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RIEMANN–HILBERT PROBLEM FOR FIRST-ORDER ELLIPTIC SYSTEMS WITH CONSTANT LEADING COEFFICIENTS ON THE PLANE

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Abstract. In a finite domain D of the complex plane bounded by a smooth contour Γ, we consider the Riemann–Hilbert boundary-value problem $\operatorname{Re} CU^+ = f$ for the first-order elliptic system

$$\frac{\partial U}{\partial y} - A\frac{\partial U}{\partial x} + a(z)U(z) + b(z)\overline{U(z)} = F(z)$$

with constant leading coefficients. Here $+$ means the boundary value of the function U on Γ, the constant matrices $A_1, A_2 \in \mathbb{C}^{l\times l}$ and the $(l\times l)$-matrix coefficients a and b belong to the Hölder class C^μ, $0<\mu<1$, and $(l\times l)$-matrix function C belongs to the class $C^\mu(\Gamma)$. We prove that in the class $U \in C^\mu(\overline{D})\cap C^1(D)$, this problem is a Fredholm problem and its index is given by the formula

$$\varkappa = -\sum_{j=1}^{m}\frac{1}{\pi}\big[\arg\det G\big]_{\Gamma_j} + (2-m)l.$$

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In a finite domain D of the complex plane bounded by a smooth contour Γ we consider the first-order elliptic system

$$A_1\frac{\partial U}{\partial x} + A_2\frac{\partial U}{\partial x} + a(z)U(z) + b(z)\overline{U(z)} = F(z), \quad z\in D,$$

where the constant matrices $A_1, A_2 \in \mathbb{C}^{l\times l}$ and $(l\times l)$-matrix coefficients a and b belong to the Hölder class $C^\mu(D)$, $0<\mu<1$. The ellipticity condition is as follows: both matrices A_j are invertible and the matrix $A = -A_2^{-1}A_1$ has no real eigenvalues. Multiplying the system by A_2^{-1} and introducing new notation, we can write it in the form

$$\frac{\partial U}{\partial y} - A\frac{\partial U}{\partial x} + a(z)U(z) + b(z)\overline{U(z)} = F(z). \tag{1}$$

It is convenient to represent the set of eigenvalues of the matrix A as the union $\sigma_1\cup\overline{\sigma}_2$, where both sets σ_1 and σ_2 lie in the upper half-plane $\operatorname{Im}\nu>0$ and $\overline{\sigma}_2 = \{\bar\nu,\ \nu\in\sigma_2\}$. Then the matrix A can be reduced to the following Jordan form:

$$\widetilde{B}^{-1}A\widetilde{B} = \widetilde{J}, \quad \widetilde{J} = \operatorname{diag}(J_1, \overline{J}_2), \tag{2}$$

where $J_k \in \mathbb{C}^{l_k\times l_k}$, $k=1,2$, consists of Jordan cells with eigenvalues $\nu\in\sigma_k$. Surely, $l = l_1 + l_2$ and the cases $l_1 = 0$ or $l_2 = 0$, when one of the sets σ_k is empty, are not excluded. Thus, in this representation the eigenvalues of the matrix J_k form the set σ_k. According to (2), we represent the matrix $\widetilde{B}$ in the block form

$$\widetilde{B} = (B_1, \overline{B}_2), \quad B_k \in \mathbb{C}^{l\times l_k}. \tag{3}$$

For a given $(l\times l)$-matrix-valued function $C\in C^\mu(\Gamma)$, we consider the Riemann–Hilbert boundary-value problem for the system (1):

$$\operatorname{Re} CU^+ = f, \tag{4}$$

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where + means the boundary value of the function U on Γ. Assuming that $F \in C^\mu(\overline{D})$ and $f \in C^\mu(\Gamma)$, it is natural to consider this problem in the class of classical solutions $C^\mu(\overline{D}) \cap C^1(D)$ of the system (1).

