \tag{23} $$

holds on the spectrum $\sigma(B)$ of the bounded operator B.

Proof. By Theorem 1, problem (20)–(22) can be reduced to the problem of finding a function $u(t)$ and an element $p \in X$ such that

$$ u(t) = (\ln t)^{\alpha-1} E_{\alpha,\alpha}((\ln t)^{\alpha} B) u_0 + \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\left(\ln \frac{t}{s}\right)^{\alpha} B\right) (\ln s)^{k-1} p \, \frac{ds}{s}. \tag{24} $$

From relation (24) and the boundary condition (22), for the unknown element p, we obtain the equation

$$ \lim_{t \to e} {}^A I^{\beta}_{1+} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\left(\ln \frac{t}{s}\right)^{\alpha} B\right) (\ln s)^{k-1} p \, \frac{ds}{s} $$
$$ = u_1 - \lim_{t \to e} {}^A I^{\beta}_{1+} ((\ln t)^{\alpha-1} E_{\alpha,\alpha}((\ln t)^{\alpha} B) u_0). \tag{25} $$

By taking into account the semigroup property of the fractional integration operation, we obtain

$$ \lim_{t \to e} {}^A I^{\beta}_{1+} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\left(\ln \frac{t}{s}\right)^{\alpha} B\right) (\ln s)^{k-1} p \, \frac{ds}{s} $$
$$ = \frac{\Gamma(k)}{\Gamma(k+\beta)} \sum_{j=0}^{\infty} \int_0^1 (1-\tau)^{k+\beta-1} \tau^{j\alpha+\alpha-1} \frac{B^j p}{\Gamma(j\alpha+\alpha)} \, d\tau $$
$$ = \Gamma(k) \sum_{j=0}^{\infty} \frac{B^j p}{\Gamma(j\alpha + k + \alpha + \beta)} = \Gamma(k) E_{\alpha, k+\alpha+\beta}(B) p. \tag{26} $$

Likewise, we have $\lim_{t\to e} {}^A I_{1+}^{\beta}((\ln t)^{\alpha-1}E_{\alpha,\alpha}((\ln t)^{\alpha}B)u_0) = E_{\alpha,\alpha+\beta}(B)u_0$.

Then Eq. (25) admits the operator representation

$$G_0 p = q_0, \tag{27}$$

where

$$G_0 p = E_{\alpha,k+\alpha+\beta}(B)p, \qquad q_0 = \frac{1}{\Gamma(k)}(u_1 - E_{\alpha,\alpha+\beta}(B)u_0). \tag{28}$$

Problem (20)–(22) with bounded operator B is uniquely solvable if and only if Eq. (27) is solvable, i.e., if the spectrum $\sigma(G_0)$ of the operator G_0 does not contain the point $\lambda = 0$. By virtue of (28), the operator G_0 is an analytic function of the operator B. By the spectral mapping theorem for bounded operators, we have

$$\sigma(G_0) = \sigma(E_{\alpha,k+\alpha+\beta}(B)) = E_{\alpha,k+\alpha+\beta}(\sigma(B)).$$

Consequently, $\lambda = 0$ is not a point of spectrum of the operator G_0 only if the function $E_{\alpha,k+\alpha+\beta}(z)$ is nonzero on the spectrum of B. The proof of the theorem is complete.

It follows from Theorem 7 that the position of zeros of the function $E_{\alpha,k+\alpha+\beta}(z)$ specifies the unique solvability of problem (20)–(22) with a bounded operator B. As was mentioned in [9], for a first-order equation with an unbounded operator B, a condition of the form (23) is not sufficient for unique solvability, although the arrangement of zeros is still important. Therefore, we present some results in [11] on their arrangement. It was proved in Theorem 1 in [11] that, for $\alpha \in (0,1)$, $k+\alpha+\beta > 0$, and an appropriate numbering, all zeros μ_n, $n \in \mathbf{Z}\backslash\{0\}$, of the function $E_{\alpha,k+\alpha+\beta}(z)$ with sufficiently large absolute values are simple, and the asymptotics

$$\mu_n^{1/\alpha} = 2\pi n i + (k+\beta-1)\left(\ln 2\pi|n| + \frac{\pi i}{2}\operatorname{sgn} n\right) + \ln\frac{\alpha}{\Gamma(k+\beta)} + o(1), \qquad n \to \pm\infty, \tag{29}$$

holds as $n \to \pm\infty$.

