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AN ANALOGUE OF THE SCHWARZ PROBLEM FOR THE MOISIL–TEODORESCU SYSTEM IN A MULTIPLY CONNECTED DOMAIN

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ABSTRACT

An analogue of the Schwarz problem for the Moisil – Teodorescu system is considered in a multiply connected domain D. It is shown that this problem is Fredholm in the Holder class C (D).

Keywords: Moisil – Teodorescu system, Schwarz problem, Cauchy type integral, singular integral equation

Let a domain D be bounded by the connected smooth surface $\Gamma \in C^{1,\neg}$, $0 < \neg < 1$. Let us consider the Moisil – Teodorescu system [1] in this domain for a forth-component vector (u_1, v), $v = (v_1, v_2, v_3)$ written in the form

$$div\ v = 0, \quad rot\ v + grad\ u_1 = 0 \tag{1}$$

It is well known that these components u_1 and v_i are harmonic functions. An analogue of the Schwarz problem is the following: to find a solutions $(u_1, v) \in C(\overline{D})$ of (1) under boundary value conditions

$$u_1^+ = f_1, \quad v^+ n = g, \tag{2}$$

Where the sign + is point out the boundary value, $n = (n_1, n_2, n_3)$ is the unit external vector and $v^+ n$ denotes the inner product.

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If the domain D is homeomorphic to a ball then this problem is investigated in detail by V.I.Shevchenko [2, 3]. Another approach which is based on an integral representation of special type was described in [4, 5]. In particular the following result is valid in this case. The homogeneous problem (1), (2) has only trivial solutions, but the nonhomogeneous one is solved if and only if the right- hand side (f_1; g) satisfies to one orthogonality condition.

The case of multiply connected domain has its own characteristics. It is well known [6], that every connected smooth closed surface is homeomorphic to sphere with m handles. In particular the domain *D* is homeomorphic to a ball for $m = 0$ ant it is homeomorphic to a torus for $m = 1$.

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Theorem 1. *Let the surface* $\Gamma = \partial D$ *is homeomorphic to sphere with m handles and belong to the class* $C^{1,\neg}$. *Then a dimension of the space of solutions of homogeneous problem (1), (2) is equal to m.*

Proof. Let (u_1, v) be a solution of homogeneous problem (1), (2). Since the function u_1 is harmonic in the domain *D*, then $u_1 = 0$ and the second equality (2) becomes *rot v* = 0. Hence, in each simply-connected subdomain $D_0 \subseteq D$ the function *v* can be defined as grad w_0 of some function w_0, which is harmonic by virtue of the first equality of (2). If D_1 is another simply-connected subdomain D with the correspondence representation *v* = grad w_1, then $w_0 - w_1$ is a local constant function on the open set $D_0 \cap D_1$. On the whole multiply – connected domain *D* the harmonic function *w* such that *v* = *grad w* is multiply – valued function. It follows from the second equality of (2) that

$$\frac{\partial w^+}{\partial n} = 0. \qquad (3)$$

To avoid the multiply connectedness of the domain *D* let us consider its cuts. By definition the cut is a simply – connected smooth surface $R \subseteq \overline{D}$ with smooth boundary ∂L, such that $R \cap \Gamma = \partial L$. Under assumption there exist disjoint cuts $\overline{R_1}$,…, R_m, such that the set

$$\mathrm{D_R = D \setminus R,\ R = R_1 \cup ... R_m},$$

is a simply connected domain. In this domain the function *w* is a single-valued and its boundary values satisfy the relation

$$(w^+ - w^-)|_{R_i} = c_i, \quad 1 \le i \le m, \qquad (4)$$

with some constants c_i. Nevertheless equalities $c_1 = ... c_m = 0$ indicate that *w* is univalent function. So it is harmonic in the whole domain *D*, while in a view of (3) this is possible only if *w* is constant. These arguments prove that the space of solutions of the homogeneous problem is finite dimensional space and its dimension doesn't exceed *m*. In fact this dimension is equal to *m* exactly. Indeed it is suffciently to prove that the problem (3), (4) with additional condition

