Three-Dimensional Analog of the Cauchy Type Integral

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Abstract—On a smooth closed surface, we consider integrals of the Cauchy type with kernel depending on the difference of arguments. They cover both double-layer potentials for second-order elliptic equations and generalized integrals of the Cauchy type for first-order elliptic systems. For the functions described by such integrals, we find sufficient conditions providing their continuity up to the boundary surface. We obtain the corresponding formulas for their limit values.

Let $D \subseteq \mathbb{R}^3$ be a given finite domain bounded by a smooth surface $\Gamma = \partial D$, and let $Q(x, y; \xi)$ be a continuous function of the variables $x \in \overline{D}$, $y \in \Gamma$, and $\xi \in \mathbb{R}^3$, $\xi \neq 0$, homogeneous of degree -2 and odd with respect to ξ; i.e., $Q(r\xi) = r^{-2}Q(\xi)$, $r > 0$, and $Q(-\xi) = -Q(\xi)$. In the domain D, consider the integral

$$\phi(x) = \int\limits_{\Gamma} Q(x, y; y - x)\varphi(y)\, ds_y, \qquad x \in D, \tag{1}$$

where ds_y is the surface area element on Γ. Such integrals arise as double layer potentials for second-order elliptic equations [1, Chap. II]. For example, the double layer potential

$$u(x) = \int\limits_{\Gamma} \frac{\partial}{\partial n(y)} \left[\frac{1}{|y - x|}\right] \varphi(y)\, ds_y$$

for the Laplace equation, where $n(y)$, $y \in \Gamma$, is the unit outward normal, can be represented in the form (1) with

$$Q(y, \xi) = [n_1(y)\xi_1 + n_2(y)\xi_2 + n_3(y)\xi_3]|\xi|^{-3}.$$

Another example is provided by Cauchy type integrals corresponding to the Moisil–Theodoresco system [2]

$$D\left(\frac{\partial}{\partial x}\right) u = 0, \qquad D(\xi) = \begin{pmatrix} 0 & \xi_1 & \xi_2 & \xi_3 \\ \xi_1 & 0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & 0 & -\xi_1 \\ \xi_3 & -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

for a four-component vector function $u = (u_1, u_2, u_3, u_4)$. By [3, p. 243], the function

$$u(x) = \int\limits_{\Gamma} M[y - x, n(y)]\varphi(t)\, ds_t$$

with matrix kernel

$$Q(y, \xi) = M[\xi, n(y)], \qquad M(\xi, \eta) = -D^{\mathrm{T}}\left(\frac{\partial}{\partial \xi}\right) \frac{1}{|\xi|} D(\eta)$$

is a solution of this system.

In the above-represented examples, under the assumption of certain smoothness of the density φ and the surface Γ, the function $u(x)$ can be extended by continuity to Γ, and the corresponding formula holds for its limit values. In the present paper, we consider a similar problem for general integrals of the form (1). The relation for the limit values $\phi^+(y_0)$ of the function (1) at the boundary points y_0 will contain the singular integral

$$\phi^*(y_0) = \int_\Gamma Q(y_0, y; y - y_0)\varphi(y)\, ds_y, \qquad y_0 \in \Gamma, \tag{2}$$

which is understood in the sense of principal value as the limit of integrals over $\Gamma \cap \{|y - y_0| \geq \varepsilon\}$ as $\varepsilon \to 0$.

Note that the problem on the continuity of the function ϕ up to the boundary for the case in which the kernel Q depends only on ξ was considered in [4], where the following assertion was proved.

Lemma 1. *Let $Q(\xi)$, $\xi \neq 0$, be a continuously differentiable odd function homogeneous of degree -2; next, let P be a given plane with unit normal n, and let G be the half-space with boundary $\partial G = P$ for which n is the inward normal. Then the singular integral*

$$q = \int_P Q(y - x)\, ds_y, \qquad x \in G, \tag{3}$$

treated in the sense of principal value at infinity as the limit of integrals over $P \cap \{|y| \leq R\}$ as $R \to \infty$ exists and is independent of $x \in G$. One also has the relation

$$q = \int_{P\cap\{|y-y_0|\leq 1\}} Q(y - y_0 - n)\, ds_y + \int_{P\cap\{|y-y_0|\geq 1\}} [Q(y - y_0 - n) - Q(y - y_0)]\, ds_y, \tag{3$'$}$$

where $y_0 \in P$ and integrals exist in the ordinary sense.

All forthcoming considerations will be carried out in the framework of the Hölder classes C^ν. Note that, in this class, the norm of a function φ defined on a set E is given by the relation $|\varphi|_{\nu,E} = |\varphi|_{0,E} + [\varphi]_{\nu,E}$, where

$$|\varphi|_{0,E} = \sup_{x\in E} |\varphi(x)|, \qquad [\varphi]_{\nu,E} = \sup_{x\neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\nu}, \qquad 0 < \nu \leq 1.$$

If $[\varphi]_{1,E} < \infty$, then one says that the function φ satisfies the Lipschitz condition. If the set E is a closed domain, then one can introduce the class $C^{1,\nu}(E)$ of continuously differentiable functions φ by the conditions $\varphi, \varphi' \in C^\nu(E)$, where φ' stands for any of the first partial derivatives of φ.

