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Generalized Cauchy Integrals on the Plane

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Abstract. The integrals with homogeneous-difference kernels are considered on a smooth contour. The boundary properties of the integrals are described in the Hölder space. An analogue of the known Sokhotski–Plemelj formula is obtained. Moreover, the differentiation formula of these integrals is also given.

Let $D \subset \mathbb{C}$ be a plane domain with a smooth boundary Γ and the function $Q(t, \xi)$, $t \in \Gamma$, be odd with respect to $\xi \in \mathbb{C}$ and homogeneous of degree -1. We call the integral

$$(I\varphi)(z) = \int_\Gamma Q(t, t - z)\varphi(t) d_1 t, \quad z \in D,$$

by generalized Cauchy type integral. This form permits to represent the classical Cauchy type integrals [5], the corresponding integrals for solutions of first order elliptic systems [2, 7], the double layer potential in the theory of elliptic equations of second order [1, 4]. These also occur in applications [3].

Let $C^\mu(G)$, $0 < \mu \leq 1$, be the usual Hölder functional space on the set $G \subseteq \mathbb{C}$ with Hölder exponent μ and the corresponding norm

$$|\varphi|_{\mu,G} = |\varphi|_{0,G} + [\varphi]_{\mu,G} \quad [\varphi]_\mu = \sup_{z_1,z_2} \frac{|\varphi(z_1) - \varphi(z_2)|}{|z_1 - z_2|^\mu}.$$

We denote by $C^{n,\mu}(G)$, $n \geq 1$, the corresponding space of continuously differentiable functions φ, for which $\varphi' = (\partial\varphi/\partial x, \partial\varphi/\partial y) \in C^{n-1,\mu}(G)$. The class $C^{1,\mu}$ of smooth contours is defined with respect to their parametrization.

We have also to introduce notations for homogeneous functions. Let us denote by $\mathcal{H}_\lambda \subseteq C^\infty(\mathbb{C} \setminus 0)$ the class of functions $Q(\xi)$, $\xi = \xi_1 + i\xi_2$, which are homogeneous of degree λ. We define norms in this class by

$$|Q|_{(n)} = \max_{0 \leq i \leq n} |Q^{(i)}_\xi|_{0,\Omega}, \quad n = 0, 1, \ldots,$$

where Ω is the unit circle $\{|\xi| = 1\}$. Note that

$$|Q|_{1,\Omega} \leq M_1 |Q|_{(1)}, \tag{1}$$

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where M_1 depends only on λ.

Let $C^{\mu(n)}(G,\mathcal{H}_\lambda)$ be the class of functions $Q(t,\xi)\in\mathcal{H}_\lambda$, for which $Q^{(i)}_\xi(t,\xi)\in C^\mu(G)$, $i\le n$, uniformly with respect to $|\xi|=1$. The analogous class $C^{1,\nu(n)}(G,\mathcal{H}_\lambda)$ corresponds to $C^{1,\mu}(G)$. Note that differentiation $Q\to\partial Q/\partial\xi_i$ acts $C^{\mu(n)}(G,\mathcal{H}_\lambda)\to C^{\mu(n-1)}(G,\mathcal{H}_{\lambda-1})$, $n\ge 1$.

It follows from these definitions the following properties.

Lemma 1. *(a) If $Q\in\mathcal{H}_\lambda$ then for all $\xi,\eta\in\mathbb{C}$ the inequality*

$$|Q(\xi)-Q(\eta)|\le M|Q|_{(1)}(|\xi|^{\lambda-1}+|\eta|^{\lambda-1})|\xi-\eta|, \tag{2}$$

is valid, where $M>0$ doesn't depend only on λ.

(b) Let a set G be bounded, the kernel $Q(t_0,t;\xi)\in C^{\mu(1)}(G\times G,\mathcal{H}_0)$ and $Q(t,t,\xi)\equiv 0$. Then the function $q(t_0,t)=Q(t_0,t;t-t_0)$ belongs to $C^\mu(G\times G)$ and $q(t,t)=0$.

(c) Let a smooth contour $\Gamma\subseteq\mathbb{C}$ belongs to $C^{1,\mu}$, so that the unit tangent vector $e(t)$, $t\in\Gamma$ belongs to $C^\mu(\Gamma)$. Let a kernel $Q_0(t_0,t;\xi)\in C^{\mu(1)}(\Gamma\times\Gamma,\mathcal{H}_0)$ be even with respect to ξ.