Let us prove a necessary condition for the uniqueness of the solution of the inverse problem (20)–(22) with an unbounded operator B.

Theorem 8. *Let B be a linear closed operator in X, and let the inverse problem (20)–(22) have a solution $(u(t), p)$. For that solution be unique, it is necessary that no zero μ_n of the entire function $E_{\alpha,k+\alpha+\beta}(z)$ be an eigenvalue of the operator B.*

Proof. Suppose that some zero μ_n in a countable set of zeros of the function $E_{\alpha,k+\alpha+\beta}(z)$ is an eigenvalue of the operator B with an eigenvector $h_n \neq 0$. Consider the function $w(t) = \psi(t)h_n$ and take a function $\psi(t)$ such that the function $w(t)$ satisfies Eq. (20) for $p = h_n$ and the zero initial condition (21).

One can readily see that the function $\psi(t)$ can be found from the problem

$${}^A D_{1+}^{\alpha}\psi(t) = \mu_n \psi(t) + (\ln t)^{k-1}, \qquad \lim_{t\to 1} {}^A I_{1+}^{1-\alpha}\psi(t) = 0. \tag{30}$$

By Theorem 1, problem (30) has a unique solution, which can be represented in the form

$$\psi(t) = \int_1^t \left(\ln\frac{t}{x}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\mu_n\left(\ln\frac{t}{x}\right)^{\alpha}\right)(\ln x)^{k-1}\,\frac{dx}{x}.$$

Since μ_n is a zero of the function $E_{\alpha,k+\alpha+\beta}(z)$, we have, by analogy with (26),

$$\lim_{t\to e} {}^A I_{1+}^{\beta}\psi(t) = \Gamma(k)\lim_{t\to e} {}^A I_{1+}^{\beta+k}((\ln t)^{\alpha-1}E_{\alpha,\alpha}(\mu_n(\ln t)^{\alpha})) = \Gamma(k)E_{\alpha,k+\alpha+\beta}(\mu_n) = 0.$$

The function $w(t) = \psi(t)h_n$ satisfies Eq. (20) for $p = h_n$ and the zero conditions (21) and (22), which contradicts the assumption of the solution uniqueness, since the pair $(u(t) + w(t), p + h_n)$ is a solution of problem (20)–(22) as well. The proof of the theorem is complete.

Next, let us prove the unique solvability of problem (20)–(22) with an unbounded operator B satisfying Condition 1. Following the proof of Theorem 7, with regard of (10), we reduce problem (20)–(22) to the operator equation

$$Gp = q, \tag{31}$$

$$Gp = \lim_{t\to e} {}^A I_{1+}^{k+\beta}\, {}^A T_\alpha(t)p, \qquad G:\ X \to X, \tag{32}$$

where ${}^A T_\alpha(t)$ is given by formula (8) and

$$q = \frac{1}{\Gamma(k)}(u_1 - \lim_{t\to e} {}^A I_{1+}^{\beta}({}^A T_\alpha(t)u_0)), \qquad q \in D(B). \tag{33}$$

Therefore, the unique solvability of problem (20)–(22) can be reduced to the problem on the existence of the inverse operator defined on some subset of the Banach space X for the bounded operator G given by relation (32). To clarify the latter, we represent the operator G on the narrower set $D(B)$ dense in X in a more convenient form with the use of the resolvent $R(z) = (zI - B)^{-1}$.

Theorem 9. *Let Condition* 1 *be satisfied. Then the representation*

$$Gp = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} E_{\alpha,k+\alpha+\beta}(z)R(z)p\,dz, \qquad \sigma > \omega, \tag{34}$$

holds for each $p \in D(B)$.