Let H^n_{-2} be the class of functions $Q(\xi) \in C^n$, $\xi \neq 0$, homogeneous of degree -2. One can readily see [4] that the functions $Q \in H^1_{-2}$ satisfy the Lipschitz condition outside a neighborhood of the point $\xi = 0$. More precisely, they admit the estimate

$$|Q(\xi) - Q(\eta)| \leq C(|\xi|^{-3} + |\eta|^{-3})|\xi - \eta|, \tag{4}$$

where $C > 0$ is a constant independent of Q. In particular, this estimate implies that the second integral on the right-hand side in (3$'$) exists in the ordinary sense.

Just as in (1), functions $Q(\xi) = Q(u, \xi)$ depending on the parameter $u \in E$ are most interesting. By $C^\nu(E; H^n_{-2})$ we denote the class of functions $Q(u; \xi)$ that belong to H^n_{-2} for each u and, treated as functions of the first variable, belong to $C^\nu(E)$ together with their partial derivatives with respect to ξ; moreover, the following norm is finite:

$$|Q|^{(n)}_\nu = \sup_{|\xi|=1,\, k\leq n} |Q^{(k)}|_{\nu,E}, \tag{5}$$

where $Q^{(k)}$ stands for the set of all partial derivatives of $Q(u,\xi)$ of order k with respect to the variables ξ_1, ξ_2, and ξ_3.

We introduce the class L^1 of continuously differentiable homeomorphic mappings of the space $\mathbb{R}^3$ into itself that, together with their inverses, satisfy the Lipschitz condition. Such mappings α are referred to as Lipschitz mappings; by definition, for some constant $M > 0$, they admit the two-sided estimate

$$M^{-1}|x-y| \le |\alpha(x)-\alpha(y)| \le M|x-y|, \qquad x, y \in \mathbb{R}^3. \tag{6}$$

Consider the Jacobi matrix $D\alpha$, whose columns are the partial derivatives $\partial\alpha/\partial x_i$. Obviously,

$$\alpha(x)-\alpha(y) = [A(x,y)](x-y), \qquad A(x,y) = \int_0^1 (D\alpha)[x(1-\tau)+y\tau]\,d\tau, \tag{7}$$

where the matrix A in brackets acts on the vector $x-y$ in the standard way. By substituting this expression into (6) and by setting $x = y + r\xi$, $r > 0$, with a fixed vector ξ, we obtain the similar estimate

$$M^{-1}|\xi| \le |[(D\alpha)(x)]\xi| \le M|\xi| \tag{8}$$

uniformly with respect to x and ξ in the limit as $r \to 0$. In particular, the inverse mapping α^{-1} is continuously differentiable as well, and the Jacobi matrices $D(\alpha^{\pm 1})$ are uniformly bounded. The converse is also true: if the mappings $\alpha^{\pm 1}$ are continuously differentiable and their Jacobi matrices are uniformly bounded, then $\alpha \in L^1$. In what follows, we consider Lipschitz mappings α with the property $D\alpha \in C^\nu(\mathbb{R}^3)$; the class of such mappings is denoted by $L^{1,\nu}$. Obviously, the inverse mapping $\beta = \alpha^{-1}$ belongs to that class together with α.

Consider the relationship between Lipschitz transformations and the above-introduced functions $Q \in H^n_{-2}$.

Lemma 2. *Let $Q(x,y;\xi) \in C^\nu(E_1 \times E_2; H^2_{-2})$, where $E_j \subseteq \mathbb{R}^3$, and let $\alpha \in L^{1,\nu}$ be a given Lipschitz transformation. Then there exists a function $\tilde{Q}(x,y;\xi) \in C^\nu(\tilde{E}_1 \times \tilde{E}_2; H^1_{-2})$, where $\tilde{E}_j = \alpha^{-1}(E_j)$, such that*

$$k(x,y) = Q[\alpha(x),\alpha(y);\alpha(y)-\alpha(x)] - \tilde{Q}(x,y;y-x) \in C^\nu(\tilde{E}_1 \times \tilde{E}_2); \tag{9}$$

moreover,

$$\tilde{Q}(x,x;\xi) = Q[\alpha(x),\alpha(x);(D\alpha)(x)\xi]. \tag{10}$$

Proof. If $x = y$, then the matrix $A(x,y)$ occurring in (7) coincides with the Jacobi matrix $(D\alpha)(x)$, which, by assumption, belongs to the class $C^\nu(\mathbb{R}^3)$. Therefore, we have the estimate

$$|A(x,y)-A(x,x)| \le C|x-y|^\nu,$$

where the operator norm in $\mathbb{R}^{3\times 3} = \mathcal{L}(\mathbb{R}^3)$ is taken on the left-hand side. Consequently, this, together with (8), implies the two-sided inequality

$$(2M)^{-1}|\xi| \le |A(x,y)\xi| \le 2M|\xi| \quad \text{if} \quad |x-y| \le \delta, \tag{11}$$

where δ, $0 < \delta \le 1$, is chosen from the condition $2MC\delta \le 1$.