Then the function $q_0(t_0,t)=Q_0(t_0,t;t-t_0)$, extended by $q_0(t_0,t_0)=Q_0[t_0,t_0;e(t_0)]$ at $t=t_0$, belongs to $C^\mu(\Gamma\times\Gamma)$. Particularly, if a kernel $Q(t_0,t;\xi)\in C^{\mu(1)}(\Gamma\times\Gamma,\mathcal{H}_{-1})$ is odd with respect to ξ, then $Q(t_0,t;t-t_0)=q(t_0,t)(t-t_0)^{-1}$ with a function $q\in C^\mu(\Gamma\times\Gamma)$.

Proof. (a) It is obviously that (1) is equivalent to

$$|Q(\xi')-Q(\eta')|\le M|Q|_{(1)}(|\xi'|^{\lambda-1}+|\eta'|^{\lambda-1})|\xi'-\eta'|$$

with respect to $\xi'=\xi/|\xi|$, and $\eta'=\eta/|\xi|$. So we can put $|\xi|=1$. Then

$$|Q(\xi)-Q(\eta)|=\left|Q(\xi)-|\eta|^\lambda Q\left(\frac{\eta}{|\eta|}\right)\right|\le[Q]_{1,\Omega}\left|\xi-\frac{\eta}{|\eta|}\right|+|Q|_{0,\Omega}|1-|\eta|^\lambda|.$$

It is obviously

$$\left|\xi-\frac{\eta}{|\eta|}\right|\le|\xi-\eta|+\left|1-\frac{1}{|\eta|}\right||\eta|\le 2|\xi-\eta|,$$

taking into account that $|1-|\eta||=||\xi|-|\eta||\le|\xi-\eta|$. Analogously we have

$$|1-|\eta|^\lambda|\le|\lambda|\max(1,|\eta|^{\lambda-1})|1-|\eta||\le|\lambda|(1+|\eta|^{\lambda-1})|\xi-\eta|.$$

It follows from these inequalities that

$$|Q(\xi)-Q(\eta)|\le(2[Q]_{1,\Omega}+|\lambda||Q|_{0,\Omega})(1+|\eta|^{\lambda-1})|\xi-\eta|,$$

and the last with (1) gives (2), where $|\xi|=1$.

(b) By definition

$$|q_0(t_0,t)|\le|Q|_{C^{\mu(0)}}|t_0-t|^\mu\le M|Q|_{C^{\mu(0)}},$$

and it is sufficient to estimate $\Delta=q_0(t_1,t)-q_0(t_2,t)$ and $\Delta=q_0(t_0,t_1)-q_0(t_0,t_2)$. Let us consider, for example, the first one. Putting $\delta=|t_1-t_2|$ the cases $|t_1-t|\le 2\delta$ $|t_1-t|\ge 2\delta$ consider separately. For the first case $|t_2-t|\le 3\delta$ and, therefore,

$$|\Delta|\le|Q|_{C^{\mu(0)}}(|t_1-t|^\mu+|t_2-t|^\mu)\le(2^\mu+3^\mu)|Q|_{C^{\mu(0)}}\delta^\mu. \tag{3}$$

For the second case by virtue of the inequality $|t-t_1|-\delta\le|t-t_2|\le|t-t_1|+\delta$ we have

$$\delta\le|t-t_2|\le 2|t-t_1|. \tag{4}$$

We can write

$$|\Delta|\le|Q(t_1,t,t-t_1)-Q(t_2,t,t-t_1)|+|Q(t_2,t,t-t_1)-Q(t_2,t,t-t_2)|=$$

$$= |t_1 - t|^\mu \tilde{Q}_1(t - t_1) + |t_2 - t|^\mu [\tilde{Q}_2(t - t_1) - \tilde{Q}_2(t - t_2)],$$

where

$$\tilde{Q}_1(\xi) = \frac{Q(t_1, t, \xi) - Q(t_2, t, \xi)}{|t_1 - t_2|^\mu}, \quad \tilde{Q}_2(\xi) = \frac{Q(t_2, t, \xi) - Q(t, t, \xi)}{|t_2 - t|^\mu} \in \mathcal{H}_0.$$

By virtue of (2) it follows

$$|\Delta| \le |Q|_{C^{\mu(0)}} \delta^\mu + M|Q|_{C^{\mu(1)}} \delta |t_2 - t|^\mu (|t_1 - t|^{-1} + |t_2 - t|^{-1}). \tag{5}$$

Taking into account (4) we have:

$$\delta |t_2 - t|^\mu (|t_1 - t|^{-1} + |t_2 - t|^{-1}) \le 3\delta |t_2 - t|^{\mu - 1} \le 3\delta^\mu.$$

Together with (3), (5) we complete the proof.

