On Prime Numbers of Special Kind on Short Intervals

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Abstract—Suppose that the Riemann hypothesis holds. Suppose that

$$\psi_1(x) = \sum_{\substack{n \le x \\ \{(1/2)n^{1/c}\} < 1/2}} \Lambda(n),$$

where c is a real number, $1 < c \le 2$. We prove that, for $H > N^{1/2+10\varepsilon}$, $\varepsilon > 0$, the following asymptotic formula is valid:

$$\psi_1(N+H) - \psi_1(N) = \frac{H}{2}\left(1 + O\left(\frac{1}{N^{\varepsilon}}\right)\right).$$

Key words: *prime number, Riemann hypothesis, Chebyshev function, zeta function, Abel integral transformation.*

In 1940, to illustrate the application of his method, Vinogradov [1] studied the problem of the distribution of numbers of the form $\{f\sqrt{p}\}$, where p is a prime, $p \le N$, and f is an arbitrary positive constant.

In 1945, Linnik [2] proposed another approach to the solution of this problem related to the application of the explicit formula for the Chebyshev function and of theorems on the density of the distribution of zeros of the zeta function in the critical strip.

In 1979, Kaufman [3] applied Linnik's approach to the solution of a similar problem.

In 1986, Gritsenko [4] obtained an asymptotic formula for $\pi_c(M)$, the number of primes p such that $p \le M$, $\{(1/2)p^{1/c}\} < 1/2$, where c is an arbitrary real number, $1 < c \le 2$. Obviously, $\pi_c(M)$ is the number of primes not exceeding M and belonging to the intervals $[(2x)^c, (2x+1)^c)$ $x = 1, 2, \dots$. Note that the lesser is c, the shorter are the intervals. It is not clear whether each of these intervals contains at least one prime. For $1 < c \le 4/3$, Gritsenko proved that

$$\pi_c(M) = \frac{\pi(M)}{2} + O(M^{1/2+1/2c+\varepsilon}), \qquad \varepsilon > 0. \tag{1}$$

If Linnik's method is applied and the Riemann hypothesis, instead of the density theorems, is used, then it is not possible to obtain a remainder in this asymptotic formula better than $O(M^{1/2+1/2c+\varepsilon})$, without invoking additional considerations.

For $H < N^{1/2+1/2c}$, it does not follows from the asymptotic formula (1) that the closed interval $[N, N+H]$ contains at least one prime satisfying $\{(1/2)p^{1/c}\} < 1/2$.

In the present paper, we solve the problem of the number of primes satisfying the condition $\{(1/2)p^{1/c}\} < 1/2$ on a short interval. The main result is the following theorem.

Theorem 1. *Suppose that* $H > N^{1/2+10\varepsilon}$, $\varepsilon > 0$. *If the Riemann hypothesis is valid, then the following asymptotic formula holds*:

$$\psi_1(N+H) - \psi_1(N) = \frac{H}{2}\left(1 + O\left(\frac{1}{N^{\varepsilon}}\right)\right), \qquad \textit{where} \quad \psi_1(x) = \sum_{\substack{n \le x \\ \{(1/2)n^{1/c}\} < 1/2}} \Lambda(n)$$

and c *is a real number,* $1 < c \le 2$.

Note that, for $c > 2$, the theorem is no longer valid, because the length of an interval of the form $[(2x)^c, (2x+1)^c)$ is greater than H.

Proof. Let us introduce the notation

$$\psi_2(x) = \sum_{\substack{n \le x \\ \{1/2n^{1/c}\} \ge 1/2}} \Lambda(n).$$

Then $\psi_1(x) + \psi_2(x) = \psi(x)$, where $\psi(x)$ is the Chebyshev function.