Proof. First, suppose that p belongs to $D(B^2)$; then $p = R^2(\lambda)p_0$, $p_0 \in X$, where $\lambda \in \varrho(B)$, $\varrho(B)$ is the resolvent set of the operator B, and $\operatorname{Re}\lambda > \sigma > \omega$. By using the representation of the semigroup $T(t)$ via the resolvent of the generator, from (32), we obtain

$$\begin{aligned} Gp &= \frac{1}{\Gamma(k+\beta)}\lim_{t\to e}\int_1^t \left(\ln\frac{t}{s}\right)^{k+\beta-1}\frac{ds}{s}\int_0^\infty f_{\tau,\alpha}(\ln s)T(\tau)p\,d\tau \\ &= \frac{1}{\Gamma(k+\beta)}\lim_{t\to e}\int_1^t \left(\ln\frac{t}{s}\right)^{k+\beta-1}\frac{ds}{s}\int_0^\infty f_{\tau,\alpha}(\ln s)\,d\tau\,\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\tau}R(z)R^2(\lambda)p_0\,dz. \end{aligned} \tag{35}$$

By using the Hilbert identity for the expression $R(z)R^2(\lambda)$ in relation (35), we obtain

$$R(z)R^2(\lambda) = \frac{R(z) - R(\lambda)}{\lambda - z}R(\lambda) = \frac{R(z)}{(\lambda-z)^2} - \frac{R(\lambda)}{(\lambda-z)^2} - \frac{R^2(\lambda)}{\lambda - z}$$

and since, by the Jordan lemma, the integrals of functions of the form $\dfrac{\exp(z\tau)R^j(\lambda)p_0}{(\lambda-z)^{3-j}}$, $j = 1, 2$, over the line $\operatorname{Re} z = \sigma$ are zero, we have

$$Gp = \frac{1}{\Gamma(k+\beta)}\lim_{t\to e}\int_0^{\ln t}(\ln t - s)^{k+\beta-1}\,ds\int_0^\infty f_{\tau,\alpha}(s)\,d\tau\,\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\exp(z\tau)R(z)p_0}{(\lambda-z)^2}\,dz.$$

By expressing $f_{\tau,\alpha}(s)$ by the formula $f_{\tau,\alpha}(t) = t^{-1}e^{1,0}_{1,\alpha}(-\tau t^{-\alpha})$ and by using the formula for the Riemann–Liouville fractional integration of a Wright type function (see [6, formula (1.2.12)]),

$$\frac{1}{\Gamma(k+\beta)}\int\limits_0^{\eta}(\eta-\varrho)^{k+\beta-1}\varrho^{-1}e^{1,0}_{1,\alpha}(-\tau\varrho^{-\alpha})\,d\varrho = \eta^{k+\beta-1}e^{1,k+\beta}_{1,\alpha}(-\tau\eta^{-\alpha}),$$

we obtain

$$\begin{aligned} Gp &= \int\limits_0^{\infty} e^{1,k+\beta}_{1,\alpha}(-\tau)\,d\tau\,\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{\exp(z\tau)R(z)p_0}{(\lambda-z)^2}\,dz \\ &= \frac{1}{(2\pi i)^2}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{R(z)p_0\,dz}{(\lambda-z)^2}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\xi^{-k-\beta}e^{\xi}\,d\xi\int\limits_0^{\infty}\exp(z\tau)\exp(-\tau\xi^{\alpha})\,d\tau \\ &= \frac{1}{(2\pi i)^2}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{R(z)p_0\,dz}{(\lambda-z)^2}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{\xi^{-k-\beta}e^{\xi}}{\xi^{\alpha}-z}\,d\xi = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}E_{\alpha,k+\alpha+\beta}(z)\frac{R(z)p_0}{(\lambda-z)^2}\,dz \\ &= \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}E_{\alpha,k+\alpha+\beta}(z)\,\frac{R(z)((\lambda-z)I+(zI-B))(\lambda I-B)p}{(\lambda-z)^2}\,dz \\ &= \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{E_{\alpha,k+\alpha+\beta}(z)}{\lambda-z}R(z)(\lambda I-B)p\,dz, \quad p\in D(B^2); \end{aligned} \tag{36}$$

