ON PERTURBATIONS OF ABSTRACT FRACTIONAL DIFFERENTIAL EQUATIONS BY NONLINEAR OPERATORS

H. K. Avad and A. V. Glushak

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ABSTRACT. We prove the unique solvability of a Cauchy-type problem for an abstract parabolic equation containing fractional derivatives and a nonlinear perturbation term. The result is applied to establish the solvability of the inverse coefficient problem for a fractional-order equation.

Introduction. Setting the Problem

We consider the following Cauchy-type problem in a Banach space E:

$$D^{\alpha}u(t) = Au(t) + F(t, B(t)u(t)), \quad t > 0, \tag{1}$$

$$\lim_{t\to 0} D^{\alpha-1}u(t) = u_0, \tag{2}$$

where $0 < \alpha < 1$,

$$D^{\alpha-1}u(t) = I^{1-\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int\limits_0^t (t-s)^{-\alpha}u(s)\,ds$$

is the left-side fractional Riemann–Liouville integral of order $1-\alpha$ (for $\alpha = 1$, we assume that $I^{1-\alpha}$ is the identity operator), $D^{\alpha}u(t) = \dfrac{d}{dt}I^{1-\alpha}u(t)$ is the left-side fractional Riemann–Liouville derivative of order α, $\Gamma(\cdot)$ is the gamma-function, A is a linear closed densely defined operator, $B(t)$ is a linear closed densely defined operator depending on t (no assumptions about the boundedness of $B(t)$ are imposed), and $F(t, w)$ is a nonlinear operator acting in E for any $t \geq 0$; the latter operator is treated as a perturbation of the operator A.

The results presented below are related to the perturbation theory of generators of semigroups (see [8, Chap. 9]). We add to problem (1), (2) a term with a nonlinear operator subordinated in a way to the operator A and investigate how this affects the solvability of the problem. Sufficient conditions providing the solvability of the problem with the perturbed operator A are presented.

In [5], solvability results for equations with fractional Riemann–Liouville derivatives perturbed by a linear closed operator $B(t)$ are obtained. The study of problems of that kind is motivated by

numerous applications of fractional-order differential equations in physics and mathematical modelling (see, e.g., [18, Chap. 8], [13, Chap. 5], and [9, Chap. 8]).

Apart from problem (1), (2), consider the following problem without perturbations (assuming that $\beta \geq \alpha$):

$$D^{\beta}u(t) = Au(t), \quad t > 0, \tag{3}$$

$$\lim_{t\to 0} D^{\beta-1}u(t) = u_0. \tag{4}$$

Definition 1. A function $u(t)$ is called a *solution of problem* (3), (4) if it is continuous for $t > 0$, takes values in $D(A)$ (here $D(A)$ is the domain of the operator A), $I^{1-\beta}u(t)$ is continuously differentiable for $t > 0$, and $u(t)$ satisfies (3) and (4).

Definition 2. We say that problem (3), (4) is *uniformly well posed* if there exist an operator-valued function $T_\beta(t)$ defined on E and commuting with A, a positive M_1, and a real ω such that, for any $u_0 \in D(A)$, the function $T_\beta(t)u_0$ is the unique solution of problem (3), (4) and

$$\|T_\beta(t)\| \leq M_1 t^{\beta-1} e^{\omega t}. \tag{5}$$

According to Definition 2, problem (3), (4) is uniformly well posed if its solution exists, is unique, and continuously depends on the initial data uniformly with respect to t from any compact set of $(0, \infty)$. The latter property follows from (5). Apart from those standard requirements, Definition 2 includes additional information about the behavior of the solution as $t \to 0$ and $t \to \infty$ (see inequality (5)).

Condition 1. *There exists $\beta \in [\alpha, 1]$ such that problem* (3), (4) *is uniformly well posed and u_0 belongs to $D(A)$.*

In [4, 6, 10], the uniform well-posedness of problem (3), (4) is studied for $0 < \beta < 1$. If $\beta = 1$, then the uniform well-posedness of the Cauchy problem requires that the operator A be a generator of a C_0-semigroup.

Condition 2. (i) *The domain of the operator $B(t)$, denoted as D, does not depend on t and $D(A) \subset D$.*

(ii) *Let $x \in D$. Then either the function $w(t) = B(t)x$ belongs to $C((0, \infty), E)$, is absolutely integrable at the origin, and takes values in $D(A)$ and the function $Aw(t)$ belongs to $C((0, \infty), E)$ and is absolutely integrable at the origin or the function $I^{1-\alpha}w(t) = I^{1-\alpha}B(t)x$ is continuous for $t \geq 0$, continuously differentiable for $t > 0$, and $D^{\alpha}w(t)$ is absolutely integrable at the origin.*

(iii) *For any $x \in E$, there exist $M_2 > 0$, $\gamma \in (0, 1)$, and $\omega \in \mathbb{R}$ such that $T_\beta(\tau)x \in D$ (the smoothing effect) and*

$$\|B(t)T_\beta(\tau)x\| \leq M_2 \tau^{-\gamma} e^{\omega\tau} \|x\|, \quad t, \tau \in (0, \infty). \tag{6}$$

Note that if the operator $-A$ is strongly positive in the sense of [11], i.e., if

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{M_3}{1 + |\lambda|}, \quad \text{Re } \lambda \geq 0, \ M_3 > 0,$$

then we can assign $\beta = 1$ in Condition 1. In this case, $\omega = 0$ and inequality (6) means that the operator $B(t)$ is subordinated to the fractional power $(-A)^{\gamma}$ (see [11, p. 298]).

If the operator $B(t)$ is bounded and the operator A satisfies Condition 1, then inequality (6) is valid for $\gamma = 1 - \beta$.

The operators A and $B(t)$ are not assumed to commute.

Condition 3. (i) *The function F acts from $(0, \infty) \times E$ to E; if a function $w(t) = B(t)x$, $x \in D$, satisfies item* (ii) *of Condition* 2, *so does $w_1(t) = F(t, w(t))$.*

(ii) *The following inequality is valid for $w = 0$: $\|F(t, 0)\| \leq C_0(1 + t^{\mu-1})$, $mu > 0$, $C_0 > 0$.*

(iii) *The operator $F(t,w)$ satisfies the following Lipschitz condition uniformly with respect to $t>0$:*

$$\|F(t,w_2)-F(t,w_1)\|\le L\|w_2-w_1\| \qquad \text{for all } w_1,w_2\in E.$$

Condition 4. *The Banach space E possesses the Radon–Nikodým property (see* [1, p.15]*), i.e., any absolutely continuous function $F:\mathbb{R}_+\longrightarrow E$ is differentiable almost everywhere.*

For example, reflexive Banach spaces possess that property (see [1, Corollary 1.2.7]), while the spaces $L_1(a,b)$, $C[a,b]$, and the space c_0 of sequences converging to zero do not (see [1, Example 1.2.8 and Propositions 1.2.9 and 1.2.10]).

We will show below that Conditions 1–4 guarantee the unique solvability of problem (1), (2).

The following function (see [7, p. 357]) is needed for the proof:

$$f_{\tau,\nu}(t)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{\sigma-i\infty}^{\sigma+i\infty}\exp\left(tz-\tau z^{\nu}\right)\,dz, & t\ge 0,\\ 0, & t<0,\end{cases} \tag{7}$$

where $\sigma>0,\ \tau>0,\ 0<\nu<1$, and the branch of the function z^{ν} is chosen to satisfy the inequality $\operatorname{Re} z^{\nu}>0$ for $\operatorname{Re} z>0$. This branch is a one-valued function on the complex z-plane cut along the negative part of the real axis. The convergence of the integral in (7) is guaranteed by the factor $\exp\left(-\tau z^{\nu}\right)$.

Below, we present certain properties of the function $f_{\tau,\nu}(t)$ (see also [7, p. 358–361, Propositions 1–3]).

