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# The homogenization of diffusion-convection equations in non-periodic structures

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**Abstract:** We consider the homogenization of diffusion-convective problems with given divergence-free velocities in nonperiodic structures defined by sequences of characteristic functions (the first sequence). The sequence of concentration (the second sequence) is uniformly bounded in the space of square-summable functions with square-summable derivatives with respect to spatial variables. At the same time, the sequence of time-derivative of product of these concentrations on the characteristic functions, that define a nonperiodic structure, is bounded in the space of square-summable functions from time interval into the conjugated space of functions depending on spatial variables, with square-summable derivatives. We prove the strong compactness of the second sequences in the space of quadratically summable functions and use this result to homogenize the corresponding boundary value problems that depend on a small parameter.

**Key words:** Diffusion-convection, homogenization, nonperiodic structures, compactness lemma

## 1. Introduction

In the present work, we establish an Aubin-type compactness lemma [3, 9] for nonperiodic structures and then apply it to find the homogenization of diffusion-convection equations for such kind of structures. By now, there exist numerous compactness results of this type [5, 11]. However, none of them seems to be applicable to the problem which we address in this note.

Let a measurable function $\chi(\boldsymbol{x},\boldsymbol{y})$ be 1-periodic in variable $\boldsymbol{y} \in Y = \left(-\frac{1}{2},\frac{1}{2}\right)^3 \subset \mathbb{R}^3$, such that $\chi(\boldsymbol{x},\boldsymbol{y}) = 1$ for $\boldsymbol{y} \in Y_f(\boldsymbol{x})$ and $\chi(\boldsymbol{x},t,\boldsymbol{y}) = 0$ for $\boldsymbol{y} \in Y_s(\boldsymbol{x})$.

Here $\overline{Y_f(\boldsymbol{x}) \cup Y_s(\boldsymbol{x})} = \overline{Y}$, $Y_f(\boldsymbol{x}) \cap Y_s(\boldsymbol{x}) = \emptyset$, $Y_f(\boldsymbol{x}) \cap Y_s(\boldsymbol{x}) = \gamma(\boldsymbol{x})$ and $\gamma(\boldsymbol{x})$ satisfies the Lipschitz condition. For example, $Y_s(\boldsymbol{x}) = \{\boldsymbol{y} \in Y : |\boldsymbol{y}| < r(\boldsymbol{x}) < \frac{1}{2}\}$, $Y_f(\boldsymbol{x}) = \{\boldsymbol{y} \in Y : |\boldsymbol{y}| > r(\boldsymbol{x})\}$ and $\gamma(\boldsymbol{x}) = \{\boldsymbol{y} \in Y : |\boldsymbol{y}| = r(\boldsymbol{x})\}$.

Now we put $\Omega_f^\varepsilon = \left\{\boldsymbol{x} : \chi(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}) = 1\right\}$, $\Omega_s^\varepsilon = \left\{\boldsymbol{x} : \chi(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}) = 0\right\}$.

We denote the pore space as $Q_f^\varepsilon = \Omega_f^\varepsilon \times (0,T)$, the solid skeleton as $Q_s^\varepsilon = \Omega_s^\varepsilon \times (0,T)$, and through $\Gamma^\varepsilon = \overline{\Omega}_f^\varepsilon \cap \overline{\Omega}_s^\varepsilon$ we denote the boundary between the pore space and the solid skeleton.

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In what follows for the given function $r(\boldsymbol{x}) : 0 < r(\boldsymbol{x}) < \frac{1}{2}$, $r \in \mathbb{W}^{1,0}_{\infty}(\Omega_T)$ we restrict ourself with two structures:

**structure I**:

$$Y_s(\boldsymbol{x}) = \{\boldsymbol{y} \in Y : |\boldsymbol{y}| < r(\boldsymbol{x})\}, \ \ Y_f(\boldsymbol{x}) = \{\boldsymbol{y} \in Y : |\boldsymbol{y}| > r(\boldsymbol{x})\},$$

$$\chi(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\mathrm{sgn}\big(r(\boldsymbol{x}) - |\boldsymbol{\varsigma}(\boldsymbol{y})|\big), \ \ 0 \leq r(\boldsymbol{x}) < \frac{1}{2},$$

where periodic function $\boldsymbol{\varsigma}(\boldsymbol{y})$ is defined by formula

$$\boldsymbol{\varsigma}(\boldsymbol{y}) = (y_1 - [|y_1|], y_2 - [|y_2|], y_3 - [|y_3|])$$

and $[|a|]$ is an integer part of the number $a$;