here we have assumed that $\operatorname{Re}\xi^{\alpha} > \operatorname{Re}z$ and used the relation [6, formula (1.1.12)]

$$e^{1,k+\beta}_{1,\alpha}(-\tau) = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\xi^{-k-\beta}\exp(\xi-\tau\xi^{\alpha})\,d\xi$$

and the formula [1, formula (1.93)]

$$\frac{1}{2\pi i}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{\xi^{-k-\beta}e^{\xi}}{\xi^{\alpha}-z}\,d\xi = E_{\alpha,k+\alpha+\beta}(z).$$

If we set $p_1 = (\lambda I - B)p$, then $p_1 \in D(B)$, $p = R(\lambda)p_1$, and relation (36) acquires the form

$$GR(\lambda)p_1 = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{E_{\alpha,k+\alpha+\beta}(z)}{\lambda-z}\,R(z)p_1\,dz, \qquad p_1\in D(B). \tag{37}$$

The left- and right-hand sides of relation (37) are bounded operators, which coincide on $D(B)$. Since $D(B)$ is dense in X, it follows that relation (37) holds for all $p_1 \in X$. But then $p = R(\lambda)p_1 \in D(B)$, and the representation

$$Gp = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{E_{\alpha,k+\alpha+\beta}(z)}{\lambda-z}\,R(z)((\lambda-z)I+(zI-B))p\,dz = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}E_{\alpha,k+\alpha+\beta}(z)R(z)p\,dz$$

is true for such p. The proof of the theorem is complete.

Now let us proceed to the proof of sufficient conditions for the unique solvability of problem (20)–(22). As follows from Theorem 8, one should require that no zero μ_n of the function $E_{\alpha,k+\alpha+\beta}(z)$ is an eigenvalue of the operator B. Moreover, to prove the solvability, we require that all zeros belong to the resolvent set $\varrho(B)$. By taking into account their asymptotics (29), we note that if $k+\beta>1$, then the condition is imposed only on finitely many zeros μ_n, $n=1,2,\dots,n_0$, with $\operatorname{Re}\mu_n^{1/\alpha}<\sigma$, since the remaining values necessarily belong to $\varrho(B)$. If $k+\beta\le 1$, then there are countably many zeros with $\operatorname{Re}\mu_n^{1/\alpha}<\sigma$.

Theorem 10. *Suppose that the operator B satisfies Condition 1, $k+\beta>1$, $\sigma>\omega$, and $u_0,u_1\in D(B^3)$. If every zero μ_n, $n=1,2,\dots,n_0$, of the function $E_{\alpha,k+\alpha+\beta}(z)$ with $\operatorname{Re}\mu_n^{1/\alpha}<\sigma$ belongs to $\varrho(B)$, then problem (20)–(22) has a unique solution.*

Proof. The existence of a unique solution of problem (20)–(22) [or the operator equation (31)] can be reduced to the existence of the inverse of the bounded operator G given by relation (32) [or (34)]. For $u_0,u_1\in D(B^3)$, it follows from the invariance of $D(B)$ with respect to $T_\alpha(t)$ that the right-hand side q of Eq. (31) belongs to $D(B^3)$. Let us show that the operator G has an inverse $G^{-1}:\ D(B^3)\to X$.

Since each zero $\mu_n^{1/\alpha}$ of the function $E_{\alpha,k+\alpha+\beta}(z^\alpha)$ with $\operatorname{Re}\mu_n^{1/\alpha}<\sigma$ belongs to $\varrho(B)$, we find that $\varrho(B)$ contains it together with some disk neighborhood Ω_n. Let Γ be a contour on the complex plane consisting of the line $\operatorname{Re}z=\sigma>\omega$ and the boundaries γ_n of the disk neighborhoods Ω_n; i.e., $\Gamma=\{\operatorname{Re}z=\sigma\}\bigcup_{\operatorname{Re}\mu_n^{1/\alpha}<\sigma}\gamma_n$.