In the space of right-hand sides $(F, f) \in C^\mu(\overline{D}) \times C^\mu(\Gamma)$, we introduce the bilinear form

$$\big\langle (F, f), (\widetilde{F}, \widetilde{f}) \big\rangle = \int_D \operatorname{Re}\big[F(z)\widetilde{F}(z)\big]\, d_2 z + \int_\Gamma f(t)\widetilde{f}(t)\, d_1 t, \tag{5}$$

where $d_2 z$ and $d_1 t$ mean the area and arclength elements, respectively, and the notation $F\widetilde{F}$ for two vectors $F = (F_1, \dots, F_l)$ and $\widetilde{F} = (\widetilde{F}_1, \dots, \widetilde{F}_l)$ means their scalar product $F_1\widetilde{F}_1 + \ldots + F_l\widetilde{F}_l$; the notation $f\widetilde{f}$ has a similar sense.

The main result of this paper consists of establishing a criterion for the Fredholm property of the problem (1), (4) in the class $U \in C^\mu(\overline{D}) \cap C^1(D)$ and the formula of its index. The Fredholm property is understood in the following sense: the space X of solutions of the homogeneous problem (i.e., problems with zero right-hand sides $F = 0$ and $f = 0$) is finite-dimensional and there exists a finite-dimensional subspace $\widetilde{X} \subseteq C^\mu(\overline{D}) \times C^\mu(\Gamma)$ such that the orthogonality conditions

$$\big\langle (F, f), (\widetilde{F}, \widetilde{f}) \big\rangle = 0, \quad (\widetilde{F}, \widetilde{f}) \in \widetilde{X},$$

are necessary and sufficient for the solvability of the inhomogeneous problem. The difference

$$\varkappa = \dim X - \dim \widetilde{X}$$

determines the index of the problem.

Theorem 1. *Let $\Gamma \in C^{1,\nu}$ and $C \in C^\nu(\Gamma)$, $\mu < \nu < 1$, so that the matrix-valued function $G(t) = (C(t)B_1, \overline{C(t)}B_2)$ (see the notation* (3)*) belongs to the class $C^\nu(\Gamma)$. Then the condition*

$$\det G(t) \neq 0, \quad t \in \Gamma, \tag{6}$$

holds if and only if the problem (1), (4) *be is a Fredholm problem in the class $C^\mu(\overline{D}) \cap C^1(D)$, and its index is given by the formula*

$$\varkappa = -\sum_{j=1}^{m} \frac{1}{\pi}[\arg \det G]_{\Gamma_j} + (2 - m)l, \tag{7}$$

where $\Gamma_1, \dots, \Gamma_m$ are simple contours composing Γ and the increment $[\ ]_{\Gamma_j}$ along Γ_j is taken in the direction of leaving the domain D to the left.

If, in addition, $C \in C^{1,\nu}(\Gamma)$, then any solution $U \in C^\mu(\overline{D}) \cap C^1(D)$ of the problem with the right-hand side $f \in C^{1,\mu}(\Gamma)$ actually belongs to $C^{1,\mu}(\overline{D})$.

Proof. According to (2), we write the vector-valued function $\widetilde{\phi} = \widetilde{B}^{-1}U$ as a pair $(\phi_1, \overline{\phi}_2)$ with l_k-vectors ϕ_k. After this substitution, (1) turns into the system

$$\frac{\partial \widetilde{\phi}}{\partial y} - \widetilde{J}\frac{\partial \widetilde{\phi}}{\partial x} + \widetilde{a}(z)\widetilde{\phi}(z) + \widetilde{b}(z)\overline{\widetilde{\phi}(z)} = \widetilde{f}^1(z)$$

with the coefficients $\widetilde{a} = \widetilde{B}^{-1}a\widetilde{B}$ and $\widetilde{b} = \widetilde{B}^{-1}b\overline{\widetilde{B}}$ and the right-hand side $\widetilde{f}^1 = \widetilde{B}^{-1}F$. We rewrite it in the componentwise block form:

$$\begin{aligned}
&\frac{\partial \phi_1}{\partial y} - J_1\frac{\partial \phi_1}{\partial x} + a_{11}\phi_1 + a_{12}\overline{\phi}_2 + b_{11}\overline{\phi}_1 + b_{12}\phi_2 = f_1^1,\\
&\frac{\partial \overline{\phi}_2}{\partial y} - \overline{J}_2\frac{\partial \overline{\phi}_2}{\partial x} + a_{21}\overline{\phi}_1 + a_{22}\phi_2 + b_{21}\phi_1 + b_{22}\overline{\phi}_2 = \overline{f}_2^1,
\end{aligned}$$