**structure II**:

$$Y_s(\boldsymbol{x}) = Y_s^1(\boldsymbol{x}) \cup Y_s^2(\boldsymbol{x}) \cup Y_s^3(\boldsymbol{x});$$

$$Y_s^1(\boldsymbol{x}) = \{\boldsymbol{y} \in Y : y_2^2 + y_3^2 < r(\boldsymbol{x})\},$$

$$Y_s^2(\boldsymbol{x}) = \{\boldsymbol{y} \in Y : y_1^2 + y_3^2 < r(\boldsymbol{x})\},$$

$$Y_s^3(\boldsymbol{x}) = \{\boldsymbol{y} \in Y : y_1^2 + y_2^2 < r(\boldsymbol{x})\}, \ \ Y_f(\boldsymbol{x}) = Y \setminus \overline{Y_s}(\boldsymbol{x});$$

$$\chi(\boldsymbol{x},\boldsymbol{y}) = \chi_1(\boldsymbol{x},\boldsymbol{y})\chi_2(\boldsymbol{x},\boldsymbol{y})\chi_3(\boldsymbol{x},\boldsymbol{y}),$$

$$\chi_1(\boldsymbol{x},\boldsymbol{y}) = \mathrm{sng}\big(r^2(\boldsymbol{x}) - y_2^2 - y_3^2\big),$$

$$\chi_2(\boldsymbol{x},\boldsymbol{y}) = \mathrm{sng}\big(r^2(\boldsymbol{x}) - y_1^2 - y_3^2\big),$$

$$\chi_3(\boldsymbol{x},\boldsymbol{y}) = \mathrm{sng}\big(r^2(\boldsymbol{x}) - y_1^2 - y_2^2\big), \ \ \chi^{\varepsilon}(\boldsymbol{x}) = \chi(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}).$$

Suppose for simplicity that $\Omega = \left(-\frac{1}{2}, \frac{1}{2}\right)^3$, $S = \partial\Omega$,

$S^{\varepsilon,\pm} = \overline{\Omega}^{\varepsilon}_f \cap \left\{x_1 = \pm\frac{1}{2}\right\}$, $S^{\pm} = \{x_1 = \pm\frac{1}{2}\}$, $S^{\varepsilon,0} = \overline{\Omega}^{\varepsilon}_f \setminus \big(\overline{S}^{\varepsilon,+} \cup \overline{S}^{\varepsilon,-}\big)$, $S^0 = \Big(\partial\Omega \setminus \big(\{x_1 = \frac{1}{2}\} \cup \{x_1 = -\frac{1}{2}\}\big)\Big)$.

We consider a diffusion-convection of some admixture with concentration $c^{\varepsilon}$ during the movement of the liquid in the pore space with given divergence-free velocity $\boldsymbol{v}^{\varepsilon}$.

The concentration $c^{\varepsilon}$ of the admixture satisfies the diffusion-convection equation

$$\frac{\partial c^{\varepsilon}}{\partial t} = \nabla \cdot (D\nabla c^{\varepsilon} - c^{\varepsilon}\,\boldsymbol{v}^{\varepsilon}) \tag{1.1}$$

in the domain $Q^{\varepsilon}_f$ and following boundary and initial conditions

$$\frac{\partial c^{\varepsilon}}{\partial n}(\boldsymbol{x},t) = 0, \ \ (\boldsymbol{x},t) \in \Gamma^{\varepsilon} \times (0,T), \tag{1.2}$$

$$\frac{\partial c^{\varepsilon}}{\partial n}(\boldsymbol{x},t)=0,\ \ (\boldsymbol{x},t)\in S^{0,\varepsilon}\times(0,T), \tag{1.3}$$

$$c^{\varepsilon}(\boldsymbol{x},t)=c^{\pm}(\boldsymbol{x},t),\ \ (\boldsymbol{x},t)\in S^{\varepsilon,\pm}\times(0,T), \tag{1.4}$$

$$c^{\varepsilon}(\boldsymbol{x},0)=c_0(\boldsymbol{x}),\ \ \boldsymbol{x}\in\Omega^{\varepsilon}_f. \tag{1.5}$$

In (1.1) – (1.5) $D$ is given positive constant and $\boldsymbol{n}$ is a normal vector to the boundary $S^{0,\varepsilon}$.

Due to [8] the problem (1.1) – (1.4) has a unique weak solution $c^{\varepsilon}\in\mathbb{W}^{1,0}_2(Q^{f,\varepsilon})\cap\mathbb{L}_{\infty}(Q^{f,\varepsilon})$ and these solutions are uniformly bounded in $\mathbb{W}^{1,0}_2(Q^{f,\varepsilon})\cap\mathbb{L}_{\infty}(Q^{f,\varepsilon})$.

Now, using results of [1, 6] we extend our solutions onto the hole domain $\Omega_T$.

Let $\widetilde{c}^{\varepsilon}$ be such extensions. Without loss of generality we may assume that the sequence $\{\widetilde{c}^{\varepsilon}\}$ weakly converges to some function $c\in\mathbb{W}^{1,0}_2(\Omega_T)\cap\mathbb{L}_{\infty}(\Omega_T)$.

As a next step we will show that the sequence $\{\chi^{\varepsilon}(\boldsymbol{x},t_0)\,\widetilde{c}^{\varepsilon}(\boldsymbol{x},t_0)\}$ weakly converges to the function $m(\boldsymbol{x},t_0)\,c(\boldsymbol{x},t_0)$ for almost all $t_0\in(0,T)$.

Here

$$m(\boldsymbol{x},t)=\int_Y\chi(\boldsymbol{x},t,\boldsymbol{y})dy \tag{1.6}$$

is a porosity of the continuum medium.