Take a smooth cutoff function $\chi(x)$ identically equal to unity for $|x| \le \delta/2$ and identically zero for $|x| \ge \delta$, and set

$$\tilde{Q}(x,y;\xi) = \chi(x-y)Q[\alpha(x),\alpha(y);A(x,y)\xi]. \tag{12}$$

Obviously, this function satisfies condition (10). Let us show that it belongs to the class $C^\nu(\tilde{E}_1 \times \tilde{E}_2; H^1_{-2})$. By the choice of the function χ, it suffices to verify this assertion for the set $\tilde{E} \subseteq \tilde{E}_1 \times \tilde{E}_2$ distinguished by the condition $|x-y| \le \delta$.

By (4), the function $Q[\alpha(x),\alpha(y);\eta]$ satisfies the Lipschitz condition in the spherical layer $(2M)^{-1}\le|\eta|\le 2M$ uniformly with respect to x and y. By (11), the vector $A(x,y)\xi$ varies in this layer for $|\xi|=1$ and $(x,y)\in\tilde E$. Therefore, in view of the condition $A\in C^\nu(\mathbb{R}^3\times\mathbb{R}^3)$, from this we find that $Q[\alpha(x),\alpha(y);A(x,y)\xi]\in C^\nu(\tilde E)$ uniformly with respect to $|\xi|=1$. In a similar way, one can also prove this fact for the partial derivatives,

$$\frac{\partial\tilde Q}{\partial\xi_i}=\chi(x-y)\sum_{k=1}^{3}A_{k\,i}(x,y)\frac{\partial Q}{\partial\eta_k}[\alpha(x),\alpha(x);A(x,y)\xi],$$

where the A_{ki} are the entries of the matrix A.

By (7) and (12), for the function $k(x,y)$ in (9), we have the expression

$$k(x,y)=[1-\chi(x-y)]Q[\alpha(x),\alpha(y);\alpha(y)-\alpha(x)].$$

Since $1-\chi(x-y)=0$ for $|x-y|\le\delta/2$, it suffices to show that $k(x,y)\in C^\nu(\tilde F)$ on the set $\tilde F=\{(x,y)\in\tilde E_1\times\tilde E_2,\ |x-y|\ge\delta/2\}$. By (6), we have the estimate $|\alpha(x)-\alpha(y)|\ge\delta/(2M)$ for $(x,y)\in\tilde F$. Therefore, by virtue of (4), the function $Q[\alpha(x),\alpha(y);\eta]$ satisfies the Lipschitz condition in the domain $|\eta|\ge\delta/(2M)$ uniformly with respect to $(x,y)\in E$. It follows that the function k belongs to the class $C^\nu(\tilde F)$, which completes the proof of the lemma.

Throughout the following, we assume that Γ is a Lyapunov surface and belongs to the class $C^{1,\nu}$. The latter means the following: for each $a\in\Gamma$, there exists a homeomorphic mapping $y=\gamma(t)=(\gamma_1(t),\gamma_2(t),\gamma_3(t))$ of class $C^{1,\nu(B)}$ of the unit disk $B=\{t\in\mathbb{R}^2,\ |t|\le1\}$ onto some neighborhood of the point a on the surface Γ such that the rank of the Jacobi matrix $(D\gamma)(t)$ is equal to 2 at each point; i.e., the tangent vectors

$$c_i(t)=\frac{\partial\gamma}{\partial t_i},\qquad i=1,2,\tag{13}$$

are linearly independent.

Note that the surface area element on Γ is given by the formula

$$ds_y=|[c_1(t),c_2(t)]|\,dt_1\,dt_2,$$

where $[\cdot\,,\cdot]$ stands for the vector product. In other words, the integral of a function $\varphi\in C(\Gamma)$ over the surface $\gamma(B)\subseteq\Gamma$ is given by the formula

$$\int\limits_{\gamma(B)}\varphi(x)\,ds_x=\int\limits_{B}\varphi[\gamma(t)]|[c_1(t),c_2(t)]|\,dt.\tag{14}$$

We denote the tangent plane to the smooth surface Γ at a point a by $(d\Gamma)(a)$. The surface Γ is flat in a neighborhood of that point if there exists a ball B centered at a such that $B\cap\Gamma=B\cap(d\Gamma)(a)$. By definition, a Lipschitz transformation β rectifies the surface Γ in a neighborhood of a if the image $\beta(\Gamma)$ is flat in a neighborhood of the point $\beta(a)$.

