MATHEMATICAL MODELS OF A DIFFUSION-CONVECTION IN POROUS MEDIA

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Abstract. Mathematical models of a diffusion-convection in porous media are derived from the homogenization theory. We start with the mathematical model on the microscopic level, which consist of the Stokes system for a weakly compressible viscous liquid occupying a pore space, coupled with a diffusion-convection equation for the admixture. We suppose that the viscosity of the liquid depends on a concentration of the admixture and for this nonlinear system we prove the global in time existence of a weak solution. Next we rigorously fulfil the homogenization procedure as the dimensionless size of pores tends to zero, while the porous body is geometrically periodic. As a result, we derive new mathematical models of a diffusion-convection in absolutely rigid porous media.

1. Introduction

The present paper is devoted to a correct description of diffusion - convection processes in porous media. As a rule, this phenomena describes by diffusion-convection equation

$$\frac{\partial c}{\partial t} + \boldsymbol{v} \cdot \nabla c = \alpha_D \Delta c \tag{1.1}$$

for the concentration c of an admixture with a given velocity field $\boldsymbol{v}$ [5, 20].

It is clear that the velocity field must be defined as a solution to some dynamic equations. For many diffusion-convection processes this dynamics depends on the concentration c of the admixture, and here we look for such a coupled system.

There are different types of mathematical models, but we are interested only in some of the fundamental models of continuum mechanics (such as, for example, Stokes equations for a slow motion of a viscous liquid, or Lamé's equations for displacements of an elastic solid body), or models asymptotically close to above mentioned ones. For that reason we follow the fundamental method, suggested by Burridge and Keller [13].

To explain ideas let us consider for the moment only the dynamics in porous media. The most famous model here is the Darcy system of a filtration. It is well-known that this system is an asymptotic limit of the Stokes system for an incompressible viscous liquid in the domain Ω^ε, when dimensionless pore size ε

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tends to zero (see [13, 26]). But this Stokes system on the microscopic level is the particular case ($\alpha_\tau = 0$, $\alpha_p^\varepsilon = \infty$, $\alpha_\lambda = \infty$) of a more general system

$$\alpha_\tau \rho^\varepsilon \frac{\partial^2 \boldsymbol{w}}{\partial t^2} - \rho^\varepsilon \boldsymbol{F} = \nabla \cdot \Big(\chi^\varepsilon \alpha_\mu \mathbb{D}(x, \frac{\partial \boldsymbol{w}}{\partial t}) + (1-\chi^\varepsilon)\alpha_\lambda \mathbb{D}(x, \boldsymbol{w}) + \big(\alpha_\nu \chi^\varepsilon \nabla \cdot (\frac{\partial \boldsymbol{w}}{\partial t}) - p\big)\mathbb{I}\Big), \tag{1.2}$$

$$\alpha_p^\varepsilon p + \nabla \cdot \boldsymbol{w} = 0, \tag{1.3}$$

for the displacement $\boldsymbol{w}$ and the pressure p of the continuum medium Ω [13, 26, 21, 24]. The microscopic system (1.2), (1.3) describes the joint motion of the viscous liquid in a pore space $\Omega^\varepsilon \subset \Omega$ and an elastic solid skeleton $\Omega \backslash \overline{\Omega}^\varepsilon$ and is understood in the sense of distributions. Roughly speaking, this system contains the Stokes system for the viscous liquid in the pore space, the Lamé's system for the solid skeleton and the boundary condition (the continuity of the normal stresses) on the common boundary "solid skeleton - pore space".

In (1.2) $\mathbb{D}(x, \boldsymbol{v})$ is the symmetric part of $\nabla \boldsymbol{v}$, χ^ε is the characteristic function of the pore space Ω^ε, $\varepsilon = l/L$ is the dimensionless pore size, l is an average size of pore, L is a characteristic size of the domain in consideration,

$$\alpha_\tau = \frac{L}{g\tau^2}, \quad \alpha_\mu = \frac{2\mu}{\tau L g \rho_0}, \quad \alpha_\lambda = \frac{2\lambda}{L g \rho_0}, \quad \alpha_\nu = \frac{2\nu}{\tau L g \rho_0},$$

$$\alpha_p^\varepsilon = \alpha_{p,f}\chi^\varepsilon + \alpha_{p,s}(1-\chi^\varepsilon), \quad \rho^\varepsilon = \rho_f \chi^\varepsilon + \rho_s(1-\chi^\varepsilon),$$

$$\alpha_{p,f} = \frac{\rho_f}{Lg\,\bar{c}_f^2}, \quad \alpha_{p,s} = \frac{\rho_s}{Lg\,\bar{c}_s^2},$$

where τ is a characteristic time of the process, ρ_f and ρ_s are the respective mean dimensionless densities of the liquid in pores and the solid skeleton correlated with the mean density of water ρ_0, g is the value of acceleration of gravity, μ is the shear viscosity of liquid, ν is the bulk viscosity of liquid, λ is the elastic Lamé's constant, and $\bar{c}_f$ and $\bar{c}_s$ are speed of sound in the liquid and in the solid respectively.

Theoretically the microscopic system (1.2), (1.3) with corresponding initial and boundary conditions is one of the most adequate mathematical model, describing the joint motion of the viscous liquid in the pore space and the elastic solid skeleton. But this model has no practical significance, since it is necessary to solve the problem in the physical domain of a few hundred meters, while the coefficients oscillate on the scale of a few tens of microns. The practical significance of the model appears only after homogenization. So, we have to let all dimensionless criteria α_τ, α_μ and α_λ to be variable functions, depending on the small parameter ε, and find all limiting regimes as $\varepsilon \to 0$.

First of all note, that in the present paper we consider only filtration processes, where the characteristic time τ of processes is about several months. Thus, we may suppose that

$$\alpha_\tau \to 0 \quad \text{as } \varepsilon \to 0. \tag{1.4}$$

Next we note, that for almost all physical processes $\alpha_\mu \sim 0$ and α_λ is sufficiently large. It is known, that the asymptotic limit of (1.2), (1.3) under the conditions

$$\alpha_\mu \sim O(\varepsilon^2), \quad \alpha_\lambda \to \infty \quad \text{as } \varepsilon \searrow 0,$$

is the Darcy system of a filtration [13, 26, 21]. We say that the Darcy system of a filtration is the *first level of approximation* of the microscopic system (1.2), (1.3).

It should be noted that the condition $\alpha_\mu \sim O(\varepsilon^2)$ does not mean that for this case we consider only sufficiently small shear viscosity. This viscosity is fixed for the given physical processes. Therefore, to let α_μ be variable, we just use a representation

$$\alpha_\mu = \frac{2\mu}{\tau L g \rho_0} \cdot \frac{L^2}{l^2} \cdot \varepsilon^2 = \mu_1 \varepsilon^2, \quad \mu_1 = \frac{2\mu}{\tau L g \rho_0} \cdot \frac{L^2}{l^2},$$

with a fixed constant μ_1.

The *second level of approximation* of (1.2), (1.3) is the Terzaghi - Biot system of a poroelasticity [13, 26, 21, 8, 27] and corresponds to the conditions

$$\alpha_\mu \sim O(\varepsilon^2), \quad \alpha_\lambda \sim O(1),$$

Finally, even for sufficiently small α_μ and sufficiently large α_λ we always may suppose that

$$0 < \lambda_0, \quad \mu_0 < \infty,$$

where

$$\lambda_0 = \lim_{\varepsilon \searrow 0} \alpha_\lambda, \quad \mu_0 = \lim_{\varepsilon \searrow 0} \alpha_\mu.$$

After homogenization procedure we arrive at the system of equations of a viscoelastic filtration ([13], [21], [24]), which is the *third level of approximation* of (1.2), (1.3).

All different asymptotic models of the system (1.2), (1.3) describe the same physical process, but with different degrees of approximation. The choice of the model depends on aims of the researcher.

The same method we may apply to the diffusion-convection problem in porous media. On the microscopic level the process is described by the system (1.1) - (1.3), where $\boldsymbol{v} = \partial \boldsymbol{w}/\partial t$, and α_μ in (1.2) depends on the concentration c.

The first level of approximation of this system is a mathematical model of *a diffusion-convection in absolutely rigid porous media*, consisting of the Darcy system of a filtration with variable viscosity coupled with a homogenized diffusion-convection equation.

The second level of approximation of the system (1.1)–(1.3) is a mathematical model of *a diffusion-convection in poroelastic media*, consisting of the Terzaghi-Biot system of a poroelasticity with a variable viscosity coupled with a corresponding homogenized diffusion-convection equation.