Next we prove that there exists some subsequence $\{\varepsilon_k\}$, such that for almost all $t_0\in(0,T)$

$$\lim_{\varepsilon\to 0}\varepsilon_k^2\int_{\Omega}|\nabla\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t_0)|^2dx=0, \tag{1.7}$$

and for almost all $t_0\in(0,T)$ the sequence $\{\widetilde{c}^{\varepsilon_k}(\boldsymbol{x},t_0)\}$ converges weakly and two-scale to the function $c(\boldsymbol{x},t_0)$.

Finally, as a last step, we prove that the sequence $\{\widetilde{c}^{\varepsilon_k}\}$ converges strongly in $\mathbb{L}_2(\Omega_T)$ to the function $c(\boldsymbol{x},t)$.

Throughout the text we use notations of [8, 9] for the functional spaces and norms there.

## 2. Auxiliary statements

In this section, we define the notion of two-scale convergence in $\mathbb{L}_2(\Omega_T)$ and formulate basic results from [1, 6, 14].

**Definition 2.1** A sequence $\{u^{\varepsilon}(\boldsymbol{x},t)\}$, $u^{\varepsilon}\in L_2(\Omega_T)$, two-scale converges as $n\to\infty$ to 1-periodic in $\boldsymbol{y}\in Y=(0,1)^3\subset\mathbb{R}^3$ function $U(\boldsymbol{x},t,\boldsymbol{y})$ if

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\Psi(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon})u^{\varepsilon}(\boldsymbol{x},t)dxdt=\int_{\Omega_T}\left(\int_Y\Psi(\boldsymbol{x},t,\boldsymbol{y})U(\boldsymbol{x},t,\boldsymbol{y})dy\right)dxdt$$

for any smooth 1-periodic in $\boldsymbol{y}$ function $\Psi(\boldsymbol{x},t,\boldsymbol{y})$.

**Notation:** $u^{\varepsilon}(\boldsymbol{x},t)\overset{two-sc.}{\longrightarrow}U(\boldsymbol{x},t,\boldsymbol{y})$.

**Theorem 1** *(Gabriel Nguetseng)*

1) Any bounded in $L_2(\Omega_T)$ sequence $\{u^\varepsilon\}$ two-scale converges in $L_2(\Omega_T)$ (up to some subsequence) to some 1-periodic in $\boldsymbol{y}$ function $U(\boldsymbol{x},t,\boldsymbol{y}) \in L_2(\Omega_T \times Y)$:

$$u^\varepsilon(\boldsymbol{x},t) \stackrel{two-sc.}{\longrightarrow} U(\boldsymbol{x},t,\boldsymbol{y}).$$

2) Let the sequence $\{u^\varepsilon\}$ be bounded in $W_2^{1,0}(\Omega_T)$. Then the sequences $\{u^\varepsilon\}$ and $\{\nabla u^\varepsilon\}$ two-scale converges (up to some subsequence) to some functions $u(\boldsymbol{x},t)$ and $\nabla u(\boldsymbol{x},t) + \nabla_y U(\boldsymbol{x},t,\boldsymbol{y})$ correspondingly, where $u \in W_2^{1,0}(\Omega_T)$ and $\nabla_y U \in L_2(\Omega_T \times Y)$:

$$u^\varepsilon(\boldsymbol{x},t) \stackrel{two-sc.}{\longrightarrow} u(\boldsymbol{x},t),$$

$$\nabla u^\varepsilon(\boldsymbol{x},t) \stackrel{two-sc.}{\longrightarrow} \nabla u(\boldsymbol{x},t) + \nabla_y U(\boldsymbol{x},t,\boldsymbol{y}).$$

**Definition 2.2** A sequence $\{v^\varepsilon(\boldsymbol{x},t)\}$, $v^\varepsilon \in L_2(\Omega_T)$, weakly converges as $n \to \infty$ to some function $v(\boldsymbol{x},t)$, $v \in L_2(\Omega_T)$ if

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} \varphi(\boldsymbol{x},t) v^\varepsilon(\boldsymbol{x},t) dx dt = \int_{\Omega_T} \varphi(\boldsymbol{x},t) v(\boldsymbol{x},t) dx dt$$

for any smooth function $\varphi(\boldsymbol{x},t)$.

**Notation:** $v^\varepsilon(\boldsymbol{x},t) \rightharpoonup v(\boldsymbol{x},t)$.

**Theorem 2** *[7]*

*Any bounded in $L_2(\Omega_T)$ sequence $\{v^\varepsilon\}$ contains weakly convergent in $L_2(\Omega_T)$ subsequence.*

A limiting procedure in perforated domain needs some extension of functions defined in domains $Q_f^\varepsilon$ onto domain $\Omega_T$. To do that we use results for nonperiodic structures, similar to results of [1, 6] proved for periodic structures. Due to the special type of structures I and II, especially for the structure I (soft inclusion), proofs in [1, 6] also serve for our cases. More precisely, holds true the following lemma.

**Lemma 2.1** *Let $c^\varepsilon \in \mathbb{W}_2^{1,0}(Q_f^{\varepsilon)}$. Then there exists an extension $\widetilde{c}^\varepsilon \in \mathbb{W}_2^{1,0}(\Omega_T)$ of this function from $Q_f^\varepsilon$ onto $\Omega_T$, such that*

$$\int_{\Omega_T} |\widetilde{c}^\varepsilon|^2 dx dt \leq M \int_{Q_f^\varepsilon} |c^\varepsilon|^2 dx dt, \int_{\Omega_T} |\nabla \widetilde{c}^\varepsilon|^2 dx dt \leq M \int_{Q_f^\varepsilon} |\nabla c^\varepsilon|^2 dx dt. \tag{2.1}$$

*Through $M$ here and in what follows we denote any constant independent of $\varepsilon$.*

### 3. Main results

For the sake of simplicity we suppose that holds true such conditions.