We define $Q(N, H)$ by the equality

$$Q(N, H) = \sum_{\frac{1}{2}N^{1/c} < x \le \frac{1}{2}(N+H)^{1/c}} \{\psi((2x+2)^c) - 2\psi((2x+1)^c) + \psi((2x)^c)\}.$$

Obviously,

$$Q(N, H) = [\psi_2(N+H) - \psi_2(N)] - [\psi_1(N+H) - \psi_1(N)] + O(N^{1-1/c+\varepsilon}).$$

As is well known, if the Riemann hypothesis holds, then

$$\psi(N+H) - \psi(N) = H + O(N^{1/2+\varepsilon}).$$

Therefore, to prove the theorem, it suffices to estimate $Q(N, H)$ as $O(H/N^{\varepsilon})$. The idea of an estimate (presented below) of a quantity similar to $Q(N, H)$ belongs to Linnik [2].

Let us use the representation of the Chebyshev function as the sum over nontrivial zeros ρ of the zeta function, choosing $T = N^{1/c+\varepsilon}\ln^2 N$ (see for example, [5, p. 54]):

$$\sum_{\frac{1}{2}N^{1/c} < x \le \frac{1}{2}(N+H)^{1/c}} \Bigg\{(2x+2)^c - 2(2x+1)^c + (2x)^c - \sum_{|\gamma| \le T} \frac{(2x+2)^{c\rho} - 2(2x+1)^{c\rho} + (2x)^{c\rho}}{\rho} + O\left(\frac{N\ln^2 N}{T}\right)\Bigg\}.$$

Accepting the Riemann hypothesis, we express ρ as $1/2 + i\gamma$. Let us estimate the contribution of the leading terms:

$$\sum_{\frac{1}{2}N^{1/c} < x \le \frac{1}{2}(N+H)^{1/c}} \{(2x+2)^c - 2(2x+1)^c + (2x)^c\} \ll \sum_{\frac{1}{2}N^{1/c} < x \le \frac{1}{2}(N+H)^{1/c}} x^{c-2} \ll \frac{H}{N^{1/c}} = O\left(\frac{H}{N^{\varepsilon}}\right).$$

Let us estimate the contribution of the remainder:

$$\sum_{\frac{1}{2}N^{1/c}<x\le\frac{1}{2}(N+H)^{1/c}} O\left(\frac{N\ln^2 N}{T}\right) \ll \frac{H}{N^{\varepsilon}}.$$

It remains to estimate the sum

$$\sum_{\frac{1}{2}N^{1/c}<x\le\frac{1}{2}(N+H)^{1/c}} \sum_{|\gamma|\le T} \frac{(2x+2)^{c\rho}-2(2x+1)^{c\rho}+(2x)^{c\rho}}{\rho}.$$

Denote

$$Q_1 = \sum_{\frac{1}{2}N^{1/c}<x\le\frac{1}{2}(N+H)^{1/c}} \sum_{|\gamma|\le T_1} \frac{(2x+2)^{c\rho}-2(2x+1)^{c\rho}+(2x)^{c\rho}}{\rho},$$

where $T_1 = N^{1/c-\varepsilon}$, and

$$Q_2 = \sum_{\frac{1}{2}N^{1/c}<x\le\frac{1}{2}(N+H)^{1/c}} \sum_{T_1<|\gamma|\le T} \frac{(2x+2)^{c\rho}-2(2x+1)^{c\rho}+(2x)^{c\rho}}{\rho}.$$

Let us estimate Q_1. To do this, we use the fact that, in this sum, the order of γ is less than that of x. Factor out $(2x)^{c\rho}/\rho$; then the expression in parentheses will be of order $O(|\rho^2|/x^2)$. Let us apply the integral Abel transformation:

$$Q_1 \ll \int_{\frac{1}{2}N^{1/c}}^{\frac{1}{2}(N+H)^{1/c}} \sum_{|\gamma|\le T_1} |\mathbf{C}(x)|\,|\gamma| x^{c/2-3}\,dx + \sum_{|\gamma|\le T_1} \left|\mathbf{C}\left(\frac{1}{2}(N+H)^{1/c}\right)\right| |\gamma| N^{1/2-2/c},$$

where

$$\mathbf{C}(x) = \sum_{\frac{1}{2}N^{1/c}<n\le x} n^{ic\gamma}.$$

To the sum $\mathbf{C}(x)$, we apply the well-known lemma on its approximation by an integral (see, for example, [5, p. 44]). We obtain

$$\mathbf{C}(x) \ll \frac{N^{1/c}}{|\gamma|}, \qquad Q_1 \ll \sum_{|\gamma|\le T_1} N^{1/2-1/c} \ll N^{1/2+2\varepsilon}.$$