Lemma 3. (a) *Let $\Gamma\in C^{1,\nu}$ and $a\in\Gamma$. Then there exists a Lipschitz transformation $\beta\in L^{1,\nu}$ rectifying Γ in a neighborhood of a.*

(b) *Let a Lipschitz transformation $y=\alpha(x)\in L^{1,\nu}$ take a smooth surface $\tilde\Gamma\in C^{1,\nu}$ to Γ. Then the following change-of-variables formula holds:*

$$\int\limits_{\Gamma}\varphi(y)\,ds_y=\int\limits_{\tilde\Gamma}\varphi[\alpha(x)]J(x)\,ds_x,\qquad J(x)=|[(D\alpha)e_1(x),(D\alpha)e_2(x)]|,\tag{15}$$

where e_1 and e_2 are unit mutually orthogonal vectors lying in the tangent plane $(d\tilde\Gamma)(x)$. In addition, the coefficient $J(x)$ belongs to $C^\nu(\tilde\Gamma)$.

(c) *Let* $\Gamma \in C^{1,\nu}$, *and let a function* $Q(y_0, y; \xi)$ *belong to* $C^{\nu}(\Gamma \times \Gamma; H^2_{-2})$ *and be odd with respect to* ξ *for* $y = y_0$. *Then there exists a singular integral*

$$\int_{\Gamma} Q(y_0, y; y - y_0)\, ds_y = \lim_{\varepsilon \to 0} \int_{\Gamma \setminus B_\varepsilon} Q(y_0, y; y - y_0)\, ds_y, \tag{16}$$

where B_ε *is the ball centered at* $y_0 \in \Gamma$ *with radius* ε.

Proof. (a) Let $a = (a_1, a_2, a_3) \in \Gamma$, and let a homeomorphic mapping $y = \gamma(t)$ of the disk $B = \{|t| \leq 1\} \subseteq \mathbb{R}^2$ onto a neighborhood of the point a on the surface Γ belong to the class $C^{1,\nu}(B)$; moreover, the 3×2 Jacobi matrix $D\gamma$ has rank 2 at each point $t \in B$, and $a = \gamma(0)$. Then one of its second-order minors, for example, that in the first and second rows, is nonzero. Therefore, by the inverse function theorem, there exists a mapping $t = \omega(y_1, y_2)$ of a neighborhood G of the point (a_1, a_2) onto some disk $|t| \leq \varepsilon$ of the class $C^{1,\nu}(\overline{G})$ that is the inverse of $y_i = \gamma_i(t)$, $i = 1, 2$. By setting $f = \gamma_3 \circ \omega$, we obtain a real function $f \in C^{1,\nu}(\overline{G})$ whose graph coincides with Γ in a neighborhood of a.

Let a smooth function χ be identically equal to unity in a neighborhood of the point (a_1, a_2) and identically zero outside some compact set lying in G. Then the function $g(y_1, y_2) = f(y_1, y_2)\chi(y_1, y_2)$ belongs to the class $C^{1,\nu}(\mathbb{R}^2)$, and its graph coincides with Γ in a neighborhood of a. Therefore, the transformation $\tilde{y} = \beta(y)$ given by the relations

$$\tilde{y}_1 = y_1, \qquad \tilde{y}_2 = y_2, \qquad \tilde{y}_3 = y_3 - g(y_1, y_2),$$

satisfies all assumptions of the lemma.

(b) Consider a neighborhood Γ_0 of some point a of the surface Γ represented parametrically by the equation $y = \gamma(t)$, $|t| \leq 1$, of the class $C^{1,\nu}$ occurring in the proof of (a). Then $\tilde{\gamma} = \alpha^{-1} \circ \gamma$ is a parametrization of $\tilde{\Gamma}_0 = \alpha^{-1}(\Gamma_0)$ of the same type. Let $\tilde{c}_i$ be defined by analogy with (13) on the basis of $\tilde{\gamma}_i$. Then, by virtue of (14), relation (15) written out for Γ_0 and $\tilde{\Gamma}_0$ acquires the form

$$\int_B \varphi[\gamma(t)] |[c_1(t), c_2(t)]|\, dt = \int_B \varphi[\gamma(t)] J[\tilde{\gamma}(t)] |[\tilde{c}_1(t), \tilde{c}_2(t)]|\, dt,$$

where B stands for the unit disk on the plane $\mathbb{R}^2$. Therefore, the coefficient J is given by the formula $J[\tilde{\gamma}(t)] |[\tilde{c}_1(t), \tilde{c}_2(t)]| = |[c_1(t), c_2(t)]|$. At the point $\tilde{y} = \tilde{\gamma}(t)$, the vectors $\tilde{c}_i = \partial\tilde{\gamma}/\partial t_i$ are linear combinations $\tilde{c}_i = p_{i,1}e_1 + p_{i,2}e_2$, $i = 1, 2$, of the vectors e_1 and e_2 with determinant $\det p \neq 0$. Obviously, $[\tilde{c}_1, \tilde{c}_2] = [p_{1,1}e_1 + p_{1,2}e_2, p_{2,1}e_1 + p_{2,2}e_2] = (\det p)[e_1, e_2]$. Since $c_i = \partial\gamma/\partial t_i$ is related to $\tilde{c}_i$ by the formula $c_i = (D\alpha)(\tilde{y})\, \tilde{c}_i$, where the Jacobi matrix $D\alpha$ is evaluated at the point $\tilde{y} = \tilde{\gamma}(t)$, we have