where $(a_{ij}) = \widetilde{a}$ and $(b_{ij}) = \widetilde{b}$ are block matrices and $(f_1^1, \overline{f}_2^1) = \widetilde{f}^1$. Replacing the second equation of this system by its complex conjugate, we obtain the following system for the vector $\phi = (\phi_1, \phi_2)$:

$$\frac{\partial \phi}{\partial y} - J\frac{\partial \phi}{\partial x} + c\phi + d\overline{\phi} = f^1, \tag{8}$$

where $J = \mathrm{diag}(J_1, J_2)$ is a block-diagonal matrix, the right-hand side $f^1 = (f_1^1, f_2^1) \in C^\mu(\overline{D})$, and the coefficients have the form

$$c = \begin{pmatrix} a_{11} & b_{12} \\ b_{21} & a_{22} \end{pmatrix}, \quad d = \begin{pmatrix} b_{11} & a_{12} \\ a_{21} & b_{22} \end{pmatrix} \in C^\mu(\overline{D}).$$

Since $U = \widetilde{B}\widetilde{\phi} = B_1\phi_1 + \overline{B}_2\overline{\phi}_2$, after this substitution the boundary condition (4) takes the form

$$\mathrm{Re}\, G\phi^+ = f^0, \tag{9}$$

where for uniformity with (8) we denote the function f by f^0.

The problem (8), (9) for this system (under the assumption that Γ is a simple contour) is studied in [10]. The corresponding arguments of that paper with minor changes are also suitable for the considered case of a composite contour. Keeping in mind the smoothness of solutions to the problem, we briefly recall these arguments.

Using the matrix notation $z_J = x \cdot 1 + y \cdot J$, for $z = x + iy \in \mathbb{C}$, we introduce the following Cauchy-type integral operator:

$$(I^0\varphi)(z) = \frac{1}{2\pi i}\int_\Gamma (t - z)_J^{-1} dt_J \varphi(t), \quad z \in D,$$

where $t = t_1 + it_2$ is a point on the contour Γ, which is oriented positively with respect to D, dt_J denotes the complex matrix differential $dt_1 1 + dt_2 J$, and the singular Cauchy operator

$$(S^0\varphi)(t_0) = \frac{1}{\pi i}\int_\Gamma (t - t_0)_J^{-1} dt_J \varphi(t), \quad t_0 \in \Gamma,$$

where $\varphi \in C^\mu(\Gamma)$ is a real l-vector-valued function.

Following [7], we say that the function $\phi = I^0\psi$ is *J-analytic* in the domain D, i.e., it satisfies the equation

$$\frac{\partial \phi}{\partial y} - J\frac{\partial \phi}{\partial x} = 0. \tag{10}$$

For $J = i$, this system becomes the classical Cauchy–Riemann system. As was shown in [7], all basic facts of the theory of analytic functions associated with the integral Cauchy formula can be extended to solutions of the system (14).

According to [6], the integral operator $I^0 : C^\mu(\Gamma) \to C^\mu(\overline{D})$ is bounded and the following Sokhotsk—Plemelj formula is valid:

$$2(I^0\varphi)^+(t_0) = \varphi(t_0) + (S^0\varphi)(t_0), \quad t_0 \in \Gamma. \tag{11}$$

Obviously, in the case of the scalar matrix $J = i$, the operator S^0 becomes a classical singular Cauchy operator, which we denote by S. As was shown in [6], under the assumption $\Gamma \in C^{1,\nu}$, the difference $S^0 - S$ is a compact operator in the space $C^\mu(\Gamma)$, and all principal results of the classical theory of singular operators (see [4]) can also be applied to the operator

$$2N^0\varphi = \mathrm{Re}\left[G(\varphi + S^0\varphi)\right] \tag{12}$$

acting in the space of real l-vector-valued functions $\varphi \in C^\mu(\Gamma)$.