Finally, the third level of approximation of the system (1.1)–(1.3) is a mathematical model of *a diffusion-convection in viscoelastic media*, consisting of the system of a viscoelastic filtration with a variable viscosity coupled with a corresponding homogenized diffusion-convection equation. Each level of the approximation has the proper homogenized diffusion-convection equation and it depends on the asymptotic behavior of the velocity and the concentration.

To prove the well-posedness of above mentioned mathematical models on the macroscopic level we must

(1) prove the existence of a weak solution $\{\boldsymbol{w}^\varepsilon, p^\varepsilon, c^\varepsilon\}$ to the problem (1.1)–(1.3) on the microscopic level for every fixed $\varepsilon > 0$, and

(2) fulfill the rigorous homogenization procedure as $\varepsilon \searrow 0$.

In the present paper we do it only for the mathematical model of a diffusion-convection in absolutely rigid porous media. It is clear, that the same mathematical model we obtain, if as a basic model on the microscopic level we take the Stokes system with a variable viscosity, coupled with a diffusion-convection equation (1.1) only in the liquid domain Ω^{ε}_{f}, where $\chi^{\varepsilon}(\boldsymbol{x}) = 1$.

Note, that for an incompressible liquid there is a restriction on the geometry of a solid skeleton: we may prove the correctness of the homogenization procedure only for a disconnected solid skeleton (Tartar, Appendix in [26]). To avoid this restriction we consider the Stokes system for a weakly compressible liquid

$$\alpha_{\tau}\frac{\partial \boldsymbol{v}}{\partial t} = \nabla\cdot\big((\alpha_{\mu}(c)\mathbb{D}(x,\boldsymbol{v}) + (\alpha_{\nu}\nabla\cdot\boldsymbol{v} - p)\mathbb{I}\big) + \rho_{f}\boldsymbol{F}, \tag{1.5}$$

$$\frac{\partial p}{\partial t} + c_{f}^{2}\nabla\cdot\boldsymbol{v} = 0. \tag{1.6}$$

Here α_{ν} is the dimensionless bulk viscosity and c_f is a dimensionless speed of sound in the liquid.

But this approximation results in the diffusion-convection equation an additional term containing $\nabla\cdot\boldsymbol{v}$. For an absolutely rigid skeleton the homogenization of the Stokes system has a sense if and only if the dimensionless shear viscosity α_{μ} is proportional to ε^{2}:

$$\alpha_{\mu} = \varepsilon^{2}\mu_{1}(c), 0 < \mu_{*} \leqslant \mu_{1}(c) \leqslant \mu_{*}^{-1}, \mu_{*} = \text{const.} \tag{1.7}$$

Therefore, to control the term $\nabla\cdot\boldsymbol{v}$ in (1.1) we have to suppose that α_{ν} does not depend on ε:

$$\alpha_{\nu} = \nu_{0} = \text{const.} > 0. \tag{1.8}$$

As we have mentioned above, suppositions (1.7) and (1.8) do not mean that the shear viscosity μ is much more smaller then the bulk viscosity ν. They are fixed for the given physical processes:

$$\mu_{1}(c) = \frac{2\mu(c)}{\tau L g\rho_{0}}\cdot\frac{L^{2}}{l^{2}}, \quad \nu_{0} = \frac{2\nu}{\tau L g\rho_{0}}.$$

To simplify the proofs we additionally suppose that the dimensionless coefficient of a diffusivity α_{D} also does not depend on ε:

$$\alpha_{D} = D_{0} = \text{const.} > 0. \tag{1.9}$$

In Theorem 2.3 we prove that for every $\varepsilon > 0$ the system (1.1), (1.5), (1.6) with corresponding initial and boundary conditions has at least one weak solution $\{\boldsymbol{v}^{\varepsilon}, p^{\varepsilon}, c^{\varepsilon}\}$.

The next Theorem 2.4 states that these solutions converge to the solution $\{\boldsymbol{v}, p, c\}$ of the homogenized problem

$$\begin{gathered}
\boldsymbol{v} = \frac{1}{\mu_{1}(c)}\mathbb{B}^{(f)}\big(-\frac{1}{m}\nabla q + \rho_{f}\boldsymbol{F}\big),\\
q = p + \frac{\nu_{0}}{c_{f}^{2}}\frac{\partial p}{\partial t},\\
\frac{\partial p}{\partial t} + c_{f}^{2}\nabla\cdot\boldsymbol{v} = 0,\\
m\frac{\partial c}{\partial t} + \boldsymbol{v}\cdot\nabla c = D_{0}\nabla\cdot\big(\mathbb{B}^{(c)}\nabla c\big).
\end{gathered} \tag{1.10}$$

This system with corresponding initial and boundary condition is a desired mathematical model of a diffusion-convection for a compressible liquid in absolutely rigid porous media.

In Theorem 2.5 we show that the limit in (1.10) as $c_f \to \infty$ results a mathematical model of a diffusion-convection for an incompressible liquid in absolutely rigid porous media:

$$\begin{aligned} \boldsymbol{v}^{(\infty)} &= \frac{1}{\mu_1(c^{(\infty)})}\mathbb{B}^{(f)}\big(-\frac{1}{m}\nabla p^{(\infty)} + \rho_f \boldsymbol{F}\big), \\ \nabla\cdot \boldsymbol{v}^{(\infty)} &= 0, \\ m\frac{\partial c^{(\infty)}}{\partial t} + \boldsymbol{v}^{(\infty)}\cdot\nabla c^{(\infty)} &= D_0\nabla\cdot\big(\mathbb{B}^{(c)}\nabla c^{(\infty)}\big). \end{aligned} \tag{1.11}$$

Finally, Theorem 2.6 states that after the limit in (1.10) as $\nu_0 \to 0$ we arrive at the classical Darcy system of filtration with variable viscosity for the slightly compressible liquid coupled with a diffusion-convection equation:

$$\begin{aligned} \boldsymbol{v}^{(0)} &= \frac{1}{\mu_1(c^{(0)})}\mathbb{B}^{(f)}\big(-\frac{1}{m}\nabla p^{(0)} + \rho_f \boldsymbol{F}\big), \\ \frac{\partial p^{(0)}}{\partial t} &+ c_f^2\nabla\cdot \boldsymbol{v}^{(0)} = 0, \\ m\frac{\partial c^{(0)}}{\partial t} + \boldsymbol{v}^{(0)}\cdot\nabla c^{(0)} &= D_0\nabla\cdot\big(\mathbb{B}^{(c)}\nabla c^{(0)}\big). \end{aligned} \tag{1.12}$$

In the present paper the notation for functional spaces and norms are the same as in [18, 19].

2. The problem statement and main results

To apply the well - known homogenization results [25], we must consider special domains Ω^ε and impose the following constraints.

Assumption 2.1. Let $\chi(\boldsymbol{y})$ be 1-periodic in the variable $\boldsymbol{y}$ function, such that $\chi(\boldsymbol{y}) = 1$, $\boldsymbol{y} \in Y_f \subset Y$, $\chi(\boldsymbol{y}) = 0$, $\boldsymbol{y} \in Y_s = Y\backslash\overline{Y}_f$, Y is a unit cube in $\mathbb{R}^3$.

(1) The set Y_f is an open one and $\gamma = \partial Y_f \cap \partial Y_s$ is a Lipschitz continuous surface.

(2) Let Y_f^ε be a periodic repetition in $\mathbb{R}^3$ of the elementary cell εY_f. Then Y_f^ε is a connected set with a Lipschitz continuous boundary ∂Y_f^ε.

(3) $\Omega \subset \mathbb{R}^3$ is a bounded domain with a Lipschitz continuous boundary $S = \partial\Omega$ and $\Omega^\varepsilon = \Omega \cap Y_f^\varepsilon$, $S^\varepsilon = \partial\Omega^\varepsilon$.

If $h(\boldsymbol{x})$ is a characteristic function of the domain Ω, then $\chi^\varepsilon(\boldsymbol{x}) = h(\boldsymbol{x})\chi(\boldsymbol{x}/\varepsilon)$ will be a characteristic function of the domain Ω^ε.