$$[c_1, c_2] = [p_{1,1}(D\alpha)e_1 + p_{1,2}(D\alpha)e_2, p_{2,1}(D\alpha)a_1 + p_{2,2}(D\alpha)a_2] = (\det p)[(D\alpha)e_1, (D\alpha)e_2].$$

Since $|[e_1, e_2]| = 1$, it follows that relation (15) holds for J.

It remains to show that $J \in C^{\nu}(\tilde{\Gamma})$. To this end, it suffices to choose the unit vectors $e_i(x)$ occurring in (15) from the class $C^{\nu}(\tilde{\Gamma}_0)$. By using orthogonalization, one can start from a pair of linearly independent vectors. By virtue of assertion (a) of the lemma, such a choice is always possible.

(c) First, suppose that the surface Γ is flat in a neighborhood of y_0; i.e., $B_r \cap \Gamma = B_r \cap (d\Gamma)(y_0)$ for some $r > 0$. Next, since $Q(y_0, y_0; \xi)$ is an odd function of ξ, it follows that the integral of $Q(y_0, y_0; y - y_0)$ over the circular annulus $\Gamma \cap (B_r \setminus B_\varepsilon)$, $0 < \varepsilon < r$, is zero. Consequently,

$$\int_{\Gamma \setminus B_\varepsilon} Q(y_0, y; y - y_0)\, ds_y$$

$$= \int_{\Gamma \setminus B_\varepsilon} [Q(y_0, y; y - y_0) - Q(y_0, y_0; y - y_0)]\, ds_y + \int_{\Gamma \setminus B_\varepsilon} Q(y_0, y_0; y - y_0)\, ds_y.$$

By the definition of the norm (5), we have the estimate

$$|Q(y_0,y;\xi)-Q(y_0,y_0;\xi)| \le |Q|_{\nu}^{(0)}|\xi|^{-2}|y-y_0|^{\nu};$$

thus, the bracketed expression in the preceding relation is integrable over Γ. Therefore, the limit (16) exists.

In the general case, we use assertion (a) of the lemma and consider a transformation $\beta \in L^{1,\nu}$ rectifying Γ in a neighborhood of y_0. Without loss of generality, one can assume that $\beta(y_0) = y_0$ and $(D\beta)(y_0)$ is the identity matrix. In particular,

$$\beta(y) - y = |y-y_0|\Delta(y), \qquad \lim_{y\to y_0}\Delta(y) = 0. \tag{17}$$

By (15), in the notation of Lemma 2, we have

$$\int\limits_{\Gamma\setminus B_\varepsilon} Q(y_0,y;y-y_0)\,ds_y = \int\limits_{\tilde\Gamma\setminus\beta(B_\varepsilon)} \tilde Q(y_0,y;y-y_0)J(y)\,ds_y + \int\limits_{\tilde\Gamma\setminus\beta(B_\varepsilon)} k(y_0,y)J(y)\,ds_y.$$

Obviously, the last term on the right-hand side in this relation has a limit as $\varepsilon\to 0$. The assumptions of the above-considered case hold for $\tilde\Gamma = \beta(\Gamma)$ and $\tilde Q(y_0,y;\xi)J(y)$. Therefore, it suffices to include the image $\beta(B_\varepsilon)$ of the ball B_ε in balls of close radius appropriately. By (17), for $|y-y_0| = \varepsilon$, we have the estimates

$$\varepsilon(1-\delta_\varepsilon) \le |\beta(y)-y_0| \le \varepsilon(1+\delta_\varepsilon),$$

where

$$\delta_\varepsilon = \max_{|y-y_0|=\varepsilon}|\Delta(y)| \to 0 \quad \text{as} \quad \varepsilon\to 0.$$

Consequently,

$$B_\varepsilon^- \subseteq \beta(B_\varepsilon) \subseteq B_\varepsilon^+, \quad B_\varepsilon^\pm = B_{\varepsilon\pm\varepsilon\delta_\varepsilon}. \tag{18}$$

Since the surface $\tilde\Gamma$ is flat in a neighborhood of the point y_0, it follows that, for sufficiently small ε, its intersection with $B_\varepsilon^+\setminus B_\varepsilon^-$ is a circular annulus of area

$$\pi\varepsilon^2[(1+\delta_\varepsilon)^2 - (1-\delta_\varepsilon)^2] = 4\pi\varepsilon^2\delta_\varepsilon.$$