(c) It is sufficient to prove that $q_0(t_0, t) \in C^\mu(\Gamma_0 \times \Gamma_0)$ for every arc $\Gamma_0 \subseteq \Gamma$. We suppose that the parametrization $\gamma : [0, 1] \to \Gamma_0$ belongs to the class $C^{1,\mu}[0, 1]$ and $a(s_0, s) = q_0[\gamma(s_0), \gamma(s)]$, $0 \le s, s_0 \le 1$. Since the function Q_0 is homogeneous and even we can represent the last function in the form

$$a(s_0, s) = Q_0[\gamma(s_0), \gamma(s); b(s_0, s)], \quad b(s_0, s) = \frac{\gamma(s) - \gamma(s_0)}{s - s_0}.$$

It is obvious that $b \in C^\mu([0, 1] \times [0, 1])$ and $|b(s_0, s)| \ge c$ for some $c > 0$. Then $a \in C^\mu([0, 1] \times [0, 1]$ and therefore $q_0 \in C^\mu(\Gamma_0 \times \Gamma_0)$.

The second part of (c) follows easily from the first one because $q(t_0, t) = Q_0(t_0, t, t - t_0)$ with $Q_0(t_0, t, \xi) = \xi Q(t_0, t, \xi)$. □

Theorem 2. *Let $\Gamma \in C^{1,\mu}$ and the generalized Cauchy kernel $Q(t; \xi)$ belong to $C^{\mu(2)}(\Gamma, \mathcal{H}_{-1})$.*
Then the integral operator $I : C^\mu(\Gamma) \to C^\mu(\overline{D})$ is bounded with the norm estimate $|I|_{\mathcal{L}} \le C|Q|_{C^{\mu(2)}}$.

Proof. Suppose that $\rho > 0$ is a small such that for any $t_0 \in \Gamma$ the arc $\Gamma_\rho(t_0) = \Gamma \cap \{|z - t_0| \le \rho\}$ is smooth and there exists the parametrization $\gamma : [-\rho; \rho] \to \Gamma_\rho(t_0)$ of class $C^{1,\mu}$ satisfying to conditions

$$|\gamma(s) - t_0| = |s|, \quad |s| \le \rho, \tag{6}$$

$$|\gamma'|_0 + [\gamma']_\mu \le M', \tag{7}$$

where $M' > 0$ does not depend on $t_0 \in \Gamma$.

Let $L(t_0)$ be the tangent to Γ at t_0. It is obviously that segment $L_\rho(t_0) = L(t_0) \cap \{|z - t_0| \le \rho\}$ has the parametric representation $l(s) = t_0 + \gamma'(0)s$, $|s| \le \rho$. By virtue of (7) we get the estimate

$$|\gamma(s) - l(s)| \le \int_0^s |\gamma'(\tau) - \gamma'(0)| d\tau \le M' s^{\mu + 1}. \tag{8}$$

Let us denote by $S_\rho(t_0)$ circular sector of radius ρ with top t_0 with angle θ for fixed $0 < \theta < \pi$. The symmetry axis of the sector is directed along the inner normal to Γ. Then for sufficiently small ρ we have the estimate

$$|z - t| \ge \delta(|z - t_0| + |t_0 - t|); \quad t \in \Gamma \cup L(t_0),\ z \in S_\rho(t_0), \tag{9}$$

where the constant $0 < \delta < 1$ does not depend on the point $t_0 \in \Gamma$.

Let us consider the function $\phi = I\varphi$ in the sector $S_\rho(t_0)$. For its partial derivatives Let $z \in S_\rho(t_0)$, $z = x_1 + ix_2$ and we have the expression

$$\frac{\partial \phi}{\partial x_j}(z) = \int_\Gamma P(t, t - z) d_1 t, \quad j = 1, 2,$$

with kernel

$$P(t, \xi) = \frac{\partial Q}{\partial \xi_j}(t, \xi)\varphi(t) \in C^{\mu(1)}(\Gamma, \mathcal{H}_{-2}).$$

Particularly, taking into account Lemma 1 (a)

$$|P(t,\xi)| \le M|Q|_{C^{0(1)}}|\varphi|_0|\xi|^{-2}, \tag{10}$$

$$|P(t,\xi) - Q_j(t_0,\xi)| \le M|Q|_{C^{\mu(1)}}|\varphi|_\mu|t-t_0|^\mu|\xi|^{-2}, \tag{11}$$

$$|P(t,\xi) - P(t,\eta)| \le M|Q|_{C^{0(2)}}|\varphi|_0(|\xi|^{-3}+|\eta|^{-3})|\xi-\eta|, \tag{12}$$

where constant $M > 0$ does not depend on Q and φ.