Now we are ready to complete problem (1.1), (1.5), (1.6) with boundary and initial conditions and formulate the definition of the weak solution. Namely, we suppose that

$$\boldsymbol{v}(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in S^\varepsilon, \quad t > 0, \tag{2.1}$$

$$\nabla c(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in S^\varepsilon, \quad t > 0, \tag{2.2}$$

$$c(\boldsymbol{x},0) = c_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^\varepsilon, \tag{2.3}$$

$$\boldsymbol{v}(\boldsymbol{x},0) = 0, \quad p(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \Omega^\varepsilon. \tag{2.4}$$

In (2.2) $\boldsymbol{n}$ is the unit outward normal vector to the boundary S^ε.

Definition 2.2. We say that a triple of functions $\{\boldsymbol{v}^\varepsilon, p^\varepsilon, c^\varepsilon\}$ is *a weak solution to problem* (1.1), (1.5), (1.6), (2.1)–(2.4) if

$$\boldsymbol{v}^\varepsilon \in L_2\big((0,T); \overset{\circ}{W}{}^1_2(\Omega^\varepsilon)\big), \quad c^\varepsilon \in L_2\big((0,T); W^1_2(\Omega^\varepsilon)\big), \quad \frac{\partial p^\varepsilon}{\partial t} \in L_2(\Omega^\varepsilon_T),$$

and

$$\begin{aligned}&\int_{\Omega^\varepsilon_T} \big(c^\varepsilon \frac{\partial \xi}{\partial t} + (c^\varepsilon \boldsymbol{v}^\varepsilon - D_0 \nabla c^\varepsilon)\cdot \nabla \xi + \xi\, c^\varepsilon \nabla\cdot \boldsymbol{v}^\varepsilon\big)\, dx\, dt \\ &= -\int_\Omega \chi^\varepsilon c_0(\boldsymbol{x})\, \xi(\boldsymbol{x},0)\, dx,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\int_{\Omega^\varepsilon_T} \big(\varepsilon^2 \mu_1(c^\varepsilon)\, \mathbb{D}(x,\boldsymbol{v}^\varepsilon) - (p^\varepsilon - \nu_0 \nabla\cdot \boldsymbol{v}^\varepsilon)\mathbb{I}\big) : \mathbb{D}(x,\boldsymbol{\varphi}) dx\, dt \\ &= \int_{\Omega^\varepsilon_T} \rho_f \big(\alpha_\tau \boldsymbol{v}^\varepsilon \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + \boldsymbol{F}\cdot\boldsymbol{\varphi}\big)\, dx\, dt,\end{aligned} \tag{2.6}$$

$$\int_{\Omega^\varepsilon_T} \big(\frac{\partial \psi}{\partial t} p^\varepsilon + c_f^2\, \boldsymbol{v}^\varepsilon \cdot \nabla\psi\big)\, dx\, dt = 0 \tag{2.7}$$

for any smooth functions ξ, ψ and $\boldsymbol{\varphi}$, such that $\xi(\boldsymbol{x},T) = \psi(\boldsymbol{x},T) = 0$ and $\boldsymbol{\varphi}(\boldsymbol{x},t) = 0$ for $\boldsymbol{x} \in S^\varepsilon$.

Note, that the integral identity (2.5) contains the differential equation (1.1) in the pore space, the boundary condition (2.2) on the boundary $S^{(\varepsilon)}$, and the initial condition (2.3). The boundary condition (2.1) is already included into the corresponding functional space for $\boldsymbol{v}$.

Theorem 2.3. *Let $c_0(\boldsymbol{x})$ and $\boldsymbol{F}(\boldsymbol{x},t)$ be measurable functions,*

$$0 \leqslant c_0(\boldsymbol{x}) \leqslant 1, \quad \int_{\Omega_T} |\boldsymbol{F}(\boldsymbol{x},t)|^2 dx\, dt \leqslant F^2 < \infty, \quad \mu_1 \in C^2[0,\infty),$$

and conditions (1.4), (1.7)–(1.9) *hold.*

Then problem (1.1), (1.5), (1.6), (2.1)–(2.4) *has at least one weak solution* $\{\boldsymbol{v}^\varepsilon, p^\varepsilon, c^\varepsilon\}$, *such that*

$$\int_{\Omega^\varepsilon_T} |\nabla c^\varepsilon|^2 dx\, dt \leqslant C, \tag{2.8}$$

$$\begin{aligned}&\max_{0<t<T} \alpha_\tau \int_{\Omega^\varepsilon} |\boldsymbol{v}^\varepsilon(\boldsymbol{x},t)|^2 dx \\ &+ \int_{\Omega^\varepsilon_T} \big(\varepsilon^2 |\nabla \boldsymbol{v}^\varepsilon|^2 + |\frac{\partial p^\varepsilon}{\partial t}|^2 + (\nabla\cdot\boldsymbol{v}^\varepsilon)^2\big) dx\, dt \leqslant CF^2,\end{aligned} \tag{2.9}$$

$$0 \leqslant c^\varepsilon(\boldsymbol{x},t) \leqslant 1, \boldsymbol{x} \in \Omega^\varepsilon, t > 0, \tag{2.10}$$

where C is independent of ε.

Homogenization means the limiting procedure as $\varepsilon \searrow 0$. But for our method it is possible only for functions, defined in whole domain Ω_T. So, we first extend the functions $\boldsymbol{v}^\varepsilon$, p^ε, and c^ε onto Ω_T and then apply the homogenization theory.

The functions $\boldsymbol{v}^\varepsilon$, $\nabla\cdot\boldsymbol{v}^\varepsilon$, and p^ε are extending in a trivial way by setting $\tilde{\boldsymbol{v}}^\varepsilon = \boldsymbol{v}^\varepsilon$, $\tilde{p}^\varepsilon = p^\varepsilon$ in Ω^ε_T, and $\tilde{\boldsymbol{v}}^\varepsilon = 0$, $\nabla\cdot\tilde{\boldsymbol{v}}^\varepsilon = 0$, and $\tilde{p}^\varepsilon = 0$ outside Ω^ε_T.

For the functions c^ε the extension result [1] states that there exists an extension $\tilde{c}^\varepsilon \in L_2\big((0,T); W_2^1(\Omega)\big)$ such that $c^\varepsilon(\boldsymbol{x},t) = \tilde{c}^\varepsilon(\boldsymbol{x},t)$ for $(\boldsymbol{x},t) \in \Omega_T^\varepsilon$ and

$$\int_\Omega |\tilde{c}^{\,\varepsilon}(\boldsymbol{x},t)|^2 dx \leqslant C_0 \int_{\Omega^\varepsilon} |c^\varepsilon(\boldsymbol{x},t)|^2 dx,$$

$$\int_\Omega |\nabla\tilde{c}^{\,\varepsilon}(\boldsymbol{x},t)|^2 dx \leqslant C_0 \int_{\Omega^\varepsilon} |\nabla c^\varepsilon(\boldsymbol{x},t)|^2 dx.$$

Theorem 2.4. *Under the conditions of Theorem 2.3 let $\{\boldsymbol{v}^\varepsilon, p^\varepsilon, c^\varepsilon\}$ be the solution to the problem* (1.1), (1.5), (1.6), (2.1)–(2.4). *Then*

(I) *there exists a subsequence of small parameters $\{\varepsilon > 0\}$ as $\varepsilon \searrow 0$, such that*
(1) the sequence $\{\tilde{\boldsymbol{v}}^\varepsilon\}$ converges weakly in $L_2(\Omega_T)$ to the function $\boldsymbol{v}$,
(2) the sequence $\{\nabla\cdot\tilde{\boldsymbol{v}}^\varepsilon\}$ converges weakly in $L_2(\Omega_T)$ to the function $\nabla\cdot\boldsymbol{v}$,
(3) the sequence $\{\tilde{p}^\varepsilon\}$ converges weakly in $L_2(\Omega_T)$ to the function p,
(4) the sequence $\{\tilde{c}^\varepsilon\}$ converges weakly in $L_2\big((0,T); W_2^1(\Omega)\big)$ and strongly in $L_2(\Omega_T)$ to the function c.