Consider the integral defining the function $f_{\tau,\nu}(t)$ and replace the line of integration $\operatorname{Re} z=\sigma>0$ by the contour consisting of the rays $z=r\exp(-i\theta)$ and $z=r\exp(i\theta)$, where $0<r<\infty$ and $\pi/2\le\theta\le\pi$. This yields the following representation of the function $f_{\tau,\nu}(t)$ for $t>0$:

$$f_{\tau,\nu}(t)=\frac{1}{\pi}\int_0^{\infty}\exp\left(tr\cos\theta-\tau r^{\nu}\cos\nu\theta\right)\sin\left(tr\sin\theta-\tau r^{\nu}\sin\nu\theta+\theta\right)\,dr. \tag{8}$$

The function $f_{\tau,\nu}(t)$ is nonnegative, and the following relations are valid:

$$\int_0^{\infty}f_{\tau,\nu}(t)\,dt\equiv 1 \tag{9}$$

$$\exp\left(-\tau\lambda^{\nu}\right)=\int_0^{\infty}\exp\left(-\lambda t\right)f_{\tau,\nu}(t)\,dt, \qquad \tau>0,\ \lambda>0,\ 0<\nu<1. \tag{10}$$

We also note that the function $f_{\tau,\nu}(t)$ can be expressed via the Wright function (see [9, p. 54]) for $t>0$:

$$f_{\tau,\nu}(t)=t^{-1}\phi\left(-\nu,0;-\tau t^{-\nu}\right), \qquad \phi(a,b;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(ak+b)}.$$

Another representation via the more general Wright-type function (see [16, Ch. 1]) is valid as well:

$$f_{\tau,\nu}(t)=t^{-1}e^{1,0}_{1,\nu}\left(-\tau t^{-\nu}\right), \qquad e^{\mu,\delta}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\mu)\,\Gamma(\delta-\beta k)}, \tag{11}$$

where $\beta<1,\ \delta+\beta>0,\ \max\{0;\beta\}<\alpha<2,\ \alpha+\beta<2$, and $\mu,z\in C$.

1. Cauchy-Type Problems for Fractional-Order Equations: Inhomogeneous Equations

The following theorem establishes conditions under which the uniform well-posedness of problem (3), (4) implies the uniform well-posedness of the corresponding Cauchy-type problem for the equation of order α, where $0 < \alpha < \beta \le 1$.

Theorem 1.1. *Let $\alpha < \beta \le 1$, Conditions 1 and 4 be satisfied, and $\omega = 0$ in inequality (5). Then the problem*

$$D^{\alpha}u(t) = Au(t), \quad t > 0, \tag{1.1}$$

$$\lim_{t\to 0} D^{\alpha-1}u(t) = u_0 \tag{1.2}$$

is uniformly well posed and its resolving operator is of the form

$$T_{\alpha}(t)u_0 = \int_0^{\infty} f_{\tau,\nu}(t)T_{\beta}(\tau)u_0\, d\tau, \tag{1.3}$$

where $\nu = \alpha/\beta$ and the function $f_{\tau,\nu}(t)$ is defined by relation (7).

Proof. The following is proved in [4]. If problem (3), (4) is uniformly well posed and $\omega = 0$ in inequality (5), then λ^{β} belongs to the resolvent set $\rho(A)$ of the operator A for $\operatorname{Re}\lambda > 0$, the resolvent $R(\lambda^{\beta}) = (\lambda^{\beta}I - A)^{-1}$ is representable in the form

$$R(\lambda^{\beta})x = \int_0^{+\infty} \exp(-\lambda t)T_{\beta}(t)x\, dt \tag{1.4}$$

for any $x \in E$, and the following inequalities are valid for any nonnegative integer n:

$$\left\| \frac{d^n R\left(\lambda^{\beta}\right)}{d\lambda^n} \right\| \le \frac{M\Gamma\left(n+\beta\right)}{\left(\operatorname{Re}\,\lambda\right)^{n+\beta}}, \quad \operatorname{Re}\,\lambda > 0. \tag{1.5}$$

If the Banach space E possesses the Radon–Nikodým property, then the validity of inequalities (1.5) (even for real positive λ) is a sufficient condition for the uniform well-posedness of problem (3), (4). The resolving operator for this problem is of the form (see [4, formula (13)])

$$T_{\beta}(t)u_0 = D^{1-\beta}\frac{1}{2\pi i}\int_{\omega_0-i\infty}^{\omega_0+i\infty} \lambda^{\beta-1}\exp(\lambda t)R(\lambda^{\beta})u_0\, d\lambda, \qquad \omega_0 > 0. \tag{1.6}$$

Taking into account (1.4), (10), and (5), for $\nu = \alpha/\beta$, we have

$$R\left(\mu^{\alpha}\right)x = \int_0^{\infty} \exp\left(-\mu^{\nu}t\right)T_{\beta}(t)x\, dt = \int_0^{\infty} T_{\beta}(t)x\, dt \int_0^{\infty} \exp\left(-\tau\mu\right) f_{t,\nu}\left(\tau\right) d\tau.$$

In (8), we take $\theta \in [\pi/2, \pi]$ such that $\cos\theta < 0$ and $cos\nu\theta > 0$. To achieve that, we actually take it from the interval $(\pi/2, \min\{\pi/(2\nu); \pi\})$.

Hence, by virtue of (8), (9), and the theorem on the differentiability of integrals with respect to a parameter, the following inequalities are valid:

$$\left\| \frac{d^n R\left(\mu^{\alpha}\right)x}{d\mu^n} \right\| \le M_1 \|x\| \int_0^{\infty} t^{\beta-1}\, dt \int_0^{\infty} \tau^n \exp\left(-\tau \operatorname{Re}\,\mu\right) d\tau \int_0^{\infty} \exp\left(\tau s\cos\theta - ts^{\nu}\cos\nu\theta\right) ds$$

$$= M_4 \|x\| \int_0^{\infty} \tau^n \exp\left(-\tau \operatorname{Re}\,\mu\right) d\tau \int_0^{\infty} s^{-\alpha} \exp\left(\tau s\cos\theta\right) ds$$

$$= M_5 \|x\| \int_0^\infty \tau^{n-1+\alpha} \exp\left(-\tau \mathrm{Re}\ \mu\right) d\tau = \frac{M_6 \Gamma\left(n+\alpha\right) \|x\|}{\left(\mathrm{Re}\ \mu\right)^{n+\alpha}}.$$

This proves the uniform well-posedness of problem (1.1), (1.2).

Due to (1.6), (1.4), (7), and (11), the resolving operator for this problem is of the form

$$T_\alpha\left(t\right) u_0 = D^{1-\alpha} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \lambda^{\alpha-1} \exp\left(\lambda t\right) R\left(\lambda^\alpha\right) u_0 \, d\lambda$$

$$= D^{1-\alpha} \int_0^\infty T_\beta(\tau) u_0 \, d\tau \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \lambda^{\alpha-1} \exp(\lambda t - \lambda^\nu \tau) \, d\lambda$$

$$= D^{1-\alpha} \int_0^\infty t^{-\alpha} e_{1,\nu}^{1,1-\alpha}(-\tau t^{-\nu}) T_\beta(\tau) u_0 \, d\tau. \quad (1.7)$$

The following relation (implied by [16, formula (1.1.13)]) for the Laplace transformation was used:

$$L\left[t^{-\alpha} e_{1,\nu}^{1,1-\alpha}\left(-\tau t^{-\nu}\right); \lambda\right] = \lambda^{\alpha-1} \exp\left(-\tau \lambda^\nu\right).$$

Now we use the following relation for the fractional derivatives of Wright-type functions (see [16, formula (1.2.12)]):

$$D^{1-\alpha}\left(t^{-\alpha} e_{1,\nu}^{1,1-\alpha}(-\tau t^{-\nu})\right) = t^{-1} e_{1,\nu}^{1,0}(-\tau t^{-\nu}) = f_{\tau,\nu}(t).$$

Combining it with the limit relation

$$\lim_{x\to+\infty} x e_{1,\nu}^{1,0}(-x) = -\lim_{x\to+\infty} e_{1,\nu}^{0,\nu}(-x) = 0$$

(see [16, formulas (1.2.3) and (1.2.6)] and note that this relation and the estimate in (5) guarantee the convergence of the integral in (1.3) for $\omega = 0$) and using (1.7), we obtain (1.3). This is the required representation.