The function

$$h(z) = \int_{L(t_0)} Q_j(t_0, t-z)d_1t, \; z \notin L(t_0),$$

satisfies the condition $h[z + s\gamma_j(0)] = h(z), \; s \in \mathbb{R}$. By virtue of homogeneity we have

$$h[t_0 + s(z-t_0)] = s^{-1}h(z), \; s > 0.$$

Therefore, this function is identically equal to zero. So, the function $\partial\phi/\partial x_j$ can be represented as a sum $\psi_0 + \psi_1(z) + \chi$, where

$$\psi_0(z) = \int_\Gamma [Q_j(t,t-z) - Q_j(t_0,t-z)]d_1t,$$

$$\psi_1(z) = \left(\int_{\Gamma\setminus\Gamma_\rho(t_0)} - \int_{L(t_0)\setminus L_\rho(t_0)}\right) Q_j(t_0,t-z)]d_1t,$$

and

$$\chi(z) = \left(\int_{\Gamma_\rho(t_0)} - \int_{L_\rho(t_0)}\right) Q_j(t_0,t-z)d_1t.$$

By virtue of (9), (11) we have obvious inequality

$$|\psi_0(z)| \le M\delta^{-2}|Q|_{C^{\mu(1)}}|\varphi|_\mu K, \quad K = \int_\Gamma \frac{|t-t_0|^\mu d_1t}{(|t-t_0| + |t_0 - z|)^2},$$

Taking into account (6)

$$K \le \rho^{-2}\int_{\Gamma\setminus\Gamma_\rho(t_0)} |t-t_0|^\mu d_1t + M'\int_\rho^{-\rho} \frac{|s|^\mu ds}{(|s| + |t_0-z|)^2}.$$

The last integral is less than

$$|t_0 - z|^{\mu-1}\int_{\mathbb{R}} \frac{|s|^\mu ds}{(|s|+1)^2},$$

as a result we have the estimate

$$|\psi_0(z)| \le M_0|Q|_{C^{\mu(1)}}|\varphi|_\mu|t_0 - z|^{\mu-1}, \quad z \in S_\rho(t_0), \tag{13}$$

where constant M_0 does not depend on Q and φ.

For the function $\psi_1(z)$ by virtue of (9), (10) we can write

$$|\psi_1(z)| \le MK|Q|_{C^{0(1)}}|\varphi|_0, \quad K = \left(\int_{\Gamma\setminus\Gamma_\rho(t_0)} + \int_{L(t_0)\setminus L_\rho(t_0)}\right)|t-z|^{-2}d_1t,$$

$$K \le \rho^{-2}\int_{\Gamma\setminus\Gamma_\rho(t_0)} d_1t + \delta^{-2}\int_{|s|\ge\rho} |s|^{-2}d_1s.$$

Therefore we have the estimate

$$|\psi_1(z)| \le M_1|Q|_{C^{0(1)}}|\varphi|_0|t_0 - z|^{\mu-1}, \quad z \in S_\rho(t_0). \tag{14}$$

Consider the function $\chi(z)$. According to (6) we can write

$$\chi(z) = \int_{-\rho}^{\rho} [Q_j(t_0, \gamma(s) - z)|\gamma'(s)| - Q_j(t_0, l(s) - z)]ds = \chi_0(z) + \chi_1(z)$$

with

$$\chi_0(z) = \int_{-\rho}^{\rho} Q_j(t_0, \gamma(s) - z)[|\gamma'(s)| - 1]ds,$$

$$\chi_1(z) = \int_{-\rho}^{\rho} [Q_j(t_0, \gamma(s) - z) - Q_j(t_0, l(s) - z)]ds.$$

The function $\chi_0(z)$ satisfies the analogous estimate (14). We have for the function $\chi_1(z)$ according to (8), (12)

$$|\chi_1(z)| \le MM'|Q|_{C^{0(2)}}|\varphi|_0 K, \quad K = \int_{-\rho}^{\rho} (|\gamma(s) - z|^{-3} + |l(s) - z|^{-3})|s|^{1+\mu} ds.$$