Thus, this operator is a Fredholm operator if and only if the condition (6) is satisfied, and its index is given by the formula

$$\mathrm{ind}\, N^0 = -\frac{1}{\pi}\left[\arg\det G\right]\Big|_\Gamma. \tag{13}$$

Now we introduce integral operators in the domain

$$(I^1\varphi)(z) = \frac{1}{\pi i}\int\limits_D (t-z)_J^{-1}\varphi(t)d_2t, \qquad z \in D,$$

$$(S^1\varphi)(z) = \frac{1}{\pi i}\int\limits_D (t-z)_J^{-2}\varphi(t)d_2t, \qquad z \in D.$$

The last integral is singular and is understood in the corresponding sense. It is easy to verify the following necessary condition for the existence of this integral:

$$\int\limits_{\mathbb{T}} \xi_J^{-2} d_1\xi = 0,$$

where $\mathbb{T}$ is the unit circle. We note that, due to the evenness of the function ξ_J^{-2}, the relation

$$\int\limits_{\mathbb{T}^+} \xi_J^{-2} d_1\xi = 0 \tag{14}$$

also holds, where $\mathbb{T}^+$ means any semicircle.

As was shown in [11], for $\varphi \in C^\mu(\overline{D})$ the function $I^1\varphi$ is continuously differentiable in the domain D and the following formulas are valid:

$$\frac{\partial(I^1\varphi)}{\partial x} = \sigma_1\varphi + S^1\varphi, \quad \frac{\partial(I^1\varphi)}{\partial y} = \sigma_2\varphi + JS^1\varphi, \tag{15}$$

where $\sigma_k \in \mathbb{C}^{l\times l}$ are certain matrices related by the equation $\sigma_2 = J\sigma_1$. In particular,

$$\left(\frac{\partial}{\partial y} - J\frac{\partial}{\partial x}\right) I^1\varphi = 0. \tag{16}$$

By (14), we can apply [9, Theorem 3.5.1] to the singular integral operator S^1. According to this theorem, this operator is bounded in $C^\mu(\overline{D})$. Taking into account (15), we conclude that the operator $I^1 : C^\mu(\overline{D}) \to C^{1,\mu}(\overline{D})$ is bounded.

We consider the functional class

$$\phi \in C^\mu(\overline{D}) \cap C^1(D), \quad \left(\frac{\partial}{\partial y} - J\frac{\partial}{\partial x}\right)\phi \in C^\mu(\overline{D}). \tag{17}$$

Obviously, any solution $\phi \in C^\mu(\overline{D}) \cap C^1(D)$ of Eq. (8) automatically belongs to this class.

For definiteness, we assume that the contour Γ_m encircles the remaining contours $\Gamma_1, \dots, \Gamma_{m-1}$. Then any function $\phi \in C^\mu(\overline{D})$ can be uniquely represented in the form

$$\phi = I^1\varphi^1 + I^0\varphi^0 + i\xi, \quad \xi \in \mathbb{R}^l, \tag{18}$$

with some complex l-vector-valued function $\varphi^0 \in C^\mu(\Gamma)$ and a real vector-valued function $\varphi^0 \in C^\mu(\Gamma)$ satisfying the conditions

$$\int\limits_{\Gamma_j} \varphi(t)d_1t = 0, \quad 1 \le j \le m-1. \tag{19}$$

In fact, we assume

$$\varphi^1 = \left(\frac{\partial}{\partial y} - J\frac{\partial}{\partial x}\right)\phi$$

and let $\phi^0 = \phi - I^1\varphi^1$. Then, due to (16), the function ϕ^0 is J-analytic in the domain D, i.e., it satisfies Eq. (10) and belongs to the class $C^{1,\mu}(\overline{D})$. Therefore, the problem is reduced to the representation

$$\phi^0 = I^0\varphi^0 + i\xi, \quad \xi \in \mathbb{R}^l,$$

under the conditions (19) on the real density φ, which is established in [6] (see also [8]).

Using the representation (18) and the Sokhotski–Plemelj formula (11), we can reduce the problem (8), (9) to the following equivalent system of operator equations:

$$N^1\varphi^1 + N^{10}\varphi^0 + i(c-d)\xi = f^1, \quad N^{01}\varphi^1 + N^0\varphi^0 - (\operatorname{Im} G)\xi = f^0, \tag{20}$$

where, in addition to (12), for brevity we introduce the notation

$$N^1\varphi^1 = \varphi^1 + c(I^1\varphi^1) + d(\overline{I^1\varphi^1}),$$
$$N^{10}\varphi^0 = c(I^1\varphi^0) + d(\overline{I^1\varphi^0}), \quad N^{01}\varphi^1 = \operatorname{Re} G(I^1\varphi^1)^+.$$