***Conditions A***

1) $c^\pm(\boldsymbol{x},t) = c^\pm(\boldsymbol{x})$;

2) *there exists a function* $c^0(\boldsymbol{x})$ *such that* $0 \leq c^0(\boldsymbol{x}) \leq 1$, $c^0 \in \mathbb{W}_2^1(\Omega)$, *and* $c^0$ *satisfies boundary conditions (1.3) and (1.4);*

3) *functions* $\boldsymbol{v}^\varepsilon(\boldsymbol{x},t)$ *satisfy conditions*

$$\nabla \cdot \boldsymbol{v}^\varepsilon = 0, \ (\boldsymbol{x},t) \in Q_f^\varepsilon,$$

$$\boldsymbol{v}^\varepsilon \cdot \boldsymbol{n} = 0, \ (\boldsymbol{x},t) \in (S^0 \bigcup \Gamma^\varepsilon) \times (0,T),$$

$$\int_{Q_f^\varepsilon,} \big(|\boldsymbol{v}^\varepsilon|^2 + |\nabla \boldsymbol{v}^\varepsilon|^2\big) dxdt \leq M^2;$$

4) *there exist some extensions* $\widetilde{\boldsymbol{v}}^{\,\varepsilon}$ *of functions* $\boldsymbol{v}^\varepsilon$ *from domain* $Q^{f,\varepsilon}$ *onto domain* $\Omega_T$ *such that*

$$\nabla \cdot \widetilde{\boldsymbol{v}}^{\,\varepsilon} = 0, \ (\boldsymbol{x},t) \in Q_f^\varepsilon, \ \widetilde{\boldsymbol{v}}^{\,\varepsilon} \cdot \boldsymbol{n} = 0, \ (\boldsymbol{x},t) \in (S^0 \bigcup \Gamma^\varepsilon) \times (0,T),$$

$$\int_{\Omega_T} \big(|\widetilde{\boldsymbol{v}}^{\,\varepsilon}|^2 + |\nabla \widetilde{\boldsymbol{v}}^{\,\varepsilon}|^2\big) dx \leq M^2,$$

$$\widetilde{\boldsymbol{v}}^{\,\varepsilon} \rightharpoonup \boldsymbol{v}(\boldsymbol{x},t), \ \ \boldsymbol{v} \in L_2(\Omega_T), \ \text{and} \ \widetilde{\boldsymbol{v}}^{\,\varepsilon} \stackrel{two-sc.}{\longrightarrow} \boldsymbol{v}(\boldsymbol{x},t), \ \ \boldsymbol{v} \in L_2(\Omega_T),$$

$$\nabla \cdot \boldsymbol{v} = 0, \ (\boldsymbol{x},t) \in \Omega_T, \ \boldsymbol{v}(\boldsymbol{x},t) \cdot \boldsymbol{n} = 0, \ \boldsymbol{x} \in S^0 \times (0,T).$$

*Here* $\boldsymbol{n}$ *is a normal vector to the corresponding boundaries.*

**Definition 3.1** A function $c^\varepsilon \in \mathbb{W}_2^{1,0}(Q^{f,\varepsilon})$ is called a weak solution to the problem (1.1) – (1.4) if it satisfies the integral identity

$$\int_{Q^{f,\varepsilon}} \Big( - c^\varepsilon \frac{\partial \varphi}{\partial t} + \big(D\, \nabla c^\varepsilon - c^\varepsilon\, \boldsymbol{v}^\varepsilon\big) \cdot \nabla \varphi \Big) dxdt = \int_{\Omega^{f,\varepsilon}(0)} c_0(\boldsymbol{x}) \varphi(\boldsymbol{x},0) dx \tag{3.1}$$

for any smooth function $\varphi$ vanishing at $S^{\varepsilon,\pm} \times (0,T)$ and at $\{t = T\}$.

**Lemma 3.1** *Under conditions A for almost all* $\varepsilon > 0$ *there exists a unique weak solution* $c^{\,\varepsilon}(\boldsymbol{x},t)\}$ *to the problem (1.1) – (1.4) such that*

$$\int_{Q_f^\varepsilon} |c^{\,\varepsilon}|^2 dxdt + \int_{Q_f^\varepsilon} |\nabla c^{\,\varepsilon}|^2 dxdt \leq M. \tag{3.2}$$

**Lemma 3.2** *Let* $\widetilde{c}^{\,\varepsilon}$ *be an extension of the function* $c^\varepsilon$ *from* $Q_f^\varepsilon$ *onto* $\Omega_T$ *such that*

$$\int_{\Omega_T} |\widetilde{c}^{\,\varepsilon}|^2 dxdt \leq M \int_{Q_f^\varepsilon} |c^{\,\varepsilon}|^2 dxdt, \int_{\Omega_T} |\nabla \widetilde{c}^{\,\varepsilon}|^2 dxdt \leq M \int_{Q_f^\varepsilon} |\nabla c^{\,\varepsilon}|^2 dxdt. \tag{3.3}$$

*Then under conditions A for almost all* $t_0 \in (0,T)$ *the sequence*
$\{\chi^{\,\varepsilon}(\boldsymbol{x},t_0)\, \widetilde{c}^{\,\varepsilon}(\boldsymbol{x},t_0)\}$ *converges weakly in* $\mathbb{L}_2(\Omega)$ *to the function* $m(\boldsymbol{x},t_0)\, c(\boldsymbol{x},t_0)$.