(II) *The triple of limiting functions $\{\boldsymbol{v}, p, c\}$ is a weak solution to the diffusion-convection problem for a compressible liquid in absolutely rigid porous media, which consists of the dynamic equations*

$$\boldsymbol{v} = \frac{1}{\mu_1(c)}\mathbb{B}^{(f)}\big(-\frac{1}{m}\nabla q + \rho_f \boldsymbol{F}\big),\ \boldsymbol{x}\in\Omega, t>0, \tag{2.11}$$

$$q = p + \frac{\nu_0}{c_f^2}\frac{\partial p}{\partial t},\ \boldsymbol{x}\in\Omega, t>0, \tag{2.12}$$

$$\frac{\partial p}{\partial t} + c_f^2\nabla\cdot\boldsymbol{v} = 0,\ \boldsymbol{x}\in\Omega, t>0 \tag{2.13}$$

for the velocity $\boldsymbol{v}$ and the pressure p of the slightly compressible liquid, and the diffusion-convection equation

$$m\frac{\partial c}{\partial t} + \boldsymbol{v}\cdot\nabla c = D_0\nabla\cdot\big(\mathbb{B}^{(c)}\nabla c\big),\ \boldsymbol{x}\in\Omega, t>0, \tag{2.14}$$

for the concentration c of the admixture.

The problem is endowed with the homogeneous boundary conditions

$$\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = 0, \quad \boldsymbol{x}\in S, t>0, \tag{2.15}$$

$$\nabla c(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = 0, \quad \boldsymbol{x}\in S, t>0, \tag{2.16}$$

and the initial conditions

$$p(\boldsymbol{x},0) = 0, \quad c(\boldsymbol{x},0) = c_0(\boldsymbol{x}) \quad \boldsymbol{x}\in\Omega. \tag{2.17}$$

In (2.11)–(2.17)

$$m = \langle\chi\rangle_Y = \int_Y \chi(\boldsymbol{y})dy$$

is a porosity, the symmetric and strictly positively definite constant matrix $\mathbb{B}^{(f)}$ is defined below by formula (4.6), *the symmetric and strictly positively definite constant matrix $\mathbb{B}^{(c)}$ is defined below by formula* (4.8), *and $\boldsymbol{n}$ is the unit outward normal vector to the boundary S.*

Theorem 2.5. *Let $\{\boldsymbol{v}^{(k)}, p^{(k)}, c^{(k)}\}$ be the solution to* (2.11)–(2.17) *with $c_f^2 = k$. Then there exists a subsequence $k_n \to \infty$ such that*

(1) *the sequence $\{\boldsymbol{v}^{(k_n)}\}$ converges weakly in $L_2(\Omega_T)$ to the function $\boldsymbol{v}^{(\infty)}$,*

(2) *the sequence* $\{p^{(k_n)}\}$ *converges weakly in* $L_2(\Omega_T)$ *to the function* $p^{(\infty)}$,
(3) *the sequence* $\{c^{(k_n)}\}$ *converges weakly in* $L_2\big((0,T);W_2^1(\Omega)\big)$ *and strongly in* $L_2(\Omega_T)$ *to the function* $c^{(\infty)}$;

The triple of limiting functions $\{\boldsymbol{v}^{(\infty)}, p^{(\infty)}, c^{(\infty)}\}$ *is a weak solution to the diffusion-convection problem for an incompressible liquid in absolutely rigid porous media, which consists of the Darcy system of a filtration with a variable viscosity*

$$\boldsymbol{v}^{(\infty)} = \frac{1}{\mu_1(c^{(\infty)})}\mathbb{B}^{(f)}\big(-\frac{1}{m}\nabla p^{(\infty)} + \rho_f \boldsymbol{F}\big), \quad \boldsymbol{x} \in \Omega,\ t > 0, \tag{2.18}$$

$$\nabla \cdot \boldsymbol{v}^{(\infty)} = 0, \quad \boldsymbol{x} \in \Omega,\ t > 0 \tag{2.19}$$

for the velocity $\boldsymbol{v}^{(\infty)}$ *and the pressure* $p^{(\infty)}$ *of the incompressible liquid, the diffusion-convection equation* (2.14) *with the velocity field* $\{\boldsymbol{v}^{(\infty)}\}$ *for the concentration* $c^{(\infty)}$, *boundary conditions* (2.15) *and* (2.16), *and the initial condition* (2.17) *for the concentration.*

Theorem 2.6. *Let* $\{\boldsymbol{v}^{(\delta)}, p^{(\delta)}, c^{(\delta)}\}$ *be the solution to problem* (2.11)–(2.17) *with* $\nu_0 = \delta$. *Then there exists a subsequence* $\delta_n \to 0$ *such that*

(1) *the sequence* $\{\boldsymbol{v}^{(\delta_n)}\}$ *converges weakly in* $L_2(\Omega_T)$ *to the function* $\boldsymbol{v}^{(0)}$,
(2) *the sequence* $\{p^{(\delta_n)}\}$ *converges weakly in* $L_2(\Omega_T)$ *to the function* $p^{(0)}$,
(3) *the sequence* $\{c^{(\delta_n)}\}$ *converges weakly in* $L_2\big((0,T);W_2^1(\Omega)\big)$ *and strongly in* $L_2(\Omega_T)$ *to the function* $c^{(0)}$;

The triple of limiting functions $\{\boldsymbol{v}^{(0)}, p^{(0)}, c^{(0)}\}$ *is a weak solution to the diffusion-convection problem for a slightly compressible liquid in absolutely rigid porous media, which consists of the Darcy system of a filtration with a variable viscosity*

$$\boldsymbol{v}^{(0)} = \frac{1}{\mu_1(c^{(0)})}\mathbb{B}^{(f)}\big(-\frac{1}{m}\nabla p^{(0)} + \rho_f \boldsymbol{F}\big), \quad \boldsymbol{x} \in \Omega,\ t > 0, \tag{2.20}$$

$$\frac{\partial p^{(0)}}{\partial t} + c_f^2 \nabla \cdot \boldsymbol{v}^{(0)} = 0, \quad \boldsymbol{x} \in \Omega,\ t > 0 \tag{2.21}$$

for the velocity $\boldsymbol{v}^{(0)}$ *and the pressure* $p^{(0)}$ *of the slightly compressible liquid, the diffusion-convection equation* (2.14) *with the velocity field* $\{\boldsymbol{v}^{(0)}\}$ *for the concentration* $c^{(0)}$, *boundary conditions* (2.15) *and* (2.16), *and the initial condition* (2.17) *for the concentration.*

3. Proof of Theorem 2.3

Let us divide the proof into two stages. As a first step we consider the approximate problem, where the velocity $\boldsymbol{v}$ (for the moment we omit the index ε) in the convection–diffusion equation is replaced by its approximation

$$\boldsymbol{v}_{(h)}(\boldsymbol{x},t) = \mathbb{M}^h(\boldsymbol{v}) = \frac{1}{h^4}\int_{-\infty}^{\infty} J(\frac{t-\tau}{h})\Big(\int_{\mathbb{R}^3} J(\frac{|\boldsymbol{z}-\boldsymbol{x}|}{h})\bar{\boldsymbol{v}}(\boldsymbol{z},\tau)dz\Big)d\tau. \tag{3.1}$$

In (3.1)

$$\bar{\boldsymbol{v}}(\boldsymbol{x},t) = \begin{cases} \boldsymbol{v}(\boldsymbol{x},t) & \text{if } x \in \Omega^\varepsilon,\ t > 0, \\ 0 & \text{if } x \in \mathbb{R}^3 \backslash \Omega^\varepsilon,]t > 0, \\ 0 & \text{if } x \in \Omega^\varepsilon \text{ and } t \geqslant T, \text{ or } t \leqslant 0, \end{cases}$$

and $J(s)$ is an infinitely smooth function, such that

$$J(s)=0, \text{ if } |s|>1, \quad \text{and} \int_{-\infty}^{\infty} J(s)ds \int_{\mathbb{R}^3} J(|\boldsymbol{x}|)dx = 1.$$

By the well - known properties of mollifiers $\mathbb{M}^h$ [2]

(1) $\boldsymbol{v}_{(h)} \in C^{\infty}\big(\mathbb{R}^3 \times (-\infty,\infty)\big)$;
(2) if $\boldsymbol{v} \in L_2(\Omega^{\varepsilon}_T)$, then $\boldsymbol{v}_{(h)} \to \boldsymbol{v}$ strongly in $L_2(\Omega^{\varepsilon}_T)$ as $h \to 0$;
(3) if $\boldsymbol{v} \in L_2\big((0,T); \mathring{W}^1_2(\Omega^{\varepsilon})\big)$, then $\nabla \cdot \boldsymbol{v}_{(h)} \to \nabla \cdot \boldsymbol{v}$ strongly in $L_2(\Omega^{\varepsilon}_T)$ as $h \to 0$.