We take $\lambda\in\varrho(B)$, $\operatorname{Re}\lambda>\sigma>\omega$, and consider the bounded operator

$$\Upsilon q=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{R(z)q\,dz}{E_{\alpha,k+\alpha+\beta}(z)(z-\lambda)^3},\qquad \Upsilon:\ E\to E. \tag{38}$$

Note that the integral occurring in (38) is absolutely convergent by virtue of the choice of the contour Γ, the Hille–Yosida inequality

$$\|R^n(z)\|\le M/(\operatorname{Re}z-\omega)^n,\qquad n\in N,$$

and the asymptotic behavior of the Mittag-Leffler function for $0<\alpha<2$ and $|z|\to\infty$ (see [10, p. 134]),

$$E_{\alpha,\mu}(z)=\frac{1}{\alpha}z^{(1-\mu)/\alpha}\exp(z^{1/\alpha})-\sum_{j=1}^{n}\frac{z^{-j}}{\Gamma(\mu-\alpha j)}+O\left(\frac{1}{|z|^{n+1}}\right), \tag{39}$$

$$|\arg z|\le\nu\pi,\quad \nu\in(\alpha/2,\min\{1,\alpha\}),$$

$$E_{\alpha,\mu}(z)=-\sum_{j=1}^{n}\frac{1}{\Gamma(\mu-\alpha j)z^j}+O\left(\frac{1}{|z|^{n+1}}\right),\qquad \nu\pi\le|\arg z|\le\pi. \tag{40}$$

Let $q\in D(B)$ and $\sigma<\sigma_1<\operatorname{Re}\lambda$. Then, by substituting the representation (34) into (38) and by using the Hilbert identity, we obtain the relation

$$\Upsilon Gq=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{R(z)\,dz}{E_{\alpha,k+\alpha+\beta}(z)(z-\lambda)^3}\,\frac{1}{2\pi i}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}E_{\alpha,k+\alpha+\beta}(\xi)R(\xi)q\,d\xi$$

$$=\frac{1}{(2\pi i)^2}\int\limits_{\Gamma}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{E_{\alpha,k+\alpha+\beta}(\xi)}{E_{\alpha,k+\alpha+\beta}(z)(z-\lambda)^3}\,\frac{R(z)q-R(\xi)q}{\xi-z}\,d\xi\,dz. \tag{41}$$

The integral in (41) is absolutely convergent. By changing the integration order, we obtain

$$\Upsilon G q=\frac{1}{(2\pi i)^2}\int\limits_{\Gamma}\frac{R(z)q\,dz}{E_{\alpha,k+\alpha+\beta}(z)(z-\lambda)^3}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{E_{\alpha,k+\alpha+\beta}(\xi)\,d\xi}{\xi-z}$$
$$-\frac{1}{(2\pi i)^2}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}E_{\alpha,k+\alpha+\beta}(\xi)R(\xi)q\,d\xi\int\limits_{\Gamma}\frac{dz}{E_{\alpha,k+\alpha+\beta}(z)(z-\lambda)^3(\xi-z)}. \qquad (42)$$

The inner integral in the second term in (42) is zero by virtue of the choice of the contour Γ and the Jordan lemma, and to compute the integrals in the second term, we use the Cauchy integral formula. Therefore, the relation

$$\Upsilon G q=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{R(z)q}{(z-\lambda)^3}\,dz=R^3(\lambda)q$$

holds for $q\in D(B)$.

The commuting operators Υ, G, and $R(\lambda)$ are bounded, and the domain $D(B)$ is dense in X; therefore, the relation $\Upsilon Gq=R^3(\lambda)q$ holds for $q\in X$ as well, $\Upsilon G:\ X\to D(B^3)$. Hence it follows that the operator $G^{-1}q=(\lambda I-B)^3\Upsilon q$ is the inverse of G for $q\in D(B^3)$. Indeed,

$$GG^{-1}q=G(\lambda I-B)^3\Upsilon q=G\Upsilon(\lambda I-B)^3q=R^3(\lambda)(\lambda I-B)^3q=q,\qquad q\in D(B^3),$$
$$G^{-1}Gq=(\lambda I-B)^3\Upsilon Gq=q,\qquad q\in X.$$