This system is considered with respect to the set $(\varphi^1, \varphi^0, \xi)$, subject to the conditions (19). We can write it briefly using the notation $\varphi = (\varphi^1, \varphi^0)$:

$$N\varphi + T\xi = f \tag{21}$$

with the right-hand side $f = (f^1, f^0)$ and the operator matrices

$$N = \begin{pmatrix} N^1 & N^{10} \\ N^{01} & N^0 \end{pmatrix}, \quad T = \begin{pmatrix} ic - id \\ -\operatorname{Im} G \end{pmatrix}.$$

Obviously, the space $C^\mu(\overline{D}) \times C^\mu(\Gamma) \times \mathbb{R}^l$ is an extension of the space $C^\mu(\overline{D}) \times C^\mu(\Gamma)$ to l dimensions; therefore, based on well-known properties of Fredholm operators (see [5]), we conclude that the operators (N, T) and N are Fredholm equivalent and their indices are related by the formula

$$\operatorname{ind}(N, T) = \operatorname{ind} N + l. \tag{22}$$

On the other hand, the condition (19) determines a closed subspace of codimension $l(m-1)$ in the space $C^\mu(\Gamma)$; therefore, from the same considerations, the index $\varkappa$ of the system (19), (20) is related to the index of the operator (N, T) by the formula

$$\varkappa = \operatorname{ind}(N, T) - l(m-1). \tag{23}$$

Let us consider in detail the operators appeared in (20). We will write $N_1 \sim N_2$ if the difference $N_1 - N_2$ is a compact operator. Recall that the operator I^1 is compact in $C^\mu(\overline{D})$; then we can write $N^1 \sim 1$, $N^{01} \sim 0$, and consequently,

$$N \sim M = \begin{pmatrix} 1 & N^{1,0} \\ 0 & N^0 \end{pmatrix}. \tag{24}$$

Assume that the condition (6) is satisfied. Then, as was noted above, the operator N^0 is a Fredholm operator and its index is given by the formula (13). In particular, there exists its regularizer, i.e. an operator R^0 in $C^\mu(\Gamma)$ possessing the property $R^0N^0 \sim N^0R^0 \sim 1$. One can directly verify that the operator

$$R = \begin{pmatrix} 1 & -N^{1,0}R^0 \\ 0 & R^0 \end{pmatrix}$$

is a regularizer of the operator M and hence the operator M is a Fredholm operator. This implies that the operator N is a Fredholm operator and hence the initial problem (1), (4) is a Fredholm problem.

Conversely, let the problem (1), (4) be a Fredholm problem such that N and hence M are Fredholm operators. Let R be the regularizer written in the block form:

$$R = \begin{pmatrix} R^1 & R^{10} \\ R^{01} & R^0 \end{pmatrix}.$$

Then it follows directly from the relations $MR \sim MR \sim 1$ that $N^0R^0 \sim R^0N^0 \sim 1$, so that N^0 is a Fredholm operator. As was noted above, this implies the condition (6).

In order to prove the formula (7) for the index, we introduce the operator $M(t)$ depending on the parameter $0 \le t \le 1$, which is obtained by replacing of N^{10} by tN^{10} in the definition (24) of the operator M. The same arguments show that $M(t)$ is also a Fredholm operator. Since it depends on t continuously, its index is independent of t and, in particular,

$$\operatorname{ind} M = \operatorname{ind} M(0) = \operatorname{ind} N^0.$$

Hence $\operatorname{ind} N = \operatorname{ind} N^0$, which together with (13), (22), and (23) completes the proof of the index formula (7).