**Lemma 3.3** *Under conditions A for almost all $t_0 \in (0,T)$ there exists some subsequence $\{\varepsilon_k\}$, such that*

$$\lim_{\varepsilon_k \to 0} \varepsilon_k^2 \int_{\Omega} |\nabla \widetilde{c}^{\,\varepsilon_k}(\boldsymbol{x}, t_0)|^2 dxdt = 0. \tag{3.4}$$

**Lemma 3.4** *Under conditions A for almost all $t_0 \in (0,T)$ the sequence $\{\widetilde{c}^{\,\varepsilon_k}(\boldsymbol{x}, t_0)\}$ converges weakly and two-scale in $\mathbb{L}_2(\Omega)$ to the function $c(\boldsymbol{x}, t_0)$.*

**Lemma 3.5** *Under conditions A the sequence $\{\widetilde{c}^{\,\varepsilon_k}(\boldsymbol{x}, t)\}$ converges strongly in $\mathbb{L}_2(\Omega_T)$ to the function $c(\boldsymbol{x}, t)$ from $\mathbb{W}_2^{1,0}(\Omega_T)$.*

**Theorem 3** *The limiting function $c \in \mathbb{W}_2^{1,0}(\Omega_T)$ satisfies boundary and initial conditions*

$$\big(D\,\mathbb{B} \cdot \nabla c - c\,\boldsymbol{v}\big) \cdot \boldsymbol{n} = 0, \ \ (\boldsymbol{x}, t) \in S^0 \times (0,T), \tag{3.5}$$

$$c(\boldsymbol{x}, t) = c^{\pm}(\boldsymbol{x}), \ \ (\boldsymbol{x}, t) \in S^{\pm} \times (0,T), \tag{3.6}$$

$$c(\boldsymbol{x}, 0) = c_0(\boldsymbol{x}), \ \ \boldsymbol{x} \in \Omega, \tag{3.7}$$

*and the homogenized equation*

$$\frac{\partial}{\partial t}\big(m(\boldsymbol{x}, t)\, c\big) = \nabla \cdot \big(D\,\mathbb{B} \cdot \nabla c - c\,\boldsymbol{v}\big) \tag{3.8}$$

*in the domain $\Omega_T$ in the sense of distributions as a solution of the integral identity*

$$\int_{\Omega_T} \Big( - m(\boldsymbol{x}, t)\, c\, \frac{\partial \varphi}{\partial t} + \big(D\,\mathbb{B} \cdot \nabla c - c\,\boldsymbol{v}\big) \cdot \nabla \varphi \Big) dxdt = 0 \tag{3.9}$$

*for any smooth functions $\varphi$, vanishing at $S^{\pm} \times (0,T)$.*

*In (3.5) – (3.9) $\boldsymbol{n}$ is a normal vector to the boundary $S^0$ and a symmetric and strictly positively defined matrix $\mathbb{B}$ is given by formula (4.18).*

## 4. Proof of Theorem 3

### 4.1. Proof of Lemma 3.1

The proof of lemma is straightforward and based on the a priory estimates

$$\int\int_{Q_f^{\varepsilon}} (D\,|\nabla c^{\varepsilon}|^2) dxdt \leq M \left( \int_{\Omega^{f,\varepsilon}(0)} |c_0(\boldsymbol{x})|^2 dx + \int\int_{Q_f^{\varepsilon}} |c^0(\boldsymbol{x})|^2 dxdt \right) \tag{4.1}$$

$$0 \leq c^{\varepsilon}(\boldsymbol{x}, t) \leq 1, \tag{4.2}$$

what follows from the integral identity (3.1) after substituting the function $(c^{\varepsilon} - c^0)$ instead of the test function in this identity, and from the maximum principle (see, for example [8]).

### 4.2. Proof of Lemma 3.2

Due to Lemma B.1.5 (Appendix B, [12]) the sequence $\{\widetilde{c}^{\varepsilon}\}$ two-scale converges in $\mathbb{L}_2(\Omega_T)$ to some function $c$. That is

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\widetilde{c}^{\,\varepsilon}(\boldsymbol{x},t)\,\varphi(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon})dxdt=\int_{\Omega_T}c(\boldsymbol{x},t)\left(\int_Y\varphi(\boldsymbol{x},t,\boldsymbol{y})dy\right)dxdt. \tag{4.3}$$

Let us put

$$\varphi(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon})=\chi^{\varepsilon}(\boldsymbol{x},t)\,\eta(t)\,\psi(\boldsymbol{x}),\;\; f^{\varepsilon}_{\psi}(t)=\int_{\Omega}\chi^{\varepsilon}(\boldsymbol{x},t)\,\psi(\boldsymbol{x})dx,\;\; f_{\psi}=\int_{\Omega}m(\boldsymbol{x},t)\,\psi(\boldsymbol{x})dx.$$