Remark 1.1. Consider the particular case where $\nu = \alpha/\beta = 1/2$. Then (see [7, p. 369, formula (32)])

$$f_{\tau,1/2}\left(t\right) = \frac{\tau}{2t\sqrt{\pi t}} \exp\left(-\frac{\tau^2}{4t}\right).$$

Thus, the relation in (1.3) takes the form

$$T_{\beta/2}\left(t\right) u_0 = \frac{1}{2t\sqrt{\pi t}} \int_0^\infty \tau \exp\left(-\frac{\tau^2}{4t}\right) T_\beta(\tau) u_0 \, d\tau. \quad (1.8)$$

The representation in (1.8) can provide the smoothing effect (see item (iii) in Condition 2) for the resolving operator $T_{\beta/2}\left(t\right)$ in the case where this effect for the operator $T_\beta\left(t\right)$ is absent. For example, this takes place if A and B are differential operators.

The following assertion is the solvability theorem for the Cauchy problem for the inhomogeneous equation.

Theorem 1.2. *Let $\beta < 1$, and let Condition 1 be satisfied. Let one of the following two conditions hold:*

(a) *a function $h(t)$ belongs to $C\left((0,\infty), E\right)$, is absolutely integrable at the origin, and takes values in $D(A)$ and the function $Ah(t)$ belongs to $C\left((0,\infty), E\right)$ and is absolutely integrable at the origin;*

(b) *a function $h(t)$ is such that the function $I^{1-\beta}h(t)$ is continuous for $t \geq 0$ and continuously differentiable for $t > 0$ and $D^{\beta}h(t)$ is absolutely integrable at the origin.*

Then the problem

$$D^{\beta}u(t) = Au(t) + h(t), \quad t > 0, \tag{1.9}$$

$$\lim_{t\to 0} D^{\beta-1}u(t) = u_0 \tag{1.10}$$

has a unique solution, which is defined by the relation

$$u(t) = T_{\beta}(t)u_0 + \int_0^t T_{\beta}(t-\xi)\, h(\xi)\, d\xi. \tag{1.11}$$

Proof. It suffices to check that the function

$$v(t) = \int_0^t T_{\beta}(t-\xi)\, h(\xi)\ d\xi$$

satisfies Eq. (1.9) and condition (1.10), which is the zero initial condition.

Let condition (a) be satisfied. Then, for $t > 0$, we have

$$D^{\beta}v(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-\tau)^{-\beta}\, d\tau \int_0^{\tau} T_{\beta}(\tau-\xi)\, h(\xi)\ d\xi$$

$$= \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t d\xi \int_{\xi}^t (t-\tau)^{-\beta} T_{\beta}(\tau-\xi)\, h(\xi)\ d\tau$$

$$= \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t d\xi \int_0^{t-\xi} (t-\xi-x)^{-\beta} T_{\beta}(x) h(\xi)\, dx.$$

Since the integrand (with respect to ξ) is a continuous function of the variable $t-\xi$, it follows that

$$D^{\beta}v(t) = \frac{1}{\Gamma(1-\beta)}\lim_{\xi\to t}\int_0^{t-\xi} (t-\xi-x)^{-\beta} T_{\beta}(x)\, h(\xi)\ dx$$

$$+ \frac{1}{\Gamma(1-\beta)}\int_0^t d\xi \frac{d}{dt}\int_0^{t-\xi} (t-\xi-x)^{-\beta} T_{\beta}(x)\, h(\xi)\ dx$$

$$= \lim_{t-\xi\to+0} D^{\beta-1} T_{\beta}(t-\xi)\, h(\xi) + \int_0^t D^{\beta} T_{\beta}(t-\xi)\, h(\xi)\ d\xi$$

$$= h(t) + \int_0^t T_{\beta}(t-\xi)\, Ah(\xi)\, d\xi = h(t) + Av(t).$$

Hence, the function $v(t)$ satisfies Eq. (1.9).

Further, we check that the function $v(t)$ satisfies condition (1.10). We have

$$\lim_{t\to+0} D^{\beta-1}v(t) = \frac{1}{\Gamma(1-\beta)}\lim_{t\to+0}\int_0^t (t-\tau)^{-\beta}\, d\tau \int_0^{\tau} T_{\beta}(\tau-\xi)\, h(\xi)\, d\xi.$$

Since $T_\beta(t)$ satisfies the estimate in (5), it follows that

$$\left\| \int_0^t (t-\tau)^{-\beta}\, d\tau \int_0^\tau T_\beta(\tau-\xi)\, h(\xi)\, d\xi \right\|$$

$$\leq M \int_0^t (t-\tau)^{-\beta}\, d\tau \int_0^\tau (\tau-\xi)^{\beta-1} \|h(\xi)\|\, d\xi = M\ B(\beta, 1-\beta) \int_0^t \|h(\xi)\|\, d\xi,$$

for $t \in [0,1]$, where $B(\cdot,\cdot)$ is the beta-function. Hence, the function $v(t)$ satisfies condition (1.10).

Now, let condition (b) hold. Then

$$D^\beta v(t) = D^\beta \int_0^t T_\beta(\tau) h(t-\tau)\, d\tau = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\xi)^{-\beta}\, d\xi \int_0^\xi T_\beta(\tau) h(\xi-\tau)\, d\tau$$

$$= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t T_\beta(\tau)\, d\tau \int_0^{t-\tau} (t-\tau-x)^{-\beta} h(x)\, dx$$

$$= T_\beta(t) \lim_{\tau\to t} \frac{1}{\Gamma(1-\beta)} \int_0^{t-\tau} (t-\tau-x)^{-\beta} h(x)\, dx + \frac{1}{\Gamma(1-\beta)} \int_0^t T_\beta(\tau)\, d\tau \frac{d}{dt} \int_0^{t-\tau} (t-\tau-x)^{-\beta} h(x)\, dx$$

$$= T_\beta(t)\, D^{\beta-1} h(0) + \int_0^t T_\beta(\tau)\, D^\beta h(t-\tau)\, d\tau = T_\beta(t) D^{\beta-1} h(0) + \int_0^t T_\beta(t-\xi) D^\beta h(\xi)\, d\xi. \quad (1.12)$$

On the other hand, it follows from the relation

$$I^\beta D^\beta h(x) = h(x) - \frac{I^{1-\beta} h(x)}{\Gamma(\beta)} x^{\beta-1}, \qquad 0 < \beta < 1 \quad (1.13)$$

(see [18, formula (2.61)]), that

$$v(t) = \int_0^t T_\beta(t-\xi) \left(\frac{1}{\Gamma(\beta)} D^{\beta-1} h(0) \xi^{\beta-1} + I^\beta D^\beta h(\xi) \right) d\xi = \frac{1}{\Gamma(\beta)} \int_0^t \xi^{\beta-1} T_\beta(t-\xi) D^{\beta-1} h(0)\, d\xi$$

$$+ \frac{1}{\Gamma(\beta)} \int_0^t T_\beta(t-\xi)\, d\xi \int_0^\xi (\xi-\tau)^{\beta-1} D^\beta h(\tau)\, d\tau = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} T_\beta(\tau) D^{\beta-1} h(0)\, d\tau$$

$$+ \frac{1}{\Gamma(\beta)} \int_0^t T_\beta(\tau)\, d\tau \int_0^{t-\tau} (t-\tau-\xi)^{\beta-1} D^\beta h(\xi)\, d\xi. \quad (1.14)$$

Using (1.13) and the closedness of the operator A again, we obtain

$$\frac{1}{\Gamma(\beta)} A \int_0^t (t-\tau)^{\beta-1} T_\beta(\tau) v_0\, d\tau = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} D^\beta T_\beta(\tau) v_0\, d\tau$$

$$= I^\beta D^\beta T_\beta(\tau) v_0 = T_\beta(t) v_0 - \frac{t^{\beta-1}}{\Gamma(\beta)} D^{\beta-1} T_\beta(0) v_0 = T_\beta(t) v_0 - \frac{t^{\beta-1}}{\Gamma(\beta)} v_0. \quad (1.15)$$

It follows from (1.13)–(1.15) that

$$Av(t) = T_\beta(t)D^{\beta-1}h(0) - \frac{t^{\beta-1}}{\Gamma(\beta)}D^{\beta-1}h(0) + \int\limits_0^t \left(T_\beta(t-\xi)D^\beta h(\xi) - \frac{(t-\xi)^{\beta-1}}{\Gamma(\beta)}D^\beta h(\xi)\right) d\xi$$

$$= T_\beta\,(t)\,D^{\beta-1}h(0) - \frac{t^{\beta-1}}{\Gamma(\beta)}D^{\beta-1}h(0) + \int\limits_0^t T_\beta(t-\xi)D^\beta h(\xi)\;d\xi - h(t) + \frac{t^{\beta-1}}{\Gamma(\beta)}D^{\beta-1}h(0) =$$

$$D^\beta v(t) - h(t).$$

Hence, the function $v(t)$ satisfies Eq. (1.9).