More precisely, we look for the solution $\{\boldsymbol{v}^{\varepsilon,h}, p^{\varepsilon,h}, c^{\varepsilon,h}\}$ of the differential equations

$$\rho_f \alpha_\tau \frac{\partial \boldsymbol{v}^{\varepsilon,h}}{\partial t} = \nabla \cdot \big(\varepsilon^2 \mu_1(c)\mathbb{D}(x, \boldsymbol{v}^{\varepsilon,h}) + (\nu_0 \nabla \cdot \boldsymbol{v}^{\varepsilon,h} - p^{\varepsilon,h})\mathbb{I}\big) + \rho_f \boldsymbol{F}, \tag{3.2}$$

$$\frac{\partial p^{\varepsilon,h}}{\partial t} + c_f^2 \nabla \cdot \boldsymbol{v}^{\varepsilon,h} = 0, \tag{3.3}$$

$$\frac{\partial c^{\varepsilon,h}}{\partial t} + \boldsymbol{v}^{\varepsilon,h}_{(h)} \cdot \nabla c^{\varepsilon,h} = D_0 \Delta c^{\varepsilon,h} \tag{3.4}$$

in the domain Ω^{ε}_T, satisfying the following boundary and initial conditions

$$\boldsymbol{v}^{\varepsilon,h}(\boldsymbol{x},t) = 0, \boldsymbol{x} \in S^{\varepsilon}, t>0, \tag{3.5}$$

$$\nabla c^{\varepsilon,h}(\boldsymbol{x},t) \cdot \boldsymbol{n}(\boldsymbol{x}) = 0, \boldsymbol{x} \in S^{\varepsilon}, t>0, \tag{3.6}$$

$$c^{\varepsilon,h}(\boldsymbol{x},0) = c_0^h(\boldsymbol{x}), \boldsymbol{x} \in \Omega^{\varepsilon}, \tag{3.7}$$

$$\boldsymbol{v}^{\varepsilon,h}(\boldsymbol{x},0) = 0, p^{\varepsilon,h}(\boldsymbol{x},0) = 0, \boldsymbol{x} \in \Omega^{\varepsilon}. \tag{3.8}$$

In (3.4) and (3.7) $\boldsymbol{v}^{\varepsilon,h}_{(h)} = \mathbb{M}^h(\boldsymbol{v}^{\varepsilon,h})$, and

$$c_0^h \in \mathring{C}^{\infty}(\Omega^{\varepsilon}), 0 \leqslant c_0^h(\boldsymbol{x}) \leqslant 1, c_0^h(\boldsymbol{x}) \to c_0(\boldsymbol{x}) \quad \text{as } h \to 0 \, \text{a.e. in} \, \Omega^{\varepsilon}.$$

To solve (3.2)–(3.8) we fix the set $\mathfrak{M} = \{\tilde{c} \in C(\overline{\Omega}^{\varepsilon}_T) : 0 \leqslant \tilde{c}(\boldsymbol{x},t) \leqslant 1\}$ and consider the first auxiliary problem

$$\rho_f \alpha_\tau \frac{\partial \boldsymbol{u}}{\partial t} = \nabla \cdot \big(\varepsilon^2 \mu_1(\tilde{c})\mathbb{D}(x, \boldsymbol{u}) + (\nu_0 \nabla \cdot \boldsymbol{u} - q)\mathbb{I}\big) + \rho_f \boldsymbol{F}, \tag{3.9}$$

$$\frac{\partial q}{\partial t} + c_f^2 \nabla \cdot \boldsymbol{u} = 0, \tag{3.10}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = 0, \boldsymbol{x} \in S^{\varepsilon}, t>0; \; \boldsymbol{u}(\boldsymbol{x},0) = 0, q(\boldsymbol{x},0) = 0, \boldsymbol{x} \in \Omega^{\varepsilon}. \tag{3.11}$$

For all $\tilde{c} \in \mathfrak{M}$ this problem defines a nonlinear operator

$$\boldsymbol{u} = \mathbb{A}_1(\tilde{c}), \mathbb{A}_1 : \mathfrak{M} \to L_2\big((0,T); \mathring{W}^1_2(\Omega^{\varepsilon})\big).$$

Next we consider the second auxiliary problem

$$\frac{\partial c}{\partial t} + \boldsymbol{u}_{(h)} \cdot \nabla c = D_0 \Delta c, \tag{3.12}$$

$$\nabla c(\boldsymbol{x},t) \cdot \boldsymbol{n}(\boldsymbol{x}) = 0, \boldsymbol{x} \in S^{\varepsilon}, t>0, \tag{3.13}$$

$$c(\boldsymbol{x},0) = c_0^h(\boldsymbol{x}), \boldsymbol{x} \in \Omega^{\varepsilon}, \tag{3.14}$$

where

$$\boldsymbol{u}_{(h)} = \mathbb{M}^h(\boldsymbol{u}), \boldsymbol{u} = \mathbb{A}_1(\tilde{c}).$$

The problem (3.12)–(3.14) defines a nonlinear operator $\mathbb{A}_2$, which due to the maximum principle transforms $L_2\big((0,T);\overset{\circ}{W}{}^1_2(\Omega^\varepsilon)\big)$ into the set $\mathfrak{M}$:

$$c = \mathbb{A}_2(\boldsymbol{u}), \quad \mathbb{A}_2 : L_2\big((0,T);\overset{\circ}{W}{}^1_2(\Omega^\varepsilon)\big) \to \mathfrak{M}.$$

Thus, the nonlinear operator $\mathbb{A} = \mathbb{A}_2 \cdot \mathbb{A}_1$ transforms the set $\mathfrak{M}$ into itself. It is clear that all fixed points $c^{\varepsilon,h}$ of the operator $\mathbb{A}$ define solutions $\{\boldsymbol{v}^{\varepsilon,h}, p^{\varepsilon,h}, c^{\varepsilon,h}\}$ to the problem (3.2)–(3.8). To prove the existence of at least one fixed point of $\mathbb{A}$ we have to show that $\mathbb{A}$ is a completely continuous operator.

The weak solutions to the problems (3.2)–(3.8) and (3.9)–(3.11) are defined in a same way, as a weak solution to problem (1.1), (1.5), (1.6), (2.1)–(2.4).

Lemma 3.1. *Under the conditions of Theorem 2.3 for any $\tilde{c} \in \mathfrak{M}$, problem (3.9)–(3.11) has the unique weak solution $\{\boldsymbol{u}, q\}$, such that*

$$\max_{0<t<T} \alpha_\tau \int_{\Omega^\varepsilon} |\boldsymbol{u}(\boldsymbol{x},t)|^2 dx + \int_{\Omega^\varepsilon_T} \big(\varepsilon^2 |\nabla \boldsymbol{u}|^2 + |\frac{\partial q}{\partial t}|^2 + (\nabla \cdot \boldsymbol{u})^2\big) dx\, dt \leqslant C\, F^2, \tag{3.15}$$

and for any $\tilde{c}_1, \tilde{c}_2 \in \mathfrak{M}$

$$\begin{aligned} &\max_{0<t<T} \alpha_\tau \int_{\Omega^\varepsilon} |(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^2(\boldsymbol{x},t) dx + \int_{\Omega^\varepsilon_T} \varepsilon^2 |\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^2 dx\, dt \\ &\leqslant C\, F^2 \big(|\tilde{c}_1 - \tilde{c}_2|^{(0)}_{\Omega^\varepsilon_T}\big)^2, \end{aligned} \tag{3.16}$$

where C is independent of ε and h. In (3.16) $\boldsymbol{u}_i = \mathbb{A}_1(\tilde{c}_i), i = 1, 2$.

Lemma 3.2. *Under the conditions of Theorem 2.3 let $\boldsymbol{u}_{(h)} = \mathbb{M}^h(\boldsymbol{u})$, $\boldsymbol{u} = \mathbb{A}_1(\tilde{c})$ for $\tilde{c} \in \mathfrak{M}$. Then problem (3.12)–(3.14) has a unique solution $c \in C^{2,1}(\overline{\Omega}^\varepsilon_T)$, such that*

$$\langle c \rangle^{(2,1)}_{\Omega^\varepsilon_T} \leqslant N(h), 0 \leqslant c(\boldsymbol{x},t) \leqslant 1, \tag{3.17}$$

$$\int_{\Omega^\varepsilon_T} |\nabla c|^2 dx\, dt \leqslant C, \tag{3.18}$$

where C is independent of ε and h.