The equality (4.3) means that

$$I=\lim_{\varepsilon\to 0}\int_0^T\eta(t)\,f^{\varepsilon}_{\psi}(t)dt=\int_0^T\eta(t)\,f_{\psi}(t)dt. \tag{4.4}$$

Coming back to the integral identity (3.1) in the form

$$\int_{\Omega_T}\chi^{\varepsilon}\Big(-\widetilde{c}^{\,\varepsilon}\frac{\partial\varphi}{\partial t}+\big(D\,\nabla\widetilde{c}^{\,\varepsilon}-\widetilde{c}^{\,\varepsilon}\,\widetilde{\boldsymbol{v}}^{\,\varepsilon}\big)\cdot\nabla\varphi\Big)dxdt=0 \tag{4.5}$$

with test function $\varphi\left(\boldsymbol{x},t,\dfrac{\boldsymbol{x}}{\varepsilon}\right)=\eta(t)\,\psi(\boldsymbol{x})$ we arrive at

$$\int_0^T\left(\frac{d\eta}{dt}\,f^{\varepsilon}_{\psi}+\eta\,U^{\varepsilon}\right)dt=0,$$

$$U^{\varepsilon}=\int_{\Omega}(\chi^{\varepsilon}\,D\,\nabla\widetilde{c}^{\,\varepsilon}-\widetilde{c}^{\,\varepsilon}\,\widetilde{\boldsymbol{v}}^{\,\varepsilon})\cdot\nabla\psi dx\int_0^T|U^{\varepsilon}|^2dt\leq\,M^2\,T\int_{\Omega}|\nabla\psi|^2dx.$$

Thus,

$$\frac{df^{\varepsilon}_{\psi}}{dt}=U^{\varepsilon},\;\; f^{\varepsilon}_{\psi}\in\mathbb{W}^1_2(0,T);\;\; |f^{\varepsilon}_{\psi}(t)|\leq\,M_{\psi},\;\; |f^{\varepsilon}_{\psi}(t_1)-f^{\varepsilon}_{\psi}(t_2)|\leq\,M_{\psi}\,|t_1-t_2|^{\frac{1}{2}}.$$

The Arzela-Ascoli Theorem [15] permits us to choose some subsequence $\{f^{\varepsilon_k}_{\psi}\}$, convergent in $\mathbb{C}(0,T)$ to some continuous function $\overline{f}_{\psi}$.

On the other hand due to (4.4) one has

$$I=\lim_{\varepsilon\to 0}\int_0^T\eta(t)\,f^{\varepsilon}_{\psi}(t)dt=\int_0^T\eta(t)\,\overline{f}_{\psi}(t)dt \tag{4.6}$$

for any smooth function $\eta(t)$.

Therefore, $f_{\psi}=\overline{f}_{\psi}$ almost everywhere in $(0,T)$, which proves the statement of the lemma.

### 4.3. Proof of Lemma 3.3

In fact, the uniform boundedness with respect to $\varepsilon$ of the sequence $\displaystyle\int\int_{\Omega_T}|\nabla\widetilde{c}^{\,\varepsilon}(\boldsymbol{x},t)|^2dxdt$ implies

$$\lim_{\varepsilon\to 0}\varepsilon^2\int_{\Omega_T}|\nabla\widetilde{c}^{\,\varepsilon}(\boldsymbol{x},t)|^2dxdt=0. \tag{4.7}$$

Let

$$u_\varepsilon(t_0) = \varepsilon^2 \int_\Omega |\nabla \widetilde{c}^{\,\varepsilon}(\boldsymbol{x}, t)|^2 dx. \tag{4.8}$$

Then (4.7) means that the sequence $\{u_\varepsilon\}$ converges to zero in $\mathbb{L}_1(0,T)$. In accordance with [7] (Theorem 1, § 37, Chapter X) there exists some subsequence $\{u_{\varepsilon_k}\}$ convergent to zero almost everywhere in $(0,T)$, which proves the lemma.

### 4.4. Proof of Lemma 3.4

Since the sequence $\{\widetilde{c}^{\,\varepsilon}\}$ is bounded in $\mathbb{L}_2(\Omega)$, there exists a subsequence (we leave for simplicity previous indexes) that two-scale converges to some 1-periodic in variable $\boldsymbol{y}$ function $\overline{C}(\boldsymbol{x}, t_0, \boldsymbol{y})$ from the space $\mathbb{L}_2(\Omega \times Y)$.