By virtue of (18), we have

$$\int\limits_{\tilde\Gamma\setminus\beta(B_\varepsilon)} \tilde Q J\,ds_x = \int\limits_{\tilde\Gamma\setminus B_\varepsilon^+} \tilde Q J\,ds_x + \int\limits_{B_\varepsilon^+\setminus\beta(B_\varepsilon^+)} \tilde Q J\,ds_x. \tag{19}$$

As was shown above, the first term has a limit as $\varepsilon\to 0$, and the absolute value of the second one does not exceed the quantity

$$\int\limits_{B_\varepsilon^+\setminus B_\varepsilon^-} |\tilde Q(y_0,y;y-y_0)|\,|\tilde J(y)|\,ds_y \le C\delta_\varepsilon$$

with some constant $C > 0$ independent of ε. Therefore, the left-hand side of relation (19) has a limit as $\varepsilon\to 0$, which completes the proof of the lemma.

Consider the original integral (1).

Theorem 1. *Let the surface $\Gamma \in C^{1,\nu}$ bound a finite domain D, and let $Q(x,y;\xi) \in C^{\nu}(\overline{D}\times\Gamma; H^2_{-2})$ be a given function odd with respect to ξ for $x = y \in \Gamma$. Then for $\varphi \in C^{\mu}(\Gamma)$, $0 < \mu < \nu$, the integral (1) defines a function $\phi(x)$ that can be continuously extended to Γ and belongs to the class $C^{\mu}(\overline{D})$. Its limit values satisfy the formula*

$$\phi^+(y_0) = q(y_0)\varphi(y_0) + \phi^*(y_0), \qquad y_0\in\Gamma, \tag{20}$$

where $\phi^*(y_0)$ is the integral (2) and $q(y_0)$ is defined by analogy with Lemma 1 for $Q(y_0,y_0;\xi)$, the plane $P = (d\Gamma)(y_0)$, and the inward (for D) normal $n(y_0)$.

Proof. It suffices to prove that the function ϕ belongs to the class C^μ only for boundary points $a \in \Gamma$. In other words, for each point a, there exists a ball B centered at a such that $\phi \in C^\mu(D \cap B)$. In this case, ϕ can be continuously extended to the closure $\overline{D \cap B}$ and belongs to $C^\mu(\overline{D \cap B})$.

First, suppose that the surface Γ is flat in a neighborhood of a, for example, in the intersection with the ball $B_{2r} = \{|x-a| \le 2r\}$; for B, we take the ball B_r. In this case, we prove the theorem under the less restrictive condition $Q \in C^\nu(\overline{D} \times \Gamma; H^1_{-2})$ for the kernel. Let us start from the case in which the kernel $Q = Q_0$ is independent of x; i.e., consider the function

$$\phi_0(x) = \int_\Gamma Q_0(y; y-x)\varphi(y)\, ds_y, \qquad x \in D \cap B. \tag{21}$$

The fact that this function belongs to the class C^μ was proved in [4], and the following estimate was obtained there:

$$|\phi_0|_{C^\mu(D\cap B)} \le C|Q|^{(1)}_\nu |\varphi|_{\mu,\Gamma}. \tag{22}$$

In the general case, we apply this estimate to the function

$$\phi_1(z, x) = \int_\Gamma Q(z, y; y-x)\varphi(y)\, ds_y, \qquad x, z \in D \cap B,$$

where z is treated as a parameter. Then, by (22), we have the estimate

$$|\phi_1(z, x') - \phi_1(z, x'')| \le C|x' - x''|^\mu \tag{23}$$

uniformly with respect to z. The difference quotient

$$\tilde{\phi}_0(x) = [\phi_1(z', x) - \phi_1(z'', x)]|z' - z''|^{-\mu}$$

for the function ϕ_1 with respect to the first variable can be represented in the form (21) for the kernel

$$\tilde{Q}_0(y; \xi) = \frac{Q(z', y; \xi) - Q(z'', y; \xi)}{|z' - z''|^\mu}. \tag{24}$$

Next, we use the following well-known property [5, p. 47] of functions satisfying the Hölder conditions: if some function $f(x)$ belongs to $C^\nu(E)$ and $0 < \mu < 1$, then the function

$$g(x, y) = [f(x) - f(y)]|x - y|^{-\mu}$$

defined to be zero for $x = y$ belongs to the class $C^{\nu-\mu}(E \times E)$, and the corresponding estimate $|g|_{\nu-\mu, E\times E} \le C|f|_{\nu,E}$ holds. By definition (5) applied to (24), this result permits one to claim that $\tilde{Q} \in C^{\nu-\mu}(\Gamma; H^1_{-2})$ uniformly with respect to z' and z''. By (22), where ν and $\mu < \nu$ should be replaced by $\tilde{\nu} = \nu - \mu$ and $\tilde{\mu} < \tilde{\nu}$, respectively, we obtain the estimate

$$|\phi_1(z', x) - \phi_1(z'', x)| \le C|z' - z''|^\mu,$$

which, together with (23), implies that ϕ_1 and hence $\phi(x) = \phi_1(x, x)$ belong to the class C^μ.