In the solution of problem (20)–(22), the element p belonging to X has the form $p=(\lambda I-B)^3\Upsilon q$, where $q\in D(B^3)$ is the element given by (33), the operator Υ is given by relation (38), $\lambda\in\varrho(B)$, $\operatorname{Re}\lambda>\sigma>\omega$, and the function $u(t)$ admits the representation

$$u(t)={}^{A}T_\alpha(t)u_0+\int\limits_1^t(\ln s)^{k-1}\,{}^{A}T_\alpha\left(\frac{t}{s}\right)p\,\frac{ds}{s}$$

with the operator ${}^{A}T_\alpha(t)$ given by formula (8). The proof of the theorem is complete.

As was mentioned above, if $k+\beta\le 1$, then the function $E_{\alpha,k+\alpha+\beta}(z)$ has countably many zeros μ_n with $\operatorname{Re}\mu_n^{1/\alpha}<\sigma$; therefore, we require the following condition to be satisfied.

Condition 4. Each zero μ_n, $n\in\mathbf{Z}\backslash\{0\}$, of the function $E_{\alpha,k+\alpha+\beta}(z)$ with $\operatorname{Re}\mu_n^{1/\alpha}<\sigma$ belongs to $\varrho(B)$, and there exist $\varepsilon\in[0,1)$ and $d>0$ such that $\sup_{\operatorname{Re}\mu_n^{1/\alpha}<\sigma}\|R(\mu_n)/\mu_n^{\varepsilon}\|\le d$.

Theorem 11. *Let Conditions* 1 *and* 4 *be satisfied, and let* $k+\beta\le 1$ *and* $u_0,u_1\in D(B^3)$. *Then problem* (20)–(22) *has a unique solution.*

Proof. Just as in Theorem 10, we introduce the operator Υ given by relation (38). In this case, the contour Γ contains countably many circles γ_n, and to prove the absolute convergence of the integral occurring in (38), we consider an integral over the circles γ_n, where n is a sufficiently large number ($|n|\ge n_0$). Then, by the residue theorem, we have

$$\frac{1}{2\pi i}\int\limits_{\substack{\cup\gamma_n\\|n|\ge N_0}}\frac{R(z)q\,dz}{E_{\alpha,k+\alpha+\beta}(z)(z-\lambda)^3}=\sum_{\substack{n=-\infty\\|n|\ge N_0}}^{+\infty}\frac{R(\mu_n)q}{E'_{\alpha,k+\alpha+\beta}(\mu_n)(\lambda-\mu_n)^3}; \qquad (43)$$

in addition, by using the relation (see [10, formula (1.5), p. 118])

$$E'_{\alpha,k+\alpha+\beta}(\mu_n)=\frac{1}{\alpha\mu_n}(E_{\alpha,k+\alpha+\beta-1}(\mu_n)-(k+\alpha+\beta-1)E_{\alpha,k+\alpha+\beta}(\mu_n))$$

and the asymptotics (39) and (29), we obtain

$$E'_{\alpha,k+\alpha+\beta}(\mu_n) = \frac{1}{\alpha\mu_n}\left(\frac{\mu_n^{(2-k-\beta)/\alpha-1}(2\pi|n|)^{k+\beta-2}\exp(i\operatorname{Im}\mu_n^{1/\alpha})}{\Gamma(k+\beta-1)} - \frac{1}{\Gamma(k+\beta-1)\mu_n}\right.$$
$$- \frac{(k+\alpha+\beta-1)\mu_n^{(1-k-\beta)/\alpha-1}(2\pi|n|)^{k+\beta-1}\exp(i\operatorname{Im}\mu_n^{1/\alpha})}{\Gamma(k+\beta)}$$
$$\left. + \frac{k+\alpha+\beta-1}{\Gamma(k+\beta)\mu_n} + O\left(\frac{1}{|\mu_n|^2}\right)\right)$$
$$= \frac{1}{\alpha\mu_n^{2-1/\alpha}}\left(\frac{\exp(i\operatorname{Im}\mu_n^{1/\alpha})}{\Gamma(k+\beta)}\left(\frac{2\pi|n|}{\mu_n^{1/\alpha}}\right)^{1-k-\beta} + O\left(\frac{1}{|\mu_n|^{1/\alpha}}\right)\right). \tag{44}$$