$$\left(\frac{\partial w^+}{\partial n} - \frac{\partial w^-}{\partial n}\right)\Big|_R = 0\,, \tag{5}$$

is always solved in the domain D_R. It is easily to establish with help of the Dirichlet integral. Let $W^{1,2}(\hat{D}_R)$ denote a space of all functions φ such that for every Lipschitzian subdomain $D_0 \subseteq D_R$ the restriction $\varphi|_{D_0}$ belongs to the Sobolev space $W^{1,2}(D_0)$. Obviously there exist one-sided boundary values $\varphi^{\pm} \in L^2(R)$ for elements of this space. It is proved by usual way [7] that the minimum of the integral

$$D(\varphi) = \int_D |\,grad\,\varphi\,|^2\, dx$$

in the class $\varphi \in W^{1,2}(\hat{D}_R)$, satisfying (4) gives a generalized solution of the problem (3)–(5). In fact this solution is classic that complete the proof.

Let us consider a question on Fredholm property of the problem (1), (2). This question is solved with the help of an integral operator *I* which acts for two-vector-valued functions $\varphi = (\varphi_1, \varphi_2) \in C(\Gamma)$ by the formula

$$u_1(x) = \frac{1}{2\pi}\int_\Gamma \frac{(y-x)n(y)}{|\,y-x\,|^3}\varphi_1(y)d_2y, \quad x \in D,$$

$$v(x) = \frac{1}{2\pi}\int_\Gamma \frac{y-x}{|\,y-x\,|^3}\varphi_2(y)d_2y - \frac{1}{2\pi}\int_\Gamma \frac{[y-x, n(y)]}{|\,y-x\,|^3}\varphi_1(y)d_2y, \tag{6}$$

where d_2y is the square element and brackets imply the vector product. Note that the first equality of this formula is the potential of double layer of the Laplace operator.

Lemma 1. The function $(u_1, v) = I\varphi$ is a solution of (1), under assumption $\Gamma \in C^{1,\beta}$ the operator *I* is bounded $C^{\alpha}(\Gamma) \to C^{\alpha}(\overline{D}), 0 < \alpha < \beta$, and the following boundary value formula

$$u_1^+(y_0) = \varphi_1(y_0) + \frac{1}{2\pi}\int_\Gamma \frac{(y-y_0)n(y)}{|\,y-y_0\,|^3}\varphi_1(y)d_2y, \quad y_0 \in \Gamma\,,$$

$$v^{+}(y_0) = ¨_2(y_0)n(y_0) + \frac{1}{2®}\int_\Gamma \frac{y-y_0}{|y-y_0|^3} ¨_2(y)d_2y - \frac{1}{2®}\int_\Gamma \frac{[y-y_0, n(y)]}{|y-y_0|^3} ¨_1(y)d_2y,$$

holds where the integrals of the right-hand side of the second equality are singular in the sense of the limits of the integrals over $\Gamma \cap \{|y-y_0| \geq £\}$ as $£ \to 0$.

Proof. The system (1) can be written in the form

$$M\left(\frac{\partial}{\partial x}\right)u(x) = 0, \quad M(¶) = \begin{pmatrix} 0 & ¶_1 & ¶_2 & ¶_3 \\ ¶_1 & 0 & -¶_3 & ¶_2 \\ ¶_2 & ¶_3 & 0 & -¶_1 \\ ¶_3 & -¶_2 & ¶_1 & 0 \end{pmatrix}, \tag{7}$$

for the forth component vector *u(x) = (u₁, v₁, v₂, v₃)*. It is known that the matrix-valued function $M^{T}(x)/|x|^3$; where *T* is the symbol of matrix transformation is the fundamental solution of this system. So for every forth- component vector $· = (·_1, ..., ·_4) \in C(\Gamma)$ the generalized Cauchy type integral [8]

$$(I^0 ·)(x) = \frac{1}{2®}\int_\Gamma \frac{M^{T}(y-x)n(y)}{|y-x|^3} M[n(y)]·(y)d_2y, \quad x \notin \Gamma, \tag{8}$$