By virtue of (6), (9) values $|\gamma(s) - z|$, $|l(s) - z|$ are both not less than $\delta(|s| + |z - t_0|)$ for $z \in S_\rho(t_0)$. So the integral

$$K \le 2\delta^{-3} \int_{-\rho}^{\rho} \frac{|s|^{\mu+1} ds}{(|s| + |z - t_0|)^3} \le 2\delta^{-3}|z - t_0|^{\mu-1} \int_{\mathbb{R}} \frac{|s|^\mu}{(|s| + 1)^3} ds.$$

Using inequalities (13), (14), we have the final estimate

$$|\frac{\phi\partial}{\partial x_j}(z)| \le M|Q|_{C^{\mu(2)}}|\varphi|_\mu |z - t_0|^{\mu-1}, \quad z \in S_\rho(t_0),$$

where M does not depend on Q and φ.

The distance from the point $z \in D$ to Γ is denoted by $d(z, \Gamma)$. If $d(z, \Gamma) \le \rho$ and $t_0 \in \Gamma$ such that $d(z, \Gamma) = |z - t_0|$, then $z \in S_\rho(t_0)$. Therefore the last inequality leads to the estimate

$$|\psi(z)| \le C|\varphi|_{\mu,\Gamma} d^{\mu-1}(z, \Gamma),$$

for any $z \in D$, $d(z, \Gamma) \le \rho$. Since $\psi = \partial\phi/\partial x_j$, we come to the validity of the theorem on the basis of Lemma 1 from [7]. □

Corollary 3. *Let $\Gamma \in C^{1,\nu}$, let the kernel $Q(u, t, \xi)$ depend on a parameter $u \in G \subseteq \mathbb{R}^k$ and belong to $C^{\nu(2)}(G\times\Gamma, \mathcal{H}_{-1})$. Let $\varphi \in C^\mu(\Gamma)$, $\mu < \nu < 1$.*

Then the corresponding function

$$\phi(u, z) = \int_\Gamma Q(u, t, t - z)\varphi(t) d_1 t,$$

belongs to $C^\mu(G \times \overline{D})$ with corresponding norm estimate.

Proof. Let $z, z_1, z_2 \in D$, $u, u_1, u_2 \in G$ and $z_1 \ne z_2$, $u_1 \ne u_2$. Then by Theorem 2 we have the estimate

$$|\phi(u, z)| + |\phi(u, z_1) - \phi(u, z_2)|z_1 - z_2|^{-\mu} \le M_1|Q|_{C^{\mu(2)}}|\varphi|_\mu, \tag{15}$$

where $M_1 > 0$ doesn't depend on Q and φ.

Let us write

$$[\phi(u_1, z) - \phi(u_2, z)]|u_1 - u_2]^{-\mu} = \int_\Gamma \widetilde{Q}(t, t - z)\varphi(t) d_1 t,$$

with the kernel

$$\widetilde{Q}(t, \xi) = [Q(u_1, t, \xi) - Q(u_2, t, \xi)]|u_1 - u_2]^{-\mu}.$$

It follows from the next Lemma 4 that $\widetilde{Q}(t,\xi) \in C^{\varepsilon(2)}(\Gamma)$ with $0 < \varepsilon \le \nu - \mu$ and the corresponding estimate

$$|\widetilde{Q}|_{C^{\varepsilon(2)}(\Gamma)} \le M|Q|_{C^{\varepsilon(2)}(G\times\Gamma}$$

holds. Applying Theorem 2 with respect to $\varepsilon = \min(\mu, \nu-\mu)$ we receive the estimate

$$|\phi(u_1,z) - \phi(u_2,z)||u_1-u_2]^{-\mu} \le M_2|Q|_{C^{\varepsilon(2)}}|\varphi|_\varepsilon.$$

Together with (15) it completes the proof. □

Lemma 4. *Let $G \subseteq \mathbb{R}^k$, a function $\psi(x,y) \in C^\nu(G\times G)$ and $\psi(x,y) = 0$ for $x = y$.*

Then the function $\psi_0(x,y) = |x-y|^{\mu-\nu}\psi(x,y)$, where $0 < \mu < \nu$, belongs to $C^\mu(G\times G)$ and

$$[\psi_0]_\mu \le 6[\psi]_\nu. \tag{16}$$

Proof. First of all note that

$$|\psi(x,y)| = |\psi(x,y) - \psi(x,x)| \le [\psi]_\nu|x-y|^\nu$$

and therefore $\psi_0(x,y) \to 0$ as $x - y \to 0$.