To verify that the function $v(t)$ satisfies condition (1.10) if condition (b) is satisfied, one should represent $D^\beta v(t)$ as follows:

$$D^{\beta-1}v(t) = \int\limits_0^t T_\beta(s)I^{1-\beta}h(t-s)\;ds.$$

2. Cauchy-Type Problems for Perturbed Fractional-Order Equations

We pass to the investigation of the perturbed problem (1), (2). In the sequel, we use the following function of the Mittag-Leffler type (see [2, Chap. III-IV]):

$$E_{\mu,\rho}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+\rho)}.$$

Theorem 2.1. *Let $\alpha < \beta \le 1$, Conditions 1 and 2 be satisfied, and $\omega = 0$ in inequalities (5) and (6). Let Conditions 3 and 4 be satisfied. Then problem (1), (2) has a unique solution satisfying the estimate*

$$\|u(t)\| \le \frac{M_1\Gamma(\beta)}{\Gamma(\alpha)}t^{\alpha-1}\|u_0\| + \frac{C_0M_1\Gamma(\beta)}{\Gamma(\alpha+1)}t^\alpha + \frac{C_0M_1\Gamma(\beta)\Gamma(\mu)}{\Gamma(\alpha+\mu)}t^{\alpha+\mu-1}$$
$$+ LM_1M_2\Gamma(\beta)\Gamma(\delta/\nu)\left(t^{\alpha+\delta-1}E_{\delta,\alpha+\delta}\left(LM_2\Gamma(\delta/\nu)t^\delta\right)\|u_0\|\right.$$
$$\left.+C_0t^{\alpha+\delta}E_{\delta,\alpha+\delta+1}\left(LM_2\Gamma(\delta/\nu)t^\delta\right) + C_0\Gamma(\mu)t^{\alpha+\delta+\mu-1}E_{\delta,\alpha+\delta+\mu}\left(LM_2\Gamma(\delta/\nu)t^\delta\right)\right), \quad (2.1)$$

where $\delta = \nu(1-\gamma)$.

Proof. Taking into account Theorems 1.1 and 1.2, we reduce problem (1), (2) to an integral equation. By virtue of (1.3) and (1.11), this integral equation can be written as follows:

$$u(t) = \int\limits_0^\infty f_{\tau,\nu}(t)T_\beta(\tau)u_0\,d\tau + \int\limits_0^t\int\limits_0^\infty f_{\tau,\nu}(t-s)T_\beta(\tau)F(s,B(s)u(s))\,d\tau ds, \quad (2.2)$$

where u_0, $T_\beta(\tau)u_0 \in D(A) \subset D$ and $\nu = \alpha/\beta$. Denoting $B(t)u(t)$ by $w(t)$, we obtain

$$w(t) = \int\limits_0^\infty f_{\tau,\nu}(t)B(t)T_\beta(\tau)u_0\,d\tau + \int\limits_0^t\int\limits_0^\infty f_{\tau,\nu}(t-s)B(t)T_\beta(\tau)F(s,w(s))\,d\tau ds. \quad (2.3)$$

To solve Eq. (2.3), we use the iteration method, assigning

$$w_0(t) = 0, \qquad w_1(t) = \int\limits_0^\infty f_{\tau,\nu}(t)B(t)T_\beta(\tau)u_0\,d\tau + \int\limits_0^t\int\limits_0^\infty f_{\tau,\nu}(t-s)B(t)T_\beta(\tau)F(s,0)\,d\tau ds,$$

$$w_{n+1}(t) = \int_0^\infty f_{\tau,\nu}(t)B(t)T_\beta(\tau)u_0\,d\tau + \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)B(t)T_\beta(\tau)F(s,w_n(s))\,d\tau ds, \quad n \in N.$$

Using inequality (6) and item (ii) of Condition 3, we estimate the norm

$$\|w_1(t)\| \le M_2\|u_0\|\int_0^\infty f_{\tau,\nu}(t)\tau^{-\gamma}\,d\tau + M_2C_0\int_0^t\int_0^\infty f_{\tau,\nu}(t-s)\tau^{-\gamma}(1+s^{\mu-1})\,d\tau ds. \tag{2.4}$$

Taking into account that the function $f_{\tau,\nu}(t)$ is defined by relation (7) and using [15, integrals 2.3.4.1 and 2.3.3.4], we obtain

$$\int_0^\infty f_{\tau,\nu}(t)\tau^{-\gamma}\,d\tau = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zt}\,dz\int_0^\infty \tau^{-\gamma}\exp\left(-\tau z^\nu\right)\,d\tau$$

$$= \frac{\Gamma(1-\gamma)}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zt}z^{-\nu(1-\gamma)}\,dz = \frac{\Gamma(1-\gamma)}{\Gamma(\nu(1-\gamma))}\,t^{\nu(1-\gamma)-1}, \quad t>0. \tag{2.5}$$

Applying relation (2.5) to (2.4) twice and computing the obtained integral, we have

$$\|w_1(t)\| \le M_2\|u_0\|\frac{\Gamma(1-\gamma)}{\Gamma(\nu(1-\gamma))}t^{\nu(1-\gamma)-1} + M_2C_0\frac{\Gamma(1-\gamma)t^{\nu(1-\gamma)}}{\Gamma(\nu(1-\gamma)+1)}$$
$$+ M_2C_0\frac{\Gamma(1-\gamma)\Gamma(\mu)t^{\nu(1-\gamma)+\mu-1}}{\Gamma(\nu(1-\gamma)+\mu)} \le \frac{M_2\Gamma(\delta/\nu)}{\Gamma(\delta)}\left(t^{\delta-1}\|u_0\| + \frac{C_0}{\delta}t^\delta + \frac{C_0\Gamma(\delta)\Gamma(\mu)}{\Gamma(\delta+\mu)}t^{\delta+\mu-1}\right).$$

Using item (iii) in Condition 3, we estimate (in the same way) the norm of the difference

$$\|w_2(t)-w_1(t)\| \le \int_0^t\int_0^\infty f_{\tau,\nu}(t-s)\left\|B(t)T_\beta(\tau)\left(F(s,w_1)-F(s,0)\right)\right\|\,d\tau ds$$

$$\le \frac{LM_2^2\Gamma(\delta/\nu)}{\Gamma(\delta)}\int_0^t\int_0^\infty f_{\tau,\nu}(t-s)\tau^{-\gamma}\left(s^{\delta-1}\|u_0\| + \frac{C_0}{\delta}s^\delta + \frac{C_0\Gamma(\delta)\Gamma(\mu)}{\Gamma(\delta+\mu)}s^{\delta+\mu-1}\right)\,d\tau ds$$

$$\le \frac{LM_2^2\Gamma^2(\delta/\nu)}{\Gamma(2\delta)}\left(t^{2\delta-1}\|u_0\| + \frac{C_0}{2\delta}t^{2\delta} + \frac{C_0\Gamma(2\delta)\Gamma(\mu)}{\Gamma(2\delta+\mu)}t^{2\delta+\mu-1}\right). \tag{2.6}$$

Taking into account (2.6), for $n \in N$, by induction, we obtain the inequality

$$\|w_n(t)-w_{n-1}(t)\| \le \frac{L^{n-1}M_2^n\,\Gamma^n(\delta/\nu)}{\Gamma(n\delta)}\left(t^{n\delta-1}\|u_0\| + \frac{C_0}{n\delta}t^{n\delta} + \frac{C_0\Gamma(n\delta)\Gamma(\mu)}{\Gamma(n\delta+\mu)}t^{n\delta+\mu-1}\right). \tag{2.7}$$