If $c_i = \mathbb{A}_2(\boldsymbol{u}_i)$, $\boldsymbol{u}_i = \mathbb{A}_1(\tilde{c}_i)$, $i = 1, 2$, for $\tilde{c}_1, \tilde{c}_2 \in \mathfrak{M}$, then

$$\begin{aligned} &\max_{0<t<T} \int_{\Omega^\varepsilon} |c_1(\boldsymbol{x},t) - c_2(\boldsymbol{x},t)|^2 dx + \int_{\Omega^\varepsilon_T} |\nabla(c_1 - c_2)|^2 dx\, dt \\ &\leqslant N(h) \int_{\Omega^\varepsilon_T} |(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^2 dx dt. \end{aligned} \tag{3.19}$$

Now, to prove the solvability of (3.2)–(3.8) we just apply Schauder fixed point theorem [17]. In fact, estimates (3.16), (3.19) and interpolation inequality [18]

$$|c|^{(0)}_{\Omega^\varepsilon_T} \leqslant \beta\big(\|c\|_{2,\Omega^\varepsilon_T}\big)^{1-\alpha} \big(\langle c \rangle^{(2)}_{\Omega^\varepsilon_T}\big)^\alpha, \quad 0 < \alpha < 1,$$

prove the continuity of $\mathbb{A}$. The first estimate (3.17) shows that $\mathbb{A}$ is a compact operator. Therefore $\mathbb{A}$ is completely continuous operator on the set $\mathfrak{M}$. The second estimate (3.17) shows that $\mathbb{A}$ transforms the set $\mathfrak{M}$ into itself. Finally, $\mathfrak{M}$ is a closed convex set, which enough for existence at least one fixed point of $\mathbb{A}$ in $\mathfrak{M}$.

It is clear, that all fixed points of $\mathbb{A}$ preserve estimates (3.15), (3.17), and (3.18). Thus the following lemma holds.

Lemma 3.3. *Under the conditions of Theorem 2.3 there exists at least one weak solution* $\{\boldsymbol{v}^{\varepsilon,h}, p^{\varepsilon,h}, c^{\varepsilon,h}\}$ *to the problem* (3.2)–(3.8), *such that*

$$\max_{0<t<T} \alpha_\tau \int_{\Omega^\varepsilon} |\boldsymbol{v}^{\varepsilon,h}(\boldsymbol{x},t)|^2 dx + \int_{\Omega^\varepsilon_T} \big(\varepsilon^2 |\nabla \boldsymbol{v}^{\varepsilon,h}|^2 + |\frac{\partial p^{\varepsilon,h}}{\partial t}|^2 + (\nabla \cdot \boldsymbol{v}^{\varepsilon,h})^2\big) dx dt \leqslant C F^2, \tag{3.20}$$

$$0 \leqslant c^{\varepsilon,h}(\boldsymbol{x},t) \leqslant 1, \quad \int_{\Omega^\varepsilon_T} |\nabla c^{\varepsilon,h}|^2 dx\, dt \leqslant C, \tag{3.21}$$

where C *is independent of* ε *and* h.

As a last step in the proof of Theorem 2.3 we pass to the limit as $h \to 0$ in a corresponding integral identity.

Lemma 3.4. *Under the conditions of Theorem 2.3 there exists at least one weak solution* $\{\boldsymbol{v}^{\varepsilon}, p^{\varepsilon}, c^{\varepsilon}\}$ *to problem* (1.1), (1.5), (1.6), (2.1)–(2.4), *such that*

$$\max_{0<t<T} \alpha_\tau \int_{\Omega^\varepsilon} |\boldsymbol{v}^{\varepsilon}(\boldsymbol{x},t)|^2 dx + \int_{\Omega^\varepsilon_T} \big(|\boldsymbol{v}^{\varepsilon}|^2 + \varepsilon^2 |\nabla \boldsymbol{v}^{\varepsilon}|^2 + |\frac{\partial p^{\varepsilon}}{\partial t}|^2 + (\nabla \cdot \boldsymbol{v}^{\varepsilon})^2\big) dx\, dt \leqslant C F^2, \tag{3.22}$$

$$0 \leqslant c^{\varepsilon}(\boldsymbol{x},t) \leqslant 1, \quad \int_{\Omega^\varepsilon_T} |\nabla c^{\varepsilon}|^2 dx\, dt \leqslant C, \tag{3.23}$$

where C *is independent of* ε.

Proof of Lemma 3.1. The proof of the first part of this lemma is standard. For example, it might be based upon the Galerkin method. For this case we rewrite the problem (3.9)–(3.11) in the domain Ω^ε_T as

$$\rho_f \alpha_\tau \frac{\partial \boldsymbol{u}}{\partial t} - \rho_f \boldsymbol{F} = \nabla \cdot \Big(\varepsilon^2 \mu_1(\tilde{c}) \mathbb{D}(x, \boldsymbol{u}) + \big(\nu_0 \nabla \cdot \boldsymbol{u} + c_f^2 \int_0^t \nabla \cdot \boldsymbol{u}(\boldsymbol{x},\tau) d\tau\big)\mathbb{I}\Big), \tag{3.24}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in S^\varepsilon, \quad t > 0, \quad \boldsymbol{u}(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \Omega^\varepsilon. \tag{3.25}$$

Next we choose some basis $\{\boldsymbol{\varphi}_k\}_{k=1}^\infty$ in $\overset{\circ}{W}{}^1_2(\Omega^\varepsilon)$ and look for the approximate solution in the form

$$\boldsymbol{u}^{(n)} = \sum_{k=1}^n a_k^{(n)}(t) \boldsymbol{\varphi}_k(\boldsymbol{x}),$$

where the functions $a_k^{(n)}(t)$ are defined by the system

$$\int_{\Omega^\varepsilon} \big(\rho_f (\alpha_\tau \frac{\partial \boldsymbol{u}^{(n)}}{\partial t} - \boldsymbol{F})\big) \cdot \boldsymbol{\varphi}_m dx = \int_{\Omega^\varepsilon} \Big(\varepsilon^2 \mu_1(\tilde{c}) \mathbb{D}(x, \boldsymbol{u}^{(n)}) + \big(\nu_0 \nabla \cdot \boldsymbol{u}^{(n)} + c_f^2 \int_0^t \nabla \cdot \boldsymbol{u}^{(n)}(\boldsymbol{x},\tau) d\tau\big)\mathbb{I}\Big) : \mathbb{D}(x, \boldsymbol{\varphi}_m) dx, \tag{3.26}$$

for $m = 1, 2, \ldots n$. For the approximate solutions $\boldsymbol{u}^{(n)}$ and its limit $\boldsymbol{u}$, estimates (3.15) hold. In fact, to prove it we just multiply the $m-th$ equation of the system by $a_k^{(n)}(t)$, sum results of multiplication for all $m = 1, 2, \dots, n$, and use Hölder, Korn, Friedrichs-Poincaré and Gronwall inequalities.

Note, that due to a special geometry of the domain Ω^ε the Friedrichs-Poincaré inequality has a form

$$\int_{\Omega^\varepsilon} |\boldsymbol{u}|^2 dx \leqslant C\,\varepsilon^2 \int_{\Omega^\varepsilon} |\nabla \boldsymbol{u}|^2 dx$$

[26]. We also use Korn inequality only for the domain Ω. If $\bar{\boldsymbol{v}}$ is a zero extension of $\boldsymbol{v}$ from Ω^ε onto Ω, then

$$\int_{\Omega^\varepsilon} |\nabla \boldsymbol{v}|^2 dx = \int_{\Omega} |\nabla \bar{\boldsymbol{v}}|^2 dx \leqslant C \int_{\Omega} |\mathbb{D}(x, \bar{\boldsymbol{v}})|^2 dx = C \int_{\Omega^\varepsilon} |\mathbb{D}(x, \boldsymbol{v})|^2 dx.$$

The proof of the second part of the lemma is also standard. We consider the initial–boundary value problem for the difference $\hat{\boldsymbol{u}} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, multiply the differential equation for $\hat{\boldsymbol{u}}$ by $\hat{\boldsymbol{u}}$, integrate the result by parts over domain Ω^ε, and use Hölder inequality.

Proof of Lemma 3.2. Estimate (3.17) is a consequence of the properties of mollifiers and well - known results ([18]): the solution $c(\boldsymbol{x}, t)$ of (3.12)–(3.14) is infinitely smooth and satisfy the maximum principle.