Now we turn to the last assertion of the theorem and assume that the functions C and hence G also belong to $C^{1,\mu}(\Gamma)$. Then for $f^0 \in C^{1,\mu}(\Gamma)$, the terms $N^{01}\varphi^1$ and $(\operatorname{Im} G)\xi$ in the second equation (20) also belong to $C^{1,\mu}(\Gamma)$. Thus, the function $\varphi^0 \in C^{\mu}(\Gamma)$ satisfies the equation $N^0\varphi^0 = g$ with the right-hand side $g \in C^{1,\mu}(\Gamma)$. As was shown in [1], in this case $\varphi^0 \in C^{1,\mu}(\Gamma)$. According to the differentiation formula for the Cauchy-type integral $\phi = I^0\varphi$ (see [9]), the function $I^0\varphi^0$ belongs to $C^{1,\mu}(\overline{D})$, so that the function ϕ in the representation (18) belongs to this class, and hence the solution $U = B_1\phi_1 + B_2\phi_2$ of the original problem (1), (4) also belongs to this class. □

Note that due to the last assertion of the theorem, the problem (1), (4) is a Fredholm problem in the class $C^{1,\mu}(\overline{D})$ with the same index.

The case of the elliptic system

$$\frac{\partial U}{\partial y} - A\frac{\partial U}{\partial x} + a(z)U(z) = F(z) \tag{25}$$

with real coefficients $A \in \mathbb{R}^{2l\times 2l}$ and $a(z)$ can be considered similarly. In this case, eigenvalues of the matrix A form the set $\sigma \cup \overline{\sigma}$, where $\sigma \subseteq \{\nu,\ \operatorname{Im}\nu > 0\}$. Therefore, the relations (2) and (3) take the form

$$\widetilde{B}^{-1}A\widetilde{B} = \widetilde{J}, \quad \widetilde{J} = \operatorname{diag}(J, \overline{J}), \quad \widetilde{B} = (B, \overline{B}), \quad B \in \mathbb{C}^{2l\times l}.$$

Here the boundary condition

$$CU^+ = f \tag{26}$$

with a $(l \times 2l)$-matrix $C(t)$, $t \in \Gamma$, is an analog of the Riemann–Hilbert problem; respectively, the following assertion (with the same proof) is an analog of Theorem 1.

Theorem 2. *Let $\Gamma \in C^{1,\nu}$ and an $(l \times 2l)$-matrix C belong to the class $C^{\nu}(\Gamma)$, $\mu < \nu < 1$, so that we have $G(t) = C(t)B \in C^{\nu}(\Gamma)$ (see the notation (3). The problem (25), (26) is a Fredholm problem in the class $C^{\mu}(\overline{D}) \cap C^1(D)$ if and only if the condition (6) holds, and its index is given by the formula (7).*

If, in addition, $C \in C^{1,\nu}(\Gamma)$, then any solution $U \in C^{\mu}(\overline{D}) \cap C^1(D)$ of the problem with the right-hand side $f \in C^{1,\mu}(\Gamma)$ actually belongs to $C^{1,\mu}(\overline{D})$.

We note that in a few more general classes, Theorem 2 was established by B. Bojarski in [2] (see also [3]).

Up to now, the domain D has been assumed to be finite, i.e., lying inside a certain circle. Now we consider the case where the domain D is still bounded by a contour $\Gamma \in C^{1,\nu}$ but is infinite, i.e., it contains the outer domain of a certain circle. In this case, all simple contours Γ_j, $1 \le j \le m$, which form the contour Γ, are equivalent. For simplicity, we assume that the point $z = 0$ lies outside $\overline{D}$.

We perform our considerations in the weighted Hölder space $C^{\mu}_{\delta}(\overline{D}, \infty)$, $\delta \in \mathbb{R}$, with power behavior $O(|z|^{\delta})$ at infinity (see [9]). We briefly recall its definition. The space $C^{\mu}_0(\overline{D}, \infty)$ consists of all bounded functions $\varphi(z)$, $z \in D$ with finite norm $|\varphi| = |\varphi|_0 + \{\varphi\}_{\mu}$, where

$$|\varphi|_0 = \sup_{z\in D} |\varphi(z)|, \quad \{\varphi\}_{\mu} = \sup_{z_1\neq z_2} \frac{|z_1|^{\mu}|\varphi(z_1) - \varphi(z_2)|}{|z_1 - z_2|^{\mu}}.$$

This space is a Banach algebra with respect to multiplication, and the weighted space C^{μ}_{δ} is obtained from C^{μ}_0 by multiplying its elements by $|z|^{\delta}$ (with the transferred norm). By definition, the space $C_\delta^{1,\mu}(\overline{D},\infty)$ consists of functions $\varphi \in C_\delta^\mu(\overline{D},\infty) \cap C^1(D)$ whose partial derivatives belong to the class $C_{\delta-1}^\mu(\overline{D},\infty)$:

$$\frac{\partial\varphi}{\partial x},\ \frac{\partial\varphi}{\partial y} \in C_{\delta-1}^\mu(\overline{D},\infty).$$