Integration by parts of expression $\varepsilon_k \nabla \widetilde{c}^{\,\varepsilon}(\boldsymbol{x}, t_0) \cdot \boldsymbol{\varphi}\left(\dfrac{\boldsymbol{x}}{\varepsilon_k}\right) \psi(\boldsymbol{x})$ gives us an integral identity

$$\varepsilon_k \int_\Omega \nabla \widetilde{c}^{\,\varepsilon}(\boldsymbol{x}, t_0) \cdot \boldsymbol{\varphi}\left(\frac{\boldsymbol{x}}{\varepsilon_k}\right) \psi(\boldsymbol{x}) dx =$$

$$- \varepsilon_k \int_\Omega \widetilde{c}^{\,\varepsilon}(\boldsymbol{x}, t_0) \left(\boldsymbol{\varphi}(\frac{\boldsymbol{x}}{\varepsilon_k}) \cdot \nabla \psi(\boldsymbol{x})\right) dx - \int_\Omega \widetilde{c}^{\,\varepsilon}(\boldsymbol{x}, t_0) \left(\nabla \cdot \boldsymbol{\varphi}(\frac{\boldsymbol{x}}{\varepsilon_k})\right) \psi(\boldsymbol{x}) dx$$

for any functions $\boldsymbol{\varphi} \in \mathbb{W}^1_2(Y)$ and $\psi \in \overset{\circ}{\mathbb{W}}{}^1_2(\Omega)$.

The limit as $\varepsilon_k \to 0$ results integral identity

$$\int_\Omega \psi(\boldsymbol{x}) \int_Y \overline{C}(\boldsymbol{x}, t_0, \boldsymbol{y}) \nabla \cdot \boldsymbol{\varphi}(\boldsymbol{y}) dy = 0,$$

which, due to the arbitrary choice of functions $\boldsymbol{\varphi}$ and $\psi$, is equivalent to the equality $\overline{C}(\boldsymbol{x}, t_0, \boldsymbol{y}) = \overline{c}(\boldsymbol{x}, t_0)$.

Because $\overline{c}(\boldsymbol{x}, t_0)$ is also a weak limit of the sequence $\{\widetilde{c}^{\,\varepsilon_k}(\boldsymbol{x}, t_0)\}$, then the uniqueness of the weak limit implies $\overline{c}(\boldsymbol{x}, t_0) = c(\boldsymbol{x}, t_0)$.

### 4.5. Proof of Lemma 3.5

We put

$$\mathbb{H}^1 = \overset{\circ}{\mathbb{W}}{}^1{}_2(\Omega) \subset \mathbb{H}^0 = \mathbb{L}_2(\Omega) \subset \mathbb{H}^{-1} = \overset{\circ}{\mathbb{W}}{}^{-1}{}_2(\Omega), \;\; w_k(\boldsymbol{x}, t_0) = \widetilde{c}^{\,\varepsilon_k}(\boldsymbol{x}, t_0) - c(\boldsymbol{x}, t_0)$$

and use the inequality

$$\|w_k(., t_0)\|^2_{\mathbb{H}^0} \leq \eta \, \|w_k(., t_0)\|^2_{\mathbb{H}^1} + C_\eta \, \|w_k(., t_0)\|^2_{\mathbb{H}^{-1}}$$

(the estimate (9), § 10, Chapter III, [2]).

Next we integrate the last relation with respect to time

$$\int_0^T \|w_k(., t)\|^2_{\mathbb{H}^0} dt \leq \eta \int_0^T \|w_k(., t)\|^2_{\mathbb{H}^1} dt + C_\eta \int_0^T \|w_k(., t)\|^2_{\mathbb{H}^{-1}} dt,$$

and apply a compact embedding of the space $\mathbb{H}^0$ into the space $\mathbb{H}^{-1}$ [9, 13]: the weak convergence of the sequence $\{w_k(., t)\}$ in $\mathbb{H}^0(\Omega)$ implies the strong convergence of this sequence in $\mathbb{H}^{-1}(\Omega)$.

That is $\lim\limits_{k\to\infty}\int_0^T \|w_k(.,t)\|^2_{\mathbb{H}^{-1}}dt = 0$.

The last relation and the choice of small $\eta$ imply the equality

$\lim\limits_{k\to\infty}\int_0^T \|w_k(.,t)\|^2_{\mathbb{H}^0}dt = 0$, which proves the lemma.

### 4.6. Proof of Theorem 3

Now we may pass to the limit as $\varepsilon \to 0$ in the integral identity (4.5) and get the desired homogenization of the problem (1.2) – (1.5). Theorem 1 permits to extract some subsequence of the sequence $\{\widetilde{c}^{\,\varepsilon}\}$ (for simplicity we will leave the same indices) such that

$$\widetilde{c}^{\,\varepsilon}(\boldsymbol{x},t)\overset{two-sc.}{\longrightarrow} c(\boldsymbol{x},t),\ \nabla_x\widetilde{c}^{\,\varepsilon}(\boldsymbol{x},t)\overset{two-sc.}{\longrightarrow}\nabla_x c(\boldsymbol{x},t)+\nabla_y C(\boldsymbol{x},t,\boldsymbol{y}) \tag{4.9}$$

with some 1-periodic in $\boldsymbol{y}$ function $C(\boldsymbol{x},t,\boldsymbol{y})$, $\nabla_y C\in L_2(Q\times Y)$.

By virtue of conditions $A$ the sequence $\widetilde{\boldsymbol{v}}^{\,\varepsilon}$ weakly converges in $L_2(\Omega_T)$ to some function $\boldsymbol{v}\in L_2(\Omega_T)$, such that

$$\nabla\cdot\boldsymbol{v}=0,\ \ (\boldsymbol{x},t)\in\Omega_T,\ \ \boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x},t)=0,\ \ (\boldsymbol{x},t)\in S^0. \tag{4.10}$$

First of all we consider as a test-function in (4.5) an arbitrary function $\varphi=\varphi_0(\boldsymbol{x},t)$ vanishing at $S^{\pm}\times(0,T)$.