Further, let us estimate Q_2. Consider the sum

$$W = \sum_{T_1<|\gamma|\le T} \sum_{\frac{1}{2}N^{1/c}<x\le\frac{1}{2}(N+H)^{1/c}} \frac{(2x+\theta)^{c\rho}}{\rho},$$

where $\theta = 0, 1, 2$. Applying to it the integral Abel transformation, we find

$$W \ll \int_{1/2N^{1/c}}^{1/2(N+H)^{1/c}} \sum_{T_1<|\gamma|\le T} |\mathbf{C}(x)| \frac{x^{c/2-1}}{|\gamma|}\,dx + \sum_{T_1<|\gamma|\le T} \left|\mathbf{C}\left(\frac{1}{2}(N+H)^{1/c}\right)\right| \frac{N^{1/2}}{|\gamma|},$$

where

$$\mathbf{C}(x) = \sum_{\frac{1}{2}N^{1/c}<n\le x} n^{ic\gamma}.$$

Suppose that, for x from the interval $\left[\frac{1}{2}N^{1/c}, \frac{1}{2}(N+H)^{1/c}\right]$, the expression

$$\sum_{T_1<|\gamma|\le T} \frac{|\mathbf{C}(x)|}{|\gamma|}$$

takes the largest value at $x = x_0$. We obtain

$$W \ll \sum_{T_1<|\gamma|\le T} |\mathbf{C}(x_0)| \frac{N^{1/2}}{|\gamma|}.$$

Denote

$$f(x) = \frac{c\gamma}{2\pi} \ln x.$$

Choose k so that $\left\|f'\left(\frac{1}{2}N^{1/c}\right)\right\| = \left|f'\left(\frac{1}{2}N^{1/c}\right) - k\right|$. Let $f_1(x) = f(x) - kx$. Since

$$|f_1'(x)| \le \left|f_1'\left(\frac{1}{2}N^{1/c}\right)\right| + \left|f'(x) - f'\left(\frac{1}{2}N^{1/c}\right)\right| \le \frac{1}{2} + O\left(\frac{H}{N} N^{\varepsilon} \ln^2 N\right) \le 0,51 < 1$$

on the narrow interval $\left[\frac{1}{2}N^{1/c}, \frac{1}{2}(N+H)^{1/c}\right]$, it follows that the sum over x of the exponential function on this interval can be approximated by the integral

$$\mathbf{C}(x_0) = \sum_{\frac{1}{2}N^{1/c}<x\le x_0} e^{2\pi i f_1(x)} = \int_{\frac{1}{2}N^{1/c}}^{x_0} e^{2\pi i f_1(x)}\, dx + O(1).$$

Let us split the sum W into at most $O(N^{\varepsilon}\ln^3 N)$ sums, in each of which the summation is performed over values of γ such that

$$\left|\frac{c\gamma}{\pi N^{1/c}} - k\right| \le \frac{1}{2}.$$

Let us estimate the sum

$$\sum_{\left|\frac{c\gamma}{\pi N^{1/c}}-k\right|\le\frac{1}{2}} \frac{N^{1/2}}{|\gamma|} \left|\int_{\frac{1}{2}N^{1/c}}^{x_0} e^{2\pi i f_1(x)}\, dx + O(1)\right|.$$

For $k=0$ and x from the interval, we have $\left[\frac{1}{2}N^{1/c}, \frac{1}{2}(N+H)^{1/c}\right]$,

$$|f_1'(x)| = |f'(x)| > \frac{1}{N^{\varepsilon}};$$

therefore, the integral of the exponential function can be estimated by using the first derivative (see, for example, [6, p. 73 of the Russian translation]):

$$\sum_{N^{1/c-\varepsilon}<|\gamma|\le\frac{\pi}{2c}N^{1/c}} \frac{N^{1/2}}{|\gamma|} \left|\int_{1/2N^{1/c}}^{x_0} e^{2\pi i f_1(x)}\, dx + O(1)\right|$$
$$\ll \sum_{N^{1/c-\varepsilon}<|\gamma|\le\frac{\pi}{2c}N^{1/c}} \frac{N^{1/2}}{|\gamma|}(N^{\varepsilon}+1) \ll N^{1/2+2\varepsilon}\ln N = O\left(\frac{H}{N^{\varepsilon}}\right).$$