To prove (3.18) we consider the initial - boundary value problem for the difference $\hat{c} = c_1 - c_2$, multiply the differential equation for $\hat{c}$ by $\hat{c}$, and integrate the result by parts over domain Ω^ε:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^\varepsilon} |\hat{c}|^2 dx + D_0 \int_{\Omega^\varepsilon} |\nabla \hat{c}|^2 dx = I_1 + I_2,$$

$$I_1 \equiv -\int_{\Omega^\varepsilon} \big((\boldsymbol{u}_1)_{(h)} \cdot \nabla \hat{c}\big)\hat{c} dx \leqslant \frac{D_0}{2}\int_{\Omega^\varepsilon} |\nabla \hat{c}|^2 dx + \frac{1}{2D_0} \max_{\Omega^\varepsilon} |(\boldsymbol{u}_1)_{(h)}| \int_{\Omega^\varepsilon} |\hat{c}|^2 dx$$

$$I_2 \equiv -\int_{\Omega^\varepsilon} \big(\hat{\boldsymbol{u}} \cdot \nabla c_2\big)\, \hat{c}\, dx \leqslant \int_{\Omega^\varepsilon} |\hat{\boldsymbol{u}}|^2 dx + \frac{1}{4} \max_{\Omega^\varepsilon} |\nabla c_2| \int_{\Omega^\varepsilon} |\hat{c}|^2 dx.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^\varepsilon} |\hat{c}|^2 dx + \frac{D_0}{2}\int_{\Omega^\varepsilon} |\nabla \hat{c}|^2 dx \leqslant \int_{\Omega^\varepsilon} |\hat{\boldsymbol{u}}|^2 dx + N(h)\int_{\Omega^\varepsilon} |\hat{c}|^2 dx,$$

and estimate (3.18) follows from Gronwall inequality.

Proof of Lemma 3.4. To prove the lemma we just have to find convergent subsequences and pass to the limit as $h \searrow 0$ in integral identities

$$\begin{aligned} &\int_{\Omega^\varepsilon_T} \big(c^{\varepsilon,h}\frac{\partial \xi}{\partial t} + (c^{\varepsilon,h}\boldsymbol{v}^{\varepsilon,h}_{(h)} - D_0 \nabla c^{\varepsilon,h}) \cdot \nabla \xi + \xi\, c^{\varepsilon,h}\nabla \cdot \boldsymbol{v}^{\varepsilon,h}_{(h)}\big)\, dx\, dt \\ &+ \int_{\Omega} \chi^\varepsilon c_0^h(\boldsymbol{x})\, \xi(\boldsymbol{x}, 0)\, dx = 0, \end{aligned} \tag{3.27}$$

$$\begin{aligned} &\int_{\Omega^\varepsilon_T} \Big(\big(\varepsilon^2 \mu_1(c^{\varepsilon,h})\mathbb{D}(x, \boldsymbol{v}^{\varepsilon,h}) - (p^{\varepsilon,h} - \nu_0 \nabla \cdot \boldsymbol{v}^{\varepsilon,h})\mathbb{I}\big) : \mathbb{D}(x, \boldsymbol{\varphi})\Big) dxdt \\ &= \int_{\Omega^\varepsilon_T} \rho_f \big(\alpha_\tau \boldsymbol{v}^{\varepsilon,h} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} + \boldsymbol{F} \cdot \boldsymbol{\varphi}\big)\, dxdt, \end{aligned} \tag{3.28}$$

$$\int_{\Omega^\varepsilon_T} \big(\frac{\partial \psi}{\partial t} p^{\varepsilon,h} + c_f^2\, \boldsymbol{v}^{\varepsilon,h} \cdot \nabla \psi\big)\, dx\, dt = 0 \tag{3.29}$$

for any smooth functions ξ, ψ and $\boldsymbol{\varphi}$, such that $\xi(\boldsymbol{x},T)=\psi(\boldsymbol{x},T)=0$ and $\boldsymbol{\varphi}(\boldsymbol{x},t)=0$ for $\boldsymbol{x}\in S^{\varepsilon}$.

The weak compactness of $\{p^{\varepsilon,h}\}$ and $\{\nabla\cdot\boldsymbol{v}^{\varepsilon,h}\}$ in $L_2(\Omega^{\varepsilon}_T)$ and the weak compactness of $\{c^{\varepsilon,h}\}$ and $\{\boldsymbol{v}^{\varepsilon,h}\}$ in $L_2\big((0,T);W^1_2(\Omega^{\varepsilon})\big)$ and $L_2\big((0,T);\mathring{W}^1_2(\Omega^{\varepsilon})\big)$ correspondingly follow from estimates (3.20) and (3.21). The strong compactness of $\{c^{\varepsilon,h}\}$ and $\{\boldsymbol{v}^{\varepsilon,h}\}$ in $L_2(\Omega^{\varepsilon}_T)$ follow from the same estimates and J. P. Aubin compactness lemma [6, 19]. Finally, the compactness of $\{\boldsymbol{v}^{\varepsilon,h}_{(h)}\}$ follows from the compactness of $\{\boldsymbol{v}^{\varepsilon,h}\}$ and properties of mollifiers.

4. Proof of Theorem 2.4

The main problem here is the strong compactness of $\{\tilde{c}^{\varepsilon,h}\}$ in $L_2(\Omega_T)$, and this fact follows from [4], [23] and properties of corresponding extensions.

The boundedness and the weak compactness in $L_2(\Omega_T)$ of $\{\tilde{\boldsymbol{v}}^{\varepsilon}\}$ follow from estimates (3.22), (3.23) and the Friedrichs–Poincaré inequality in periodic structure [26]. Let

$$q^{\varepsilon}=p^{\varepsilon}-\nu_0\nabla\cdot\boldsymbol{v}^{\varepsilon}=p^{\varepsilon}+\frac{\nu_0}{c^2_f}\frac{\partial p^{\varepsilon}}{\partial t},\tag{4.1}$$

and $\tilde{q}^{\varepsilon}$ be an extension of q^{ε} from Ω^{ε}_T onto Ω_T by setting $q^{\varepsilon}=0$ outside of Ω^{ε}_T.

The weak compactness of $\{\tilde{p}^{\varepsilon}\}$, $\{\tilde{q}^{\varepsilon}\}$, and $\{\nabla\cdot\tilde{\boldsymbol{v}}^{\varepsilon}\}$ in $L_2(\Omega_T)$ follow from estimates (3.22), (3.23) and properties of corresponding extensions.

Using (4.1) we rewrite the integral identity (2.6) as

$$\begin{aligned}&\int_{\Omega^{\varepsilon}_T}\big(\varepsilon^2\mu_1(c^{\varepsilon})\mathbb{D}(x,\boldsymbol{v}^{\varepsilon})-q^{\varepsilon}\mathbb{I}\big):\mathbb{D}(x,\boldsymbol{\varphi})dxdt\\&=\int_{\Omega^{\varepsilon}_T}\rho_f\big(\alpha_\tau\boldsymbol{v}^{\varepsilon}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial t}+\boldsymbol{F}\cdot\boldsymbol{\varphi}\big)\,dxdt.\end{aligned}\tag{4.2}$$

The homogenization of the dynamic equations repeats the similar result in [22]. In fact, the weak limit in the continuity equation (2.7) and in the relation (4.1) result equations (2.13), (2.12) and the boundary condition (2.15).

If $P(\boldsymbol{x},\boldsymbol{y},t)$ and $Q(\boldsymbol{x},\boldsymbol{y},t)$ are two - scale limits of $\{\tilde{p}^{\varepsilon}\}$ and $\{\tilde{q}^{\varepsilon}\}$ respectively, then the two - scale limit in (4.1) and in (4.2) with test functions $\boldsymbol{\varphi}=\varepsilon h(\boldsymbol{x},t)\boldsymbol{\varphi}_0(\boldsymbol{x}/\varepsilon)$ with 1-periodic in $\boldsymbol{y}$ functions $\boldsymbol{\varphi}_0(\boldsymbol{y})$ gives us

$$Q=P+\frac{\nu_0}{c^2_f}\frac{\partial P}{\partial t},\quad P(\boldsymbol{x},\boldsymbol{y},t)=\frac{1}{m}p(\boldsymbol{x},t)\chi(\boldsymbol{y}).$$

Finally, let $\boldsymbol{V}(\boldsymbol{x},\boldsymbol{y},t)$ be the two - scale limit of $\{\tilde{\boldsymbol{v}}^{\varepsilon}\}$. Then the two - scale limit in (4.2) with test functions $\boldsymbol{\varphi}=h(\boldsymbol{x},t)\boldsymbol{\varphi}_1(\boldsymbol{x}/\varepsilon)$ with 1 - periodic in $\boldsymbol{y}$ finite in Y_f and divergent free functions $\boldsymbol{\varphi}_1(\boldsymbol{y})$ gives us

$$\mu_1\Delta_y\boldsymbol{V}-\nabla_yQ-\frac{1}{m}\nabla q+\varrho_f\boldsymbol{F}=0,\quad\boldsymbol{y}\in Y_f.\tag{4.3}$$

The two - scale limit in (2.7) with test functions $\psi=\varepsilon\,h(\boldsymbol{x},t)\psi_0(\boldsymbol{x}/\varepsilon)$ results the microscopic continuity equation

$$\nabla\cdot\boldsymbol{V}=0,\quad\boldsymbol{y}\in Y_f.\tag{4.4}$$

The missing boundary condition

$$\boldsymbol{V}=0,\quad\boldsymbol{y}\in\gamma\tag{4.5}$$

follows from the representation

$$\tilde{\boldsymbol{v}}^{\varepsilon} = \tilde{\boldsymbol{v}}^{\varepsilon}\,\chi^{\varepsilon},$$

the boundedness of the sequence $\{\varepsilon\nabla\tilde{\boldsymbol{v}}^{\varepsilon}\}$ and Nguetseng's theorem [25].