Hence, the series

$$\sum_{n=1}^\infty (w_n(t)-w_{n-1}(t))$$

uniformly converges on any segment $[t_0,t_1]$, $0<t_0<t_1$. Therefore, $w_n(t)$ uniformly converges to a function $w(t)$ on the same segment, where $w(t)$ is continuous on $[t_0,t_1]$ and satisfies Eq. (2.3). By virtue of (2.7), the following estimate holds for that function:

$$\|w(t)\| \le \sum_{n=1}^\infty \|w_n(t)-w_{n-1}(t)\|$$

$$\le \sum_{k=0}^{\infty} \frac{L^k M_2^{k+1} \Gamma^{k+1}(\delta/\nu)}{\Gamma((k+1)\delta)} \left(t^{(k+1)\delta-1} \|u_0\| + \frac{C_0}{(k+1)\delta} t^{(k+1)\delta} + \frac{C_0 \Gamma((k+1)\delta)\Gamma(\mu)}{\Gamma((k+1)\delta+\mu)} t^{(k+1)\delta+\mu-1} \right)$$

$$\le M_2\Gamma(\delta/\nu) \left(t^{\delta-1} \|u_0\| \sum_{k=0}^{\infty} \frac{L^k M_2^k \Gamma^k(\delta/\nu) t^{k\delta}}{\Gamma((k+1)\delta)} + C_0 t^{\delta} \sum_{k=0}^{\infty} \frac{L^k M_2^k \Gamma^k(\delta/\nu) t^{k\delta}}{\Gamma((k+1)\delta+1)} \right.$$

$$\left. + C_0\Gamma(\mu)\ t^{\delta+\mu-1} \sum_{k=0}^{\infty} \frac{L^k M_2^k \Gamma^k(\delta/\nu) t^{k\delta}}{\Gamma((k+1)\delta+\mu)} \right)$$

$$= M_2\Gamma(\delta/\nu) \left(t^{\delta-1} E_{\delta,\delta} \left(LM_2\Gamma(\delta/\nu) t^{\delta} \right) \|u_0\| + C_0 t^{\delta} E_{\delta,\delta+1} \left(LM_2\Gamma(\delta/\nu) t^{\delta} \right) \right.$$

$$\left. + C_0\Gamma(\mu) t^{\delta+\mu-1} E_{\delta,\delta+\mu} \left(LM_2\Gamma(\delta/\nu) t^{\delta} \right) \right), \quad (2.8)$$

where $E_{\sigma,\rho}(\cdot)$ is a function of the Mittag-Leffler type, $t \in [t_0, t_1]$, $0 < t_0 < t_1$.

Since the segment $[t_0, t_1]$ is chosen arbitrarily, it follows that the function $w(t)$ is a solution of Eq. (2.3) continuous on $(0, \infty)$ and satisfying inequality (2.8) on $(0, \infty)$, i.e., $w(t)$ is absolutely integrable at the origin. Moreover, from relation (2.3), we conclude that the function $w(t)$ satisfies item (ii) in Condition 2.

Finally, using relation (2.2) and Theorem 1.2, we obtain the following representation of the solution $u(t)$ of problem (1), (2):

$$u(t) = \int_0^{\infty} f_{\tau,\nu}(t) T_\beta(\tau) u_0 \, d\tau + \int_0^t \int_0^{\infty} f_{\tau,\nu}(t-s) T_\beta(\tau) F(s, w(s)) \, d\tau ds.$$

By virtue of (5), (2.8), (2.5), and item (ii) in Condition 3, it satisfies the inequality

$$\|u(t)\| \le \int_0^{\infty} f_{\tau,\nu}(t) \|T_\beta(\tau) u_0\| \, d\tau$$

$$+ \int_0^t \int_0^{\infty} f_{\tau,\nu}(t-s) \|T_\beta(\tau) F(s,0)\| \, d\tau ds + \int_0^t \int_0^{\infty} f_{\tau,\nu}(t-s) \|T_\beta(\tau)(F(s,w(s)) - F(s,0)\| \, d\tau ds$$

$$\le \frac{M_1\Gamma(\beta)\ t^{\alpha-1}}{\Gamma(\alpha)} \|u_0\| + \frac{C_0 M_1 \Gamma(\beta) t^{\alpha}}{\Gamma(\alpha+1)} + \frac{C_0 M_1 \Gamma(\beta)\Gamma(\mu) t^{\alpha+\mu-1}}{\Gamma(\alpha+\mu)}$$

$$+ \frac{LM_1M_2\Gamma(\beta)\Gamma(1-\gamma)}{\Gamma(\alpha)} \|u_0\| \int_0^t (t-s)^{\alpha-1} s^{\delta-1} E_{\delta,\delta} \left(LM_2\Gamma(\delta/\nu) s^{\delta} \right) ds$$

$$+ \frac{C_0 LM_1M_2\Gamma(\beta)\Gamma(\delta/\nu)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\delta} E_{\delta,\delta+1} \left(LM_2\Gamma(\delta/\nu) s^{\delta} \right) ds$$

$$+ \frac{C_0 LM_1M_2\Gamma(\beta)\Gamma(\delta/\nu)\Gamma(\mu)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\delta+\mu-1} E_{\delta,\delta+\mu} \left(LM_2\Gamma(\delta/\nu) s^{\delta} \right) ds.$$

Therefore, the solution satisfies the estimate

$$\|u(t)\| \le \frac{M_1\Gamma(\beta) t^{\alpha-1}}{\Gamma(\alpha)} \|u_0\| + \frac{C_0 M_1 \Gamma(\beta) t^{\alpha}}{\Gamma(\alpha+1)} + \frac{C_0 M_1 \Gamma(\beta)\Gamma(\mu) t^{\alpha+\mu-1}}{\Gamma(\alpha+\mu)}$$

$$+ LM_1M_2\Gamma(\beta)\Gamma(\delta/\nu) \left(t^{\alpha+\delta-1} E_{\delta,\alpha+\delta} \left(LM_2\Gamma(\delta/\nu) t^{\delta} \right) \|u_0\| \right.$$

$$+C_0 t^{\alpha+\delta} E_{\delta,\alpha+\delta+1}\left(LM_2\Gamma(\delta/\nu)t^{\delta}\right) + C_0\Gamma(\mu)t^{\alpha+\delta+\mu-1}E_{\delta,\alpha+\delta+\mu}\left(LM_2\Gamma(\delta/\nu)t^{\delta}\right).$$

The following relation was used:

$$I^{\alpha}(t^{\rho-1}E_{\sigma,\rho}\left(ct^{\sigma}\right)) = t^{\alpha+\rho-1}E_{\sigma,\alpha+\rho}\left(ct^{\sigma}\right), \quad \alpha,\sigma,\rho>0$$

(see [18, p. 141, formula (23)]).

To establish the uniqueness of the solution of problem (1), (2), we assume, to the contrary, that there exists another solution. We denote it by $U(t)$. Then, by virtue of Theorems 1.1 and 1.2, we have

$$U(t) = \int_0^{\infty} f_{\tau,\nu}(t)T_{\beta}(\tau)u_0\,d\tau + \int_0^t\int_0^{\infty} f_{\tau,\nu}(t-s)T_{\beta}(\tau)F(s,W(s))\,d\tau ds,$$

where $W(t)$ satisfies Eq. (2.3).

Let us prove the uniqueness of the solution of Eq. (2.3) in the class of functions continuous on $(0,\infty)$ and satisfying the estimate

$$\|W(t)\| \le Mt^{\delta-1}e^{\omega t}, \quad M>0,\ \omega\ge 0, \tag{2.9}$$

where $\delta=\nu(1-\gamma)<1$. Note that the functions satisfying estimate (2.8) belong to the specified class due to the following asymptotic behavior of the Mittag-Leffler function for $0<\mu<2$ (see [2, p. 134]):

$$E_{\mu,\rho}(z) = \frac{1}{\mu}z^{(1-\rho)/\mu}\exp\left(z^{1/\mu}\right) - \sum_{j=1}^{n}\frac{1}{\Gamma(\rho-\mu j)\ z^j} + O\left(\frac{1}{|z|^{n+1}}\right), \quad z\in\mathbb{R},\ z\to+\infty. \tag{2.10}$$

Let $b>0$ and $t\in(0,b]$. Set

$$m = \sup_{t\in[0,b]}(t^{1-\delta}e^{-\omega t}\|W(t)-w(t)\|).$$

The supremum is finite because we consider the class of functions satisfying inequality (2.9).