Next, suppose that $k \neq 0$. Let us take into account the fact that

$$f'(x) = f'(1/2N^{1/c}) + O\left(\frac{H}{N} N^{\varepsilon} \ln^2 N\right).$$

First, consider the sum

$$S_1(k) = \sum_{\left|\frac{c\gamma}{\pi N^{1/c}} - k\right| \leq \frac{100HN^{\varepsilon}\ln^3 N}{N}} \frac{N^{1/2}}{|\gamma|} \left| \int_{\frac{1}{2}N^{1/c}}^{x_0} e^{2\pi i f_1(x)}\, dx + O(1)\right|.$$

Let us estimate the integral using the second derivative (see, for example, [6, p. 73 of the Russian translation]) as $O(N^{1/c}/\sqrt{|\gamma|})$. We obtain

$$S_1(k) \ll \frac{HN^{\varepsilon+1/2c-1/2} \ln^4 N}{k^{3/2}}.$$

Without loss of generality, we can set

$$N^{\varepsilon+1/2c-1/2} \ln^4 N < N^{-2\varepsilon},$$

and we obtain the estimate

$$\sum_{\substack{|k| \ll N^{\varepsilon} \ln^3 N \\ k \neq 0}} S_1(k) = O\left(\frac{H}{N^{\varepsilon}}\right).$$

Next, let us estimate the sum

$$S_2(k) = \sum_{\frac{100HN^{\varepsilon}\ln^3 N}{N} < \left|\frac{c\gamma}{\pi N^{1/c}} - k\right| \leq \frac{1}{2}} \frac{N^{1/2}}{|\gamma|} \left| \int_{N^{1/c}}^{x_0} e^{2\pi i f_1(x)}\, dx + O(1)\right|.$$

We split $S_2(k)$ into at most $O(\ln N)$ sums, where the summation is performed over values of γ such that

$$2^{m-1} \frac{100HN^{\varepsilon}\ln^3 N}{N} < \left|\frac{c\gamma}{\pi N^{1/c}} - k\right| \leq 2^m \frac{100HN^{\varepsilon}\ln^3 N}{N}.$$

On each such interval, using the first derivative, we can estimate the integral as

$$O(N/(HN^{\varepsilon} \ln^3 N 2^{m-1}))$$

and obtain

$$S_2(k) \ll \frac{N^{1/2}}{k}.$$

Finally,

$$\sum_{\substack{|k| \ll N^{\varepsilon} \ln^3 N \\ k \neq 0}} S_2(k) = O\left(\frac{H}{N^{\varepsilon}}\right). \quad \square$$

REFERENCES

1. I. M. Vinogradov, “A general property of the distribution of primes,” *Mat. Sb.* [*Math. USSR-Sb.*], (1940), no. 7, 365–372.
2. Yu. V. Linnik, “On a theorem in the theory of primes,” *Dokl. Akad. Nauk SSSR* [*Soviet Math. Dokl.*], **47** (1945), 7–8.
3. R. M. Kaufman, “On the distribution of $\{\sqrt{p}\}$,” *Mat. Zametki* [*Math. Notes*], **26** (1979), no. 4, 497–504.
4. S. A. Gritsenko, “On a problem of I. M. Vinogradov,” *Mat. Zametki* [*Math. Notes*], **39** (1986), no. 5, 625–640.
5. A. A. Karatsuba, *Foundations of the Analytic Theory of Numbers* [in Russian], Nauka, Moscow, 1983.
6. E. Titchmarsh, *The Theory of the Riemann Zeta Function*, Oxford, 1951; Russian translation: Inostr. Lit., Moscow, 1953.

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