We consider Eq. (1) in the infinite domain D in the class

$$C_\lambda^\mu(\overline{D},\infty) \cap C^1(D), \quad -1 < \lambda < 0, \tag{27}$$

assuming that

$$a, b \in C_{-1-\varepsilon}^\mu(\overline{D},\infty), \quad F \in C_{\lambda-1}^\mu(\overline{D},\infty) \tag{28}$$

with some $\varepsilon > 0$.

Consider the operators I^0 and I^1 introduced above. In the case considered, the function $(I^0\varphi)(z)$ has the following expansion in a neighborhood of ∞:

$$(I^0\varphi)(z) = \sum_{k\le -1} c_k z_J^k, \quad c_k = -\frac{1}{2\pi i}\int_\Gamma t_J^{-k-1} dt_j \varphi(t),$$

and since $-1 < \lambda < 0$, the operator $I^0 H C^\mu(\Gamma) \to C_\lambda^\mu(\overline{D},\infty)$ is bounded. We state the corresponding properties of the operator I^1.

Theorem 3. *The operator I^1 considered as an operator $C_{\lambda-1}^\mu(\overline{D},\infty) \to C_\lambda^\mu(\overline{D},\infty)$ is bounded and being considered as an operator $C_{\lambda-1}^\mu(\overline{D},\infty) \to C_{\lambda+\varepsilon}^\mu(\overline{D},\infty)$, $\varepsilon > 0$, it is compact. Moreover, any function ϕ of the class*

$$\phi \in C_\lambda^\mu(\overline{D}) \cap C^1(D), \quad \left(\frac{\partial}{\partial y} - J\frac{\partial}{\partial x}\right)\phi \in C_{\lambda-1}^\mu(\overline{D}),$$

can be uniquely represented in the form

$$\phi = I^0\varphi^0 + I^1\varphi^1$$

with a real vector-valued function $\varphi^0 \in C^\mu(\Gamma)$ satisfying the condition

$$\int_{\Gamma_j} \varphi(t) d_1 t = 0, \quad 1 \le j \le m,$$

and a complex vector-valued function $\varphi^1 \in C_{\lambda-1}^\mu(\overline{D},\infty)$.

Proof. The first statement of the theorem on the boundedness of the operator $I^1 : C_{\lambda-1}^\mu(\overline{D},\infty) \to C_\lambda^\mu(\overline{D},\infty)$ is established similarly to [9]. It was also established in [9] that the singular operator S^1 is bounded in the space $C_{\lambda-1}^\mu(\overline{D},\infty)$. Hence, taking into account (16) and the compactness property of the embedding mentioned in [9],

$$C_{\delta-\varepsilon}^{\mu+\varepsilon}(\overline{D},\infty) \subseteq C_\delta^\mu(\overline{D},\infty), \quad \varepsilon > 0,$$

we conclude that the operator I^1 is compact.

The second part of the theorem is established similarly to the case of a finite domain. □

Using Theorem 3 and following the scheme of the proof of Theorem 1, we arrive at the validity of the following assertion.

Theorem 4. *Assume that a contour Γ belongs to the class $C^{1,\nu}$, C belongs to the class $C^\nu(\Gamma)$, $\mu < \nu < 1$, and $G = (CB_1, \overline{C}B_2)$. Then under the assumption* (28), *the problem* (1), (4) *is a Fredholm problem in the class* (27) *if and only if the condition* (6) *holds. Moreover, its index is given by the formula*

$$\varkappa = -\sum_{j=1}^m \frac{1}{\pi}\big[\arg\det G\big]_{\Gamma_j} - ml.$$

If, in addition, $C \in C^{1,\nu}(\Gamma)$, *then any solution* U *of the problem of this class with right-hand side* $f \in C^{1,\mu}(\Gamma)$ *in fact belongs to* $C^{1,\mu}_{\lambda}(\overline{D}, \infty)$.

An analog of Theorem 2 for the system (1) with real coefficients can be formulated similarly.

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