Let us prove (20) at the point $y_0 = a$. By denoting $Q(y_0, y; \xi)\varphi(y)$ by $Q(y_0, y; \xi)$ again, without loss of generality, one can assume that $\varphi \equiv 1$. First, suppose that $Q(a, a; \xi) = 0$. In this case, we have

$$|Q(x, y; \xi)| = |Q(x, y; \xi) - Q(a, a; \xi)| \le |Q|^{(0)}_\nu (|x-a| + |y-a|)^\nu |\xi|^{-2}. \tag{25}$$

Let the point $x \in D$ tend to a along the inward normal to Γ. Then the vectors $x - a$ and $y - a$, $y \in B \cap \Gamma$, are orthogonal; therefore, $|x-y|^2 = |x-a|^2 + |y-a|^2 \ge (|x-a| + |y-a|)^2/2$. This, together with (25), implies that

$$|Q(x, y; y-x)| \le |Q|^{(0)}_\nu \frac{(|x-a| + |y-a|)^\nu}{|x-a|^2 + |y-a|^2} \le 2|Q|^{(0)}_\nu |y-a|^{\nu-2}.$$

Therefore, by the Lebesgue dominated convergence theorem, one can pass to the limit in the integral (1) as $x \to a$, which provides relation (20) for $a = y_0$ and $q(a) = 0$.

In the general case, set

$$Q(x, y; \xi) = [Q(x, y; \xi) - Q(a, a; \xi)] + Q(a, a; \xi).$$

The assumptions of the above-considered case hold for the integral defined by the bracketed expression. Therefore, without loss of generality, one can assume that $Q(\xi) = Q(x, y; \xi)$ is independent of x and y. In this case, $q = q(a)$ is given by relation (3), where $P = (d\Gamma)(a)$ and the point x lies in the half-space defined by the normal $n(a)$. Since $P \cap B = \Gamma \cap B$, we have

$$\phi(x) - q = \int_{\Gamma \setminus B} Q(y - x)\, ds_y - \int_{P \setminus B} Q(y - x)\, ds_y.$$

On the right-hand side of this relation, one can pass to the limit in the integrands as $x \to a$; thus,

$$\phi^+(a) - q = \int_{\Gamma \setminus B} Q(y - a)\, ds_y - \int_{P \setminus B} Q(y - a)\, ds_y.$$

Since

$$\int_{P \setminus B} Q(y - a)\, ds_y = \int_{B} Q(y - a)\, ds_y = 0,$$

it follows that $\phi^+(a) = q + \phi^*(a)$, which completes the proof of formula (20).

Thus, under the assumption that the surface Γ is flat in a neighborhood of the boundary point a, the proof of the theorem is complete. In the general case, we use assertion (a) of Lemma 3, i.e., a Lipschitz transformation $\beta \in L^{1,\nu}$ rectifying the surface Γ in a neighborhood of a. Without loss of generality, one can assume that $\beta(a) = a$ and $(D\beta)(a) = 1$. Then, by virtue of Lemma 2, formula (1) relating the functions ϕ and φ becomes the similar relation

$$\tilde{\phi}(x) = \int_{\tilde{\Gamma}} \tilde{Q}(x, y; y - x)\tilde{\varphi}(y)\, ds_y + \int_{\tilde{\Gamma}} k(x, y)\tilde{\varphi}(y)\, ds_y = \tilde{\phi}_0(x) + \tilde{\phi}_1(x), \qquad x \in \overline{D},$$

for $\tilde{\phi}(x) = \phi[\alpha(x)]$ and $\tilde{\varphi} = \varphi[\alpha(x)]J(x) \in C^\mu(\tilde{\Gamma})$, where $\alpha = \beta^{-1}$.

Obviously, $\tilde{\phi}_1 \in C^\nu(\overline{D})$, and by virtue of the preceding argument, the function $\tilde{\phi}_0$ belongs to $C^\mu(\overline{D})$, and

$$\tilde{\phi}_0^+(a) = \tilde{q}(a)\tilde{\varphi}(a) + \tilde{\phi}_0^*(a). \tag{26}$$

Obviously, this relation also holds for $\tilde{\phi}$. Relation (10), together with the assumption $(D\alpha)(a) = 1$, implies that the function $\tilde{Q}(a, a; \xi)$ coincides with $Q(a, a; \xi)$; therefore, $\tilde{q}(a) = q(a)$. By the same argument, we find that $J(a) = 1$ and $\tilde{\varphi}(a) = \varphi(a)$. It follows from the proof of assertion (c) of Lemma 3 that $\tilde{\phi}^*(a) = \phi^+(a)$; thus, relation (26) becomes formula (20) for ϕ. The proof of Theorem 1 is complete.