After the limit as $\varepsilon\to 0$ we arrive at

$$0=\lim_{\varepsilon\to 0}\int_{\Omega_T}\Big(-\chi^{\varepsilon}\,\widetilde{c}^{\varepsilon}\frac{\partial\varphi_0}{\partial t}+\big(D\,\chi^{\varepsilon}\,\nabla\widetilde{c}^{\varepsilon}\ \nabla\varphi-\chi^{\varepsilon}\,c^{\varepsilon}\,\widetilde{\boldsymbol{v}}^{\varepsilon}\cdot\nabla\varphi_0\big)\Big)dxdt=$$

$$=\int_{\Omega_T}\Big(-m\,c\,\frac{\partial\varphi_0}{\partial t}+\big(D\,(\nabla_x c+\int_Y\chi\,\nabla_y C\,dy)\cdot\nabla\varphi_0-m\,c\,\boldsymbol{v}\cdot\nabla\varphi_0\big)\Big)dxdt=0. \tag{4.11}$$

Reintegration of (4.11) gives us desired homogenized diffusion-convection equation

$$\frac{\partial}{\partial t}(m\,c)=\nabla_x\cdot\Big(D\,(\nabla_x c+(\int_Y\chi\,\nabla_y C\,dy))-m\,c\,\boldsymbol{v}\Big) \tag{4.12}$$

with unknown function $C(\boldsymbol{x},t,\boldsymbol{x})$.

To find function $C$ we now choose as a test-function in (4.5) function $\varphi=\varphi_0(\boldsymbol{x},t)\,\varphi_1(\frac{\boldsymbol{x}}{\varepsilon})$ with the same as before function $\varphi_0(\boldsymbol{x},t)$ and with an arbitrary function $\varphi_1(\boldsymbol{y})$.

The limiting procedure gives us identity

$$0=\int_{\Omega_T}\varphi_0(\boldsymbol{x},t)\Big(\int_Y\big(\nabla_x c+\nabla_y C-m\,c\,\boldsymbol{v}\big)\cdot\nabla_y\varphi_1\Big)dxdt \tag{4.13}$$

and, consequently, the boundary value problem

$$\nabla_y\cdot\big(\nabla_x c+\nabla_y C-m\,c\,\boldsymbol{v}\big)=0,\ \boldsymbol{y}\in Y_f(\boldsymbol{x}),\ \big(\nabla_x c+\nabla_y C-m\,c\,\boldsymbol{v}\big)\cdot\boldsymbol{N}=0,\ \boldsymbol{y}\in\gamma(\boldsymbol{x}), \tag{4.14}$$

where $\boldsymbol{N}=(N_1,N_2,N_3)$ is a normal vector to the boundary $\gamma(\boldsymbol{x})$.

We look for the solution $C$ of (4.14) in the form

$$C=\sum_{i=1}^{3}C_i(\boldsymbol{x},\boldsymbol{y})\,f_i(\boldsymbol{x},t),\quad f_i=\frac{\partial c}{\partial x_i}-m\,c\,v_i. \tag{4.15}$$

Substitution of this representation into (4.14) results

$$\triangle\,C_i=0,\ \boldsymbol{y}\in Y_f(\boldsymbol{x}),\ \ (\nabla_y C_i+\boldsymbol{e}_i))\cdot\boldsymbol{N}=0,\ \in\gamma(\boldsymbol{x}). \tag{4.16}$$

This problem has an unique (up to some constant) solution [4, 6, 10] and

$$\nabla_x c+\nabla_y C-m\,c\,\boldsymbol{v}=D\,\mathbb{B}^c(\boldsymbol{x})\cdot(\nabla_x c-m\,c\,\boldsymbol{v}), \tag{4.17}$$

where strictly positively defined matrix $\mathbb{B}^c(\boldsymbol{x})$ is given by formula

$$\mathbb{B}^c(\boldsymbol{x})=\mathbb{I}+\sum_{i=1}^{3}\int_{Y_f(\boldsymbol{x})}\nabla C_i(\boldsymbol{x},\boldsymbol{y})dy. \tag{4.18}$$

Finally, the homogenized diffusion-convection equation in the domain $\Omega_T$ has a form

$$\frac{\partial}{\partial t}(m\,c)=\nabla_x\big(D\,\mathbb{B}^c(\boldsymbol{x})\cdot(\nabla_x c-m\,c\,\boldsymbol{v})\big). \tag{4.19}$$

In a usual way we show that

$$c(\boldsymbol{x},t)=c^{\pm},\ \ (\boldsymbol{x},t)\in S^{\pm}\times(0,T),\ \ \frac{\partial c}{\partial n}=0,\ \ (\boldsymbol{x},t)\in S^0\times(0,T) \tag{4.20}$$

and

$$c(\boldsymbol{x},0)=c_0,\ \ \boldsymbol{x}\in\Omega. \tag{4.21}$$

## 5. Conclusion

In this paper, we prove the compactness lemma for the domains with nonperiodic structures. This result rigorously justify the homogenization of the initial-boundary value problem, describing the diffusion-convection of an admixture during the fluid filtration in non-periodic pore space with given divergent-free velocity.


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