defines a solution of (7). Here *n(y)* is the external unit normal to Γ at the point *y*. It is known [8] that if $·$ satisfies the Holder condition and Γ belongs to Lyapunov type then there exists the boundary value

$$u^{+}(y_0) = \lim_{x \to y_0, x \in D} u(x), \quad y_0 \in \Gamma,$$

and the following analogue Plemelj – Sokhotskyii formula

$$u^{+} = · + u^{*}, \quad (I^{*} ·)(y_0) = \frac{1}{2®}\int_\Gamma \frac{M^{T}(y-y_0)}{|y-y_0|^3} M[n(y)]·(y)d_2y, \tag{9}$$

is valid. As it is shown in [5] under assumption $\Gamma \in C^{1,¬}, 0 < ¬ < 1$ the operator I^0 is bounded $C^{«}(\Gamma) \to C^{«}(\overline{D}), 0 < « < ¬$.

Putting $\cdot = (¨_1, ¨_2 n)$, we can see that $M(n)\cdot = (¨_2, ¨_1 n)$ and $M^{T}(¶)M(n)\cdot = ((¶ n)¨_1, ¨_2 ¶ - ¨_1 [¶, n])$. Substituting these expressions into (8), (9), we complete the proof.

Denote by *S* the operator of the boundary value problem (1), (2). By lemma 1 the composition of this operator with *I* gives the formula

$$(SI¨)_1 = ¨_1 + K_{11} ¨_1, \quad (SI¨)_2 = ¨_2 + K_{21} ¨_1 + K_{22} ¨_2,$$

where K_{ij} are the correspondent integral operators on Γ with weak singularities. As it is proved in [9], these operators are compact in the space $C^{«}(\Gamma), 0 < « < ¬$. Since by the known Riesz theorem [10] the operator *1 + K* is Fredholm one and its index is equal to zero. In particular the image *im (1 + K) = im (SI)* of this operator in the space $C^{«}(\Gamma)$ has a finite ko-dimension. Since $S \supseteq im(SI)$ the image of the operator *S* has the same property. Together with theorem 1 it follows that the problem (1), (2) is Fredholmian.

REFERENCES

[1] Bitsadze A.V. Foundationof the theory of analytic functions of a complex vari-able. - M.: Nauka, 1972 (Russion).

[2] Shevchenko V.I. On the Riemann – Hilbert problem for a holomorphic vector, Doklady Àkad. Nauk SSSR, 1966. Vol. 169, •6. - pp. 1285-1288 (Russion)

[3] Shevchenko V.I. Some boundary value problems for holomorphic vector, Sborn "Math. Phys." Kiev. - 1970. Vol.8. - pp. 172-187 (Russion)

[4] Polunin V.A., Soldatov A.P. The Riemann- Hilbert problem for Moisil–Theodorescu system in bounded domains, Non-classical equations of mathe-matical physics. - Novosibirsk: 2010 pp. 192-201 (Russion)

[5] Polunin V.A., Soldatov A.P., On the integral representation of solutions for Moisil-Theodorescu, Vedomosty. BSU, 2011., 5(118). pp. 84-92. 2011 (Russion)

[6] Milnor J. W. Topology from the di erentiable viewpoint, Prinston University, 1965.

[7] Rektorys K. Variational methods in mathematical sciences and engineering. Dordrect, Boston, SNTL, 1980

[8] Bitsadze A.V. Spatial analogue of Cauchy type integral and some of its appli-cations, Izv. Àkad. Nauk SSSR, math., 1953, Vol. 17, 6., pp. 525-538 (Russion)

[9] VPolunin V.A., Soldatov A.P., Three-dimensional analogue of the Cauchy type integral, Di erents. Uravn., 2011, vol. 47. No.3, pp. 366-375 (Russion)

[10] Palais P. Seminar on the Atiah–Singer index theorem, Prinston, NewJersey, Princton University Press, 1965.

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