It follows from the expression (3′) for the coefficient q in Lemma 1 that, under the assumptions of Theorem 1, the function $q(y_0)$ occurring in (20) belongs to the class $C^\mu(\Gamma)$. In particular, the singular operator $\varphi \to \phi^*$ is bounded in the space $C^\mu(\Gamma)$. This fact is well known for general singular operators on smooth manifolds. In the case of kernels $Q(x, y; \xi)$ of special form, the coefficient $q(y_0)$ can be rewritten in a more explicit form.

Lemma 4. *Suppose that, under the assumptions of Theorem* 1 *for* $x = y$, *the function* $Q(x, y, \xi)$ *is given by the relation*

$$Q(y, y; \xi) = Q_1(\xi)n_1(y) + Q_2(\xi)n_2(y) + Q_3(\xi)n_3(y), \tag{27}$$

where the n_j are the components of the inward normal on Γ; moreover,

$$\frac{\partial Q_1}{\partial \xi_1} + \frac{\partial Q_2}{\partial \xi_2} + \frac{\partial Q_3}{\partial \xi_3} = 0. \tag{28}$$

Then the coefficient $q = q(y_0)$ is independent of y_0 and is given by the relation

$$q = -\frac{1}{2} \int\limits_{|\xi|=1} \left[\sum_{i=1}^{3} Q_i(\xi)\xi_i \right] ds_\xi. \tag{29}$$

Proof. By Lemma 1, we have

$$q(y_0) = \int\limits_P Q(y_0, y_0; y - x_0)\, ds_y = \lim_{R\to\infty} \int\limits_{P\cap B_R} Q(y_0, y_0; y - x_0)\, ds_y,$$

where P is the tangent plane $d\Gamma(y_0)$ and $x_0 = y_0 + n(y_0)$. Here the integrand is defined by the right-hand side of relation (27) with constant coefficients $n_j(y_0)$ of the inward (for D) normal $n(y_0)$. Let $G^\pm$ be the half-spaces for which $\pm n(y_0)$ is the inward normal. Consider the half-ball $G^- \cap B_R$, whose spherical part of the boundary will be denoted by Ω_R^-. In this part, one can use the Gauss divergence theorem for $Q(y_0, y_0; y - x_0)$. By (28), we obtain

$$\int\limits_{\partial(G^-\cap B_R)} [Q_1(y - x_0)n_1(y) + Q_2(y - x_0)n_2(y) + Q_3(y - x_0)n_3(y)]\, ds_y = 0,$$

where the $n_l(y)$ stand for the components of the outward normal (for $G^- \cap B_R$). Therefore,

$$-\int\limits_{P\cap B_R} Q(y_0, y_0; y - x_0)\, ds_y = \frac{1}{R} \int\limits_{\Omega_R^-} [Q_1(y - x_0)y_1 + Q_2(y - x_0)y_2 + Q_3(y - x_0)y_3]\, ds_\xi,$$

where we have used the fact that $n_i(y) = y_i/R$ on Ω_R^-. After the substitution $y = R\xi$ and $|\xi| = 1$, the integral on the right-hand side acquires the form

$$\int\limits_{\Omega_1^-} R^2[Q_1(R\xi - x_0)\xi_1 + Q_2(R\xi - x_0)\xi_2 + Q_3(R\xi - x_0)\xi_3]\, ds_y.$$

By virtue of homogeneity, we have $R^2 Q_j(R\xi - x_0) = Q_j(\xi - x_0/R)$; thus, the last integral converges as $R \to \infty$ to the limit

$$\int\limits_{\Omega_1^-} [Q_1(\xi)\xi_1 + Q_2(\xi)\xi_2 + Q_3(\xi)\xi_3]\, ds_\xi,$$

which, by virtue of the oddness of $Q_j(\xi)$, coincides with the right-hand side of formula (29).

Let us illustrate the lemma by the integrals considered at the beginning of the present paper. In the case of a double layer potential for the Laplace equation, we have

$$Q(y; \xi) = \frac{n(y)\xi}{|\xi|^3};$$

accordingly,

$$\sum_{i=1}^{3} Q_i(\xi)\xi_i = \frac{1}{|\xi|},$$

and formula (29) provides the value $q = -2\pi$, which is in accordance with the classical result in [6, p. 416].

For the Moisil–Theodoresco system, the kernel is given by the formula

$$Q(y;\xi) = -D^{\mathrm{T}}\left(\frac{\partial}{\partial\xi}\right)\frac{1}{|\xi|}D[n(y)].$$

In this case,

$$\sum_{i=1}^{3} Q_i(\xi)\xi_i = \frac{D^{\mathrm{T}}(\xi)D(\xi)}{|\xi|^3}.$$

Since $D^{\mathrm{T}}(\xi)D(\xi) \equiv |\xi|^2$, it follows that $q = -2\pi$, which is in accordance with the well-known result in [3, p. 248] as well.

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