By virtue of relation (44), Condition 4, and the asymptotics (29) of zeros, the series (43) and hence the integral over $\cup\gamma_n$ are absolutely convergent. But the convergence of the integral over the line $\operatorname{Re} z = \sigma$ in relation (38) follows from the Hille–Yosida inequality and the asymptotics (40).

The subsequent proof is similar to that of Theorem 10, and we omit it. The proof of Theorem 11 is complete.

Remark. It follows from formulas (10) and (17) that, in the inverse problem for an equation with a regularized Hadamard fractional derivative, the influence of the inhomogeneity on the form of the solution of the Cauchy problem in the case of the regularized Hadamard fractional derivative is determined by the same expression as in the case of the nonregularized Hadamard fractional derivative. Therefore, the solvability condition for the inverse problem for an equation with the regularized Hadamard fractional derivative for $k \geq 1$ is the same as in Theorems 10 and 11.

ACKNOWLEDGMENTS

The research was carried out in the framework of the Federal Targeted Program "Scientific and Pedagogical Staff of Innovational Russia" for 2009–2013 (state contract no. 02.740.11.0613).

REFERENCES

1. Samko, S.G., Kilbas, A.A., and Marichev, O.I., *Integraly i proizvodnye drobnogo poryadka i nekotorye ikh prilozheniya* (Fractional Integrals and Derivatives and Some of Their Applications), Minsk: Nauka i Tehnika, 1987.
2. Kilbas, A.A., Srivastava, H.M., and Trujillo, J.J., *Theory and Application of Fractional Differential Equations*, Math. Studies, vol. 204, Elsevier, 2006.
3. Kilbas, A.A., Marzan, A.A., and Tityura, A.A., Hadamard-Type Fractional Integrals and Derivatives, and Fractional Differential Equations, *Dokl. Akad. Nauk*, 2003, vol. 389, no. 6, pp. 734–738.
4. Glushak, A.V. and Manaenkova, T.V., Abstract Differential Equations Containing Hadamard Fractional Derivatives, *Dokl. Adygsk. (Cherkessk.) Mezhdunar. Akad. Nauk*, 2009, vol. 11, no. 1, pp. 17–20.
5. Glushak, A.V., On Cauchy Type Problem for Inhomogeneous Abstract Differential Equation with Fractional Derivative, *Vestn. Voronezh. Gos. Univ. Fiz. Mat.*, 2002, no. 1, pp. 121–123.
6. Pskhu, A.V., *Kraevye zadachi dlya differentsial'nykh uravnenii s chastnymi proizvodnymi drobnogo i kontinual'nogo poryadka* (Boundary Value Problems for Partial Differential Equations of Fractional and Continual Order), Nalchik, 2005.
7. Bajlekova, E., Fractional Evolution Equations in Banach Spaces, *Ph. D. Thesis, Eindhoven University of Technology*, Eindhoven. 2001.
8. Prilepko, A.I., Orlovsky, D.G., and Vasin, I.A., *Methods for Solving Inverse Problems in Mathematical Physics*, New York, 2000.
9. Eidel'man, Yu.S., Two-Point Boundary Value Problems for a Differential Equation with a Parameter, *Dokl. Akad. Nauk Ukrain. SSR. Ser. A*, 1983, no. 4, pp. 15–18.
10. Dzhrbashyan, M.M., *Integral'nye preobrazovaniya i predstavleniya funktsii v kompleksnoi oblasti* (Integral Transforms and Representations of Functions in the Complex Domain), Moscow, 1966.
11. Sedletskii, A.M., On the Zeros of a Function of Mittag–Leffler Type, *Mat. Zametki*, 2000, vol. 68, no. 5, pp. 710–724.