The difference $W(t)-w(t)$ satisfies Eq. (2.3) for $u_0=0$. Therefore, taking into account relation (2.5), we have

$$\|W(t)-w(t)\| \le \frac{LM_2\Gamma(1-\gamma)}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\|W(s)-w(s)\|\,ds$$

$$= LM_2\Gamma(1-\gamma)I^{\delta}(\|W(t)-w(t)\|). \tag{2.11}$$

Hence, the following inequality holds:

$$\|W(t)-w(t)\| \le \frac{LM_2\Gamma(1-\gamma)m}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}s^{\delta-1}e^{\omega s}\,ds = LM_2\Gamma(1-\gamma)m\ I^{\delta}(t^{\delta-1}e^{\omega t}). \tag{2.12}$$

Substituting (2.12) into (2.11), we obtain the inequality

$$\|W(t)-w(t)\| \le L^2M_2^2\Gamma^2(1-\gamma)mI^{2\delta}(t^{\delta-1}e^{\omega t}).$$

Continuing this procedure, we arrive at the inequality

$$\|W(t)-w(t)\| \le L^kM_2^k\Gamma^k(1-\gamma)mI^{k\delta}(t^{\delta-1}e^{\omega t}) = \frac{L^kM_2^k\Gamma^k(1-\gamma)m}{\Gamma(k\delta)}\int_0^t (t-s)^{k\delta-1}s^{\delta-1}e^{\omega s}\,ds$$

$$\le \frac{L^kM_2^k\Gamma^k(1-\gamma)\Gamma(\delta)}{\Gamma((k+1)\delta)}t^{(k+1)\delta-1}e^{\omega t}m \quad \text{for all } k\in N. \tag{2.13}$$

Taking the supremum, we obtain the inequality

$$m \le \frac{L^k M_2^k \Gamma^k (1-\gamma)\Gamma(\delta)}{\Gamma((k+1)\delta)} b^{k\delta} m.$$

The factor

$$\frac{L^k M_2^k \Gamma^k (1-\gamma)\Gamma(\delta)}{\Gamma((k+1)\delta)} b^{k\delta}$$

is the common term of the series defining the Mittag-Leffler function (cf. (2.8)). Therefore, it vanishes as $k \to \infty$. Thus,

$$m = \sup_{t\in[0,b]} (t^{1-\delta} e^{-\omega t} \|W(t) - w(t)\|) = 0.$$

Since the positive number b was chosen arbitrarily, it follows that $W(t) \equiv w(t)$ for $t > 0$. This completes the proof of the uniqueness.

Note that estimate (2.1) contains a detailed dependence of the solution on the data of the problem. This dependence can be used in further research. If only the behavior of solution of problem (1), (2) as $t \to 0$ and $t \to \infty$ is investigated, then, taking into account (2.10), one can represent estimate (2.1) as follows:

$$\|u(t)\| \le M t^{\alpha-1} e^{\omega_1 t} \|u_0\|, \quad M > 0,\ \omega_1 \ge 0. \tag{2.14}$$

Theorem 2.1 establishes the solvability of problem (1), (2) for any α provided that $0 < \alpha < \beta \le 1$, Conditions 1–4 are satisfied, and $\omega = 0$ in inequalities (5) and (6). Let us prove that if $0 < \alpha = \beta < 1$, then similar results can be obtained without the requirement $\omega = 0$ in inequalities (5) and (6) and without Condition 4.

Theorem 2.2. *Let Conditions* 1–3 *be satisfied, and let* $\alpha = \beta < 1$. *Then problem* (1), (2) *has a unique solution satisfying estimate* (2.14).

Proof. Taking into account Theorem 1.2, we reduce problem (1), (2) to the integral equation

$$u(t) = T_\alpha(t)u_0 + \int_0^t T_\alpha(t-s) F\left(s, B(s)u(s)\right) ds. \tag{2.15}$$

Introducing $w(t) = B(t)u(t)$, we obtain the equation

$$w(t) = T_\alpha(t)u_0 + \int_0^t B(t) T_\alpha(t-s) F(s, w(s))\, ds. \tag{2.16}$$

To solve it by the iteration method, we set

$$w_0(t) = 0, \quad w_1(t) = T_\alpha(t)u_0, \quad w_{n+1}(t) = T_\alpha(t)u_0 + \int_0^t B(t) T_\alpha(t-s) F(s, w_n(s))\, ds, \quad n \in N.$$

Using inequalities (5) and (6) and item (iii) in Condition 3, we estimate the norm of the following difference:

$$\|w_2(t) - w_1(t)\| \le L M_2 \int_0^t (t-s)^{-\gamma} e^{\omega(t-s)} \|w_1(s)\|\, ds \le L M_1 M_2 \Gamma(1-\gamma) e^{\omega t} I^{1-\gamma}(t^{\alpha-1}) \|u_0\|. \tag{2.17}$$

Taking into account (2.17), by induction, we obtain the relation

$$\|w_n(t) - w_{n-1}(t)\| \le M_1 L^{n-1} M_2^{n-1} \Gamma^{n-1}(1-\gamma) e^{\omega t} I^{(n-1)(1-\gamma)} \left(t^{\alpha-1}\right) \|u_0\|$$

$$= \frac{M_1 L^{n-1} M_2^{n-1} \Gamma(\alpha)\Gamma^{n-1}(1-\gamma)}{\Gamma(\alpha+(n-1)(1-\gamma))} t^{\alpha-1+(n-1)(1-\gamma)} e^{\omega t}\|u_0\|, \quad n \in N.$$

Further reasoning regarding the existence of a unique solution is similar to the proof of Theorem 2.1. The following estimate holds for the solution $w(t)$ of Eq. (2.16):

$$\|w(t)\| \le M_1 t^{\alpha-1} e^{\omega t}\|u_0\| + \sum_{k=1}^{\infty} \frac{M_1 L^k M_2^k \Gamma(\alpha)\Gamma^k(1-\gamma) t^{\alpha-1+k(1-\gamma)} e^{\omega t}\|u_0\|}{\Gamma(\alpha+k(1-\gamma))} \le M_0 t^{\alpha-1} e^{\omega_0 t}\|u_0\|, \tag{2.18}$$

where $M_0 > 0$ and $\omega_0 \ge \omega$.

Using (2.18), we deduce estimate (2.14) of the solution $u(t)$ of problem (1), (2) from relation (2.15).

Remark 2.1. An assertion similar to Theorem 2.2 is also valid for $\alpha = \beta = 1$, but item (ii) in Condition 2 should be replaced by the following assumption: for any $x \in D$, either the functions $B(t)x$ and $AB(t)x$ belong to $C([0,\infty), E)$ and the function $B(t)x$ takes values in $D(A)$ or the function $B(t)x$ belongs to $C^1([0,\infty), E)$.

The following assertion is a theorem on the continuous dependence of the solution of problem (1), (2) on the initial data.