We look for the solution of problem (4.3)–(4.5) in the form

$$\boldsymbol{V} = \frac{1}{\mu_1}\Big(\sum_{i=1}^{3}\boldsymbol{V}^{(i)}\otimes\boldsymbol{e}_i\Big)\cdot\big(-\frac{1}{m}\nabla q + \varrho_f\boldsymbol{F}\big),$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is a standard Cartesian basis. Then

$$\mathbb{B}^{(f)} = \sum_{i=1}^{3}\Big(\int_{Y_f}\boldsymbol{V}^{(i)}(\boldsymbol{y})dy\Big)\otimes\boldsymbol{e}_i = \sum_{i=1}^{3}\langle\boldsymbol{V}^{(i)}\rangle_{Y_f}\otimes\boldsymbol{e}_i. \tag{4.6}$$

In (4.6) $\boldsymbol{a}\otimes\boldsymbol{b}$ stands for the matrix $\mathbb{A}$, such that $\mathbb{A}\cdot\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})$ for any vector $\boldsymbol{c}$, and $\boldsymbol{V}^{(i)}$ are solutions to periodic boundary-value problems

$$\begin{aligned} \Delta_y\boldsymbol{V}^{(i)} - \nabla\Pi^{(i)} &= -\boldsymbol{e}_i, \quad \boldsymbol{y}\in Y_f,\\ \nabla\cdot\boldsymbol{V}^{(i)} &= 0, \quad \boldsymbol{y}\in Y_f,\\ \boldsymbol{V}^{(i)} &= 0, \quad \boldsymbol{y}\in\gamma. \end{aligned} \tag{4.7}$$

The homogenization of the diffusion-convection equation for c^{ε} is also standard ([7], [16], [23]) and

$$\mathbb{B}^{(c)} = m\mathbb{I} + \big(\sum_{i=1}^{3}\langle\nabla_y C^{(i)}(\boldsymbol{y})\rangle_{Y_f}\otimes\boldsymbol{e}_i\big), \tag{4.8}$$

with

$$\nabla\cdot\big(\chi(\boldsymbol{y})(\boldsymbol{e}_i + \nabla_y C^{(i)})\big) = 0, \boldsymbol{y}\in Y. \tag{4.9}$$

5. Proof of Theorem 2.5

Let

$$\boldsymbol{w}(\boldsymbol{x},t) = \int_0^t \boldsymbol{v}(\boldsymbol{x},\tau)d\tau. \tag{5.1}$$

Then (2.11)–(2.13) take the form

$$m\mu_1(c)\big(\mathbb{B}^{(f)}\big)^{-1}\cdot\boldsymbol{v} = \nabla\big(c_f^2\nabla\cdot\boldsymbol{w} + \nu_0\nabla(\nabla\cdot\boldsymbol{v})\big) + m\rho_f\boldsymbol{F}. \tag{5.2}$$

Multiplication by $\boldsymbol{v}$ and integration by parts over Ω result an energy equality

$$\begin{aligned} &\int_{\Omega}\Big(m\,\mu_1(c)\boldsymbol{v}\cdot\big(\mathbb{B}^{(f)}\big)^{-1}\cdot\boldsymbol{v} + \nu_0(\nabla\cdot\boldsymbol{v})^2 - m\rho_f\boldsymbol{F}\cdot\boldsymbol{v}\Big)dx\\ &+\frac{c_f^2}{2}\frac{d}{dt}\int_{\Omega}(\nabla\cdot\boldsymbol{w})^2dx = 0, \end{aligned} \tag{5.3}$$

and the a priori estimate

$$\int_{\Omega_T}\big(|\boldsymbol{v}|^2 + \nu_0(\nabla\cdot\boldsymbol{v})^2\big)dx\,dt + c_f^2\max_{0<t<T}\int_{\Omega}(\nabla\cdot\boldsymbol{w})^2dx \leqslant C\,F^2, \tag{5.4}$$

where C is independent of c_f^2 and ν_0.

Coming back to (2.11) and using (5.4) one has

$$\int_{\Omega_T}|\nabla q|^2dx\,dt \leqslant C\,F^2. \tag{5.5}$$

Equations (2.12), (2.13) and boundary condition (2.15) provide the equality

$$\int_{\Omega} q(\boldsymbol{x},t)dx = 0.$$

Therefore,

$$\int_{\Omega_T} |q|^2 dx\, dt \leqslant C\, F^2 \tag{5.6}$$

(see [18]). The combination of (5.4) and (5.6) gives us

$$\int_{\Omega_T} |p|^2 dx\, dt \leqslant C\, F^2. \tag{5.7}$$

Finally, for the concentration c hold true estimates (2.8) and (2.10) with a constant C that does not depend on c_f^2 and ν_0.

Now we are ready to pass to the limit as $k = c_f^2 \to \infty$. On the base of estimates (2.8), (2.10), (5.4)–(5.7) we may choose subsequences $\{\boldsymbol{v}^{(k_n)}\}$, $\{q^{(k_n)}\}$, and $\{c^{(k_n)}\}$ such that the sequence $\{\boldsymbol{v}^{(k_n)}\}$ converges weakly in $L_2(\Omega_T)$ to the function $\boldsymbol{v}^{(\infty)}$, the sequence $\{q^{(k_n)}\}$ converges weakly in $L_2(\Omega_T)$ to the function $p^{(\infty)}$, the sequence $\{c^{(k_n)}\}$ converges weakly in $L_2\big((0,T);W_2^1(\Omega)\big)$ and strongly in $L_2(\Omega_T)$ to the function $c^{(\infty)}$, the sequence $\{\nabla\cdot\boldsymbol{v}^{(k_n)}\}$ converges weakly in $L_2(\Omega_T)$ to the function $\nabla\cdot\boldsymbol{v}^{(\infty)}$, and the sequence $\{\nabla\cdot\boldsymbol{w}^{(k_n)}\}$ converges strongly in $L_2(\Omega_T)$ to zero.

It is clear that relation $\nabla\cdot\boldsymbol{w}^{(\infty)} = 0$ and relation (5.1) for $\boldsymbol{w}^{(\infty)}$ and $\boldsymbol{v}^{(\infty)}$ imply the continuity equation $\nabla\cdot\boldsymbol{v}^{(\infty)} = 0$, and that the concentration $c^{(\infty)}$ satisfies the diffusion-convection equation (2.14) with the velocity field $\{\boldsymbol{v}^{(\infty)}\}$.

To prove the Darcy law (2.18) it is sufficient to fulfill the limiting procedure as $k_n \to \infty$ in the integral identity

$$\int_{\Omega_T} \Big(\mu_1(c^{(k_n)})\boldsymbol{\varphi}\cdot\big(\mathbb{B}^{(f)}\big)^{-1}\cdot\boldsymbol{v}^{(k_n)} - \frac{1}{m}q^{(k_n)}(\nabla\cdot\boldsymbol{\varphi}) - \rho_f\boldsymbol{F}\cdot\boldsymbol{\varphi}\Big)dx\, dt = 0.$$

The proof of Theorem 2.6 follows by repeating the proof of Theorem 2.5 with obvious changes. Note, that due to (5.4) $\nu_0\nabla\cdot\boldsymbol{v}^{(\delta)} \to 0$ as $\delta \to 0$ strongly in $L_2(\Omega_T)$.

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