Theorem 2.3. *Suppose that the conditions of Theorem* 2.1 *are satisfied and $u_n(t)$ is the sequence of solutions of the problem*

$$D^\alpha u_n(t) = Au_n(t) + F\left(t, B(t)u_n(t)\right), \quad t > 0, \tag{2.19}$$

$$\lim_{t\to 0} D^{\alpha-1} u_n(t) = g_n \in D(A). \tag{2.20}$$

If $g_n \to u_0 \in D(A)$, $Ag_n \to Au_0$, and $B(t)g_n \to B(t)u_0$ uniformly with respect to $t \in (0, b]$ for any positive b, then the sequence $u_n(t)$ of the solutions of problem (2.19), (2.20) *converges to a solution $u(t)$ of problem* (1), (2) *uniformly with respect to $t \in [t_0, b]$ for any $t_0 \in (0, b)$.*

Proof. Consider the sequence $U_n(t) = u_n(t) - \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} g_n$, which satisfies the problem

$$D^\alpha U_n(t) = AU_n(t) + F\left(t, B(t)U_n(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} B(t)g_n\right) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} Ag_n, \tag{2.21}$$

$$\lim_{t\to 0} D^{\alpha-1} U_n(t) = 0. \tag{2.22}$$

By Theorems 1.1 and 1.2, the function $U_n(t)$ satisfies the integral equation

$$U_n(t) = \int_0^t \int_0^\infty f_{\tau,\nu}(t-s) T_\beta(\tau) \left(F\left(s, B(s)U_n(s) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} B(s)g_n\right) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} Ag_n\right) d\tau ds.$$

Setting $W_n(t) = B(t)U_n(t)$, we obtain (as in the proof of Theorem 2.1) that

$$U_n(t) = \int_0^t \int_0^\infty f_{\tau,\nu}(t-s) T_\beta(\tau) \left(F\left(s, W_n(s) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} B(s)g_n\right) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} Ag_n\right) d\tau ds, \tag{2.23}$$

where $W_n(t)$ satisfies the integral equation

$$W_n(t) = \int_0^t \int_0^\infty f_{\tau,\nu}(t-s) B(t) T_\beta(\tau) \left(F\left(s, W_n(s) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} B(s)g_n\right) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} Ag_n\right) d\tau ds. \tag{2.24}$$

Let n and k be sufficiently large positive integers and $\varepsilon > 0$. Taking (2.24) into account, we obtain (as in the proof of Theorem (2.13)) that

$$\|W_n(t) - W_k(t)\| \le \frac{L\ M_2\ \Gamma(\delta/\nu)}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \|W_n(s) - W_k(s)\|\, ds$$

$$+ \frac{M_2\ \Gamma(\delta/\nu)}{\Gamma(\delta)\Gamma(\alpha)} \int_0^t (t-s)^{\delta-1}\ s^{\alpha-1}\left(\|Ag_n - Ag_k\| + L\ \|B(s)g_n - B(s)g_k\|\right)\, ds$$

and

$$m = \sup_{t\in[0,b]} (t^{1-\delta} e^{-\omega t} \|W_n(t) - W_k(t)\|) \le M_0 m + \varepsilon, \quad M_0 < 1.$$

Hence, $m \le \dfrac{\varepsilon}{1 - M_0}$. Then, by virtue of the completeness of the space E, the sequence $t^{1-\delta}e^{-\omega t}W_n(t)$ converges to a function $t^{1-\delta}e^{-\omega t}W(t)$ continuous on $[0, b]$ uniformly with respect to $t \in [0, b]$. Thus, $W_n(t)$ converges to a function $W(t)$ uniformly with respect to $t \in [t_0, b]$, $0 < t_0 < b$, where $W(t)$ satisfies inequality (2.9) and item (ii) in Condition 2.

Relation (2.23) implies the uniform (with respect to $t \in [t_0, b]$) convergence of $U_n(t)$ to the function

$$U(t) = \int_0^t \int_0^\infty f_{\tau,\nu}(t-s) T_\beta(\tau) \left(F\left(s, W(s) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} B(s)u_0\right) + \frac{s^{\alpha-1}}{\Gamma(\alpha)} Au_0 \right) d\tau ds,$$

which satisfies problem (2.21), (2.22). Finally, $u_n(t)$ converges to the function $u(t) = U(t) + \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} u_0$ uniformly with respect to $t \in [t_0, b]$, while $u(t)$ satisfies problem (1), (2).

Remark 2.2. An assertion similar to Theorem 2.3 on the continuous dependence of the solution of problem (1), (2) on the initial data can also be formulated and proved for $\alpha = \beta \le 1$.

In the particular case where the operator B does not depend on t and is bounded and Condition 4 is satisfied, the part of Theorem 2.2 regarding the unique solvability contains [6, Theorem 8]. In [6], it is proved that (in the specified particular case) for $\alpha = \beta < 1$, the resolving operator $T_\alpha(t, A + B)$ for problem (1), (2) is of the form

$$T_\alpha(t, A + B) = \sum_{n=0}^{\infty} S_n(t),$$

where $S_0(t) = T_\alpha(t, A)$ is the resolving operator for problem (3), (4) for $\beta = \alpha$ and

$$S_n(t) = \int_0^t T_\alpha(t - s, A) B S_{n-1}(s)\, ds, \qquad n = 1, 2, \ldots.$$

In [3], the perturbation theorem is proved for an equation which, unlike Eq. (1), contains the Caputo fractional derivative, provided that the operator A is a generator of an analytic semigroup and $\beta = 1$. The following example is given in [3].

Example 2.1. Let $E = L_2(\mathbb{R}^n)$. Then Condition 4 is satisfied (see [1, p. 20]). We define the operator A on the set $D(A) = W_2^{2m}(\mathbb{R}^n)$ as follows:

$$Au(t, x) = \sum_{|p|=2m} a_p(x) \frac{\partial^{p_1+\cdots+p_n} u(t, x)}{\partial x_1^{p_1} \cdots \partial x_n^{p_n}},$$

where

$$\sum_{|p|=2m} a_p(x)\xi^p \geq (-1)^{m+1} M_0 |\xi|^{2m}$$

for all x, $\xi \in \mathbb{R}^n$ and the coefficients $a_p(x)$, $|p| = 2m$, satisfy the Hölder condition uniformly in $\mathbb{R}^n$. It is known from [3] that the operator A satisfies Condition 1 if $\beta = 1$ and $\omega = 0$.

We define the operator $B(t)$ on $D = W_2^{2m-1}(\mathbb{R}^n) \supset D(A)$ as follows:

$$B(t)u(t,x) = \sum_{|p|\leq 2m-1} a_p(t,x) \frac{\partial^{p_1+\cdots+p_n} u(t,x)}{\partial x_1^{p_1} \cdots \partial x_n^{p_n}} + \int\limits_{\Omega} \sum_{|p|\leq 2m-1} b_p(t,x,\xi) \frac{\partial^{p_1+\cdots+p_n} u(t,\xi)}{\partial \xi_1^{p_1} \cdots \partial \xi_n^{p_n}}\, d\xi,$$

where $\Omega \subset \mathbb{R}^n$, the coefficients $a_p(t,x)$ are continuous and bounded with respect to $x \in \mathbb{R}^n$ for any $|p| \leq 2m-1$ and any $t \geq 0$ and satisfy the Hölder condition with respect to t with power $\mu > \alpha$ uniformly with respect to $x \in \mathbb{R}^n$, the coefficients $b_p(t,x,\xi)$ are continuous,

$$\int\limits_{\mathbb{R}^n} \int\limits_{\Omega} |b_p(t,x,\xi)|^2 \, d\xi dx < +\infty,$$

and

$$\int\limits_{\mathbb{R}^n} \int\limits_{\Omega} |b_p(t_2,x,\xi) - b_p(t_1,x,\xi)|^2 \, d\xi dx \leq C\,|t_2 - t_1|^{\mu}, \quad \mu > \alpha,\ C > 0.$$

It is known from [3] that there exists $\gamma \in (0,1)$ such that the operator $B(t)$ satisfies Condition 2 for $\omega = 0$.

Let the operator $F(t,w)$ satisfy Condition 3. Then, by virtue of Theorems 2.1 and 2.3, problem (1), (2) (the Cauchy-type problem for an integrodifferential equation) is well posed and uniquely solvable for $u_0(x) \in W_2^{2m}(\mathbb{R}^n)$ and $\alpha < 1$.

3. Loaded Fractional-Order Differential Equations

Consider the following Cauchy-type problem in a Banach space E:

$$D^{\alpha} u(t) = Au(t) + g(u(t))p, \quad t > 0, \tag{3.1}$$

$$\lim_{t\to 0} D^{\alpha-1} u(t) = u_0, \tag{3.2}$$

where $0 < \alpha < 1$, g is a nonlinear continuous functional defined on E, A is a linear closed densely defined operator, and p is a fixed element of the space E.

Problem (3.1), (3.2) is a particular case of problem (1), (2) for $F(t, B(t)u(t)) = g(u(t))p$. Equation (3.1) contains the functional g depending on the sought solution $u(t)$. Hence, it is natural to call it a *loaded differential equation* (see the definition of a loaded differential equation, e.g., in [12, Chap. 2]).

Condition 3.1. (i) *If the function $I^{1-\alpha}u(t)$ is continuous for $t \geq 0$ and continuously differentiable for $t > 0$, then the function $D^{\alpha} g(u(t))$ belongs to $C((0,\infty),E)$ and is absolutely integrable at the origin.*

(ii) *For any $u, v \in E$, there exists a positive L such that*

$$|g(u) - g(v)| \leq L\|u - v\|. \tag{3.3}$$

Theorems 2.1 and 2.2 imply the validity of the following assertions.

Theorem 3.1. *Let $\alpha < \beta \leq 1$, Condition 1 be satisfied, and $\omega = 0$ in inequality (5). Let Conditions 4 and 3.1 be satisfied. Then problem (3.1), (3.2) has a unique solution satisfying the estimate*

$$\|u(t)\| \leq M t^{\alpha-1} e^{\omega_1 t} \|u_0\|, \quad M > 0, \quad \omega_1 > \omega. \tag{3.4}$$

Theorem 3.2. *Let $\alpha = \beta \leq 1$ and Conditions 1 and 3.1 be satisfied. Then problem (3.1), (3.2) has a unique solution satisfying estimate (3.4).*

The above solvability theorems for the Cauchy-type problem for loaded abstract equations can be used for the investigation of inverse coefficient problems for fractional-order equations.

4. Inverse Problems for Fractional-Order Differential Equations

Consider the problem of finding a pair $(w(t), \varphi(t))$ satisfying the following conditions:

$$D^{\beta} w(t) = Aw(t) + \varphi(t)p, \quad t > 0, \tag{4.1}$$

$$\lim_{t \to 0} D^{\beta-1} w(t) = u_0, \tag{4.2}$$

$$f(w(t)) = \psi(t), \tag{4.3}$$

where $0 < \beta < 1$, p and u_0 are fixed elements of $D(A)$, f is a linear continuous functional over E (i.e., f belongs to the adjoint space E^*), and $\psi(t)$ is a given scalar function.

For example, a problem to reconstruct the dependence of the perturbation on the time, using an additional observation at a space point, is a particular interpretation of the considered inverse problem.

Definition 4.1. A pair $(w(t), \varphi(t))$ is called a *solution of problem* (4.1)–(4.3) if $w(t)$ is an abstract function and $\varphi(t)$ is an absolutely integrable function such that $w(t)$ satisfies Eq. (4.1) and Conditions (4.2) and (4.3).

In [14], one can find a review of publications devoted to inverse problems for abstract integer-order differential equations. In [17], their particular implementations can be found. The inverse problem (4.1)–(4.3) for fractional-order equations was not considered before.

Condition 4.1. (i) $0 < \beta < 1$ *and* $p \in D(A)$, *where* $D(A)$ *is the domain of the operator* A.

(ii) $f \in E^*$ *and* $f(p) \neq 0$ (*the nondegeneracy condition*).

(iii) *The scalar function* $I^{1-\beta}\psi(t)$ *is continuous for* $t \geq 0$ *and continuously differentiable for* $t > 0$, *the fractional derivative* $D^{\beta}\psi(t)$ *is absolutely integrable at the origin, and the conjugation condition*

$$f(u_0) = \lim_{t \to 0} D^{\beta-1}\psi(t)$$

is satisfied.

In particular implementations, the nondegeneracy condition $f(p) \neq 0$ means that a reconstructable source acts at the observation point (see [17]).

Theorem 4.1. *Let Conditions* 1 *and* 4.1 *be satisfied. Then problem* (4.1)–(4.3) *has a unique solution.*

Proof. A solution of problem (4.1)–(4.3) is sought in the form

$$w(t) = \theta(t)p + u(t), \tag{4.4}$$

where

$$\theta(t) = I^{\beta}\varphi(t). \tag{4.5}$$

It is easy to verify that the function $u(t)$ satisfies the equation

$$D^{\beta} u(t) = Au(t) + \theta(t)Ap, \quad t > 0,$$

and the initial condition

$$\lim_{t \to 0} D^{\beta-1} u(t) = u_0. \tag{4.6}$$

Taking into account condition (4.3), we obtain the following linear equation for the function $\theta(t)$:

$$\psi(t) = \theta(t)f(p) + f(u(t)). \tag{4.7}$$

Thus, to solve inverse problem (4.1)–(4.3), it suffices to find a solution of the loaded equation

$$D^{\beta} u(t) = Au(t) + g(u(t))q, \quad t > 0, \tag{4.8}$$

satisfying condition (4.6), where $q = Ap \in E$ and

$$g(u(t)) = \frac{\psi(t) - f(u(t))}{f(p)}$$

is a continuous functional (its linearity is not assumed).

By assumption, the operator A satisfies Condition 1. Obviously, the functional $g(u(t))$ satisfies Condition 3.1. By virtue of Theorem 3.2, the Cauchy-type problem (4.8), (4.6) has a unique solution $u(t)$.

The function $\varphi(t)$ can be uniquely found from relations (4.5) and (4.7). It is of the form

$$\varphi(t) = D^\beta \theta(t) = \frac{1}{f(p)} \left(D^\beta \psi(t) - f\left(D^\beta u(t) \right) \right).$$

Finally, the function $w(t)$ is defined by relation (4.4).

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REFERENCES

1. W. Arendt, C. Batty, M. Hieber, and F. Neubrander, *Laplace Transforms and Cauchy Problems*, Birkhauser-Verlag, Basel–Boston–Berlin (2001).
2. M. M. Dzhrbashyan, *Integral Transformations and the Presentation of Functions in a Complex Domain* [in Russian], Nauka, Moscow (1966).
3. M. M. El-Borai, "Some probability densities and fundamental solutions of fractional evolution equations," *Chaos, Solitons and Fractals*, **14**, 433–440 (2002).
4. A. V. Glushak, "On the Cauchy-type problem for abstract fractional-order differential equations," *Vestnik Voronezh. Univ. Ser. Fiz. Mat.*, No. 2, 74–77 (2001).
5. A. V. Glushak and H. K. Avad, "On perturbations of abstract fractional differential equations," *Dokl. Adyg (Cherkes) Int. Akad. Nauk*, **10**, No. 1, 25–31 (2008).
6. A. V. Glushak and Yu. V. Povalyaeva, "On properties of solutions of Cauchy-type problems for abstract fractional differential equations," *Spectral and Evolution Problems*, **14**, 163–172 (2004).
7. K. Iosida, *Functional Analysis* [Russian translation], Mir, Moscow (1967).
8. T. Kato, *Perturbation Theory for Linear Operators* [Russian translation], Mir, Moscow (1972).
9. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Application of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam (2006).
10. V. A. Kostin, "The Cauchy problem for an abstract differential equation with fractional derivatives," *Russian Acad. Sci. Dokl. Math.*, **46**, No. 2, 316–319 (1992).
11. M. A. Krasnosel'skiĭ, P. P. Zabreĭko, E. I. Pustyl'nik, and P. E. Sobolevskĭ, *Integral Operators in Spaces of Summable Functions* [in Russian], Nauka, Moscow (1966).
12. A. M. Nakhushev, *Equations of Mathematical Biology* [in Russian], Vysshaya Shkola, Moscow (1995).
13. A. M. Nakhushev, *Fractional Calculus and Applications* [in Russian], Fizmatlit, Moscow (2003).
14. A. I. Prilepko, D. G. Orlovsky, and I. A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*, Marcel Dekker, New York–Basel (2000).
15. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series. Elementary Functions* [in Russian], Nauka, Moscow (1981).
16. A. V. Pskhu, *Boundary-Value Problems for Fractional-Order and Continual-Order Partial Differential Equations* [in Russian], Kabardino-Balkar Scientific Center, Nal'chik (2005).
17. A. A. Samarskii and P. N. Vabishchevich, *Numerical Methods for Solving Inverse Problems of Mathematical Physics*, Walter de Gruyter, Berlin (2007).

18. S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Integrals and Derivatives of Fractional Order and Some of Their Applications* [in Russian], Nauka i Tekhnika, Minsk (1987).

Hamed Kamal' Avad
Belgorod State University
Mathematical Analysis Department
Alexandr Vasil'evich Glushak
Belgorod State University
Mathematical Analysis Department
E-mail: aleglu@mail.ru