For fixed $x_0 \in G$ consider the functions $\varphi(x) = \psi(x,x_0)$, $\varphi_0(x) = \psi_0(x,x_0)$ of variable x. These functions are linked by the corresponding relation $\varphi_0(x) = |x-x_0|^{\mu-\nu}\varphi(x)$. We prove that

$$[\varphi_0]_\mu \le 3[\varphi]_\nu. \tag{17}$$

It is sufficient to establish this estimate under assumption $x_0 = 0 \in G$. Let $x, y \in G$ and for definiteness $|y| \le |x|$. Putting $\varepsilon = \nu - \mu$ we have:

$$|\varphi_0(x) - \varphi_0(y)| \le |\varphi(x)-\varphi(y)||x|^{-\varepsilon} + |\varphi(y)|\,||x|^{-\varepsilon} - |y|^{-\varepsilon}|.$$

Since $|\varphi(y)| \le [\varphi]_\nu|y|^{\mu+\varepsilon}$ we receive

$$\frac{|\varphi_0(x)-\varphi_0(y)|}{|x-y|^\mu} \le [\varphi]_\nu\Delta, \quad \Delta = \frac{|x-y|^\varepsilon}{|x|^\varepsilon} + \frac{(|x|^\varepsilon - |y|^\varepsilon)|y|^\mu}{|x-y|^\mu|x|^\varepsilon}.$$

It is obviously,

$$\Delta \le \frac{(|x|+|y|)^\varepsilon}{|x|^\varepsilon} + \frac{(|x|^\varepsilon-|y|^\varepsilon)|y|^\mu}{(|x|-|y|)^\mu|x|^\varepsilon} = (1+t)^\varepsilon + t\frac{1-t^\varepsilon}{(1-t)^\varepsilon},$$

where $t = |y|/|x| \le 1$. Since $1 - t^\varepsilon \le 1 - t \le (1-t)^\mu$, it follows $\Delta \le 3$ and hence (17) is valid.

Now it easily to prove (16). We write

$$|\psi_0(x,y) - \psi_0(x',y')| \le |\psi_0(x,y) - \psi_0(x',y)| + |\psi_0(x',y) - \psi_0(x',y')|$$

and by virtue of (17) we obtain

$$|\psi_0(x,y) - \psi_0(x',y')| \le 3[\psi]_\nu(|x-x'|^\mu + |y-y'|^\mu) \le 6[\psi]_\nu(|x-x'|^2 + |y-y'|^2)^{\mu/2}.$$

□

Corollary 5. *Let $\Gamma \in C^{1,\mu}$, the generalized Cauchy kernel $Q(t;\xi)$ belong to $C^{\mu(2)}(\Gamma, \mathcal{H}_{-1})$ and*

$$Q[t, e(t)] = 0, \;\; t \in \Gamma, \tag{18}$$

where $e(t)$ is the unit tangent vector to Γ at the point t.

Then the operator I is bounded $C(\Gamma) \to C(\overline{D})$.

Proof. With the help of (18) analogously to the proof of Theorem 2 we can establish that

$$M = \sup_{z\in D} \int_\Gamma |Q(t, t - z)| d_1 t < \infty$$

and hence

$$\sup_{z\in D} |(I\varphi)(z)| \le M|\varphi|_0, \quad \varphi \in C(\Gamma). \tag{19}$$

Let $\varphi_n \in C^\mu(\Gamma)$ and $|\varphi_n - \varphi|_0 \to 0$ as $n \to \infty$. By Theorem 2 the functions $I\varphi_n \in C^\mu(\overline{D})$. By virtue of (19) it follows that $I\varphi \in C(\overline{D})$ and hence the operator I is bounded in $C(\Gamma) \to C(\overline{D})$. □

Example 6. The double layer potential for Laplace operator is defined by the kernel

$$Q(t, \xi) = \frac{1}{\pi} \frac{\xi_1 n_1(t) + \xi_2 n_2(t)}{|\xi|^2},$$

where $n(t) \in \mathbb{C}$ is the unit outward normal, satisfies (18). It is well known that the operator I is bounded $C(\Gamma) \to C(\overline{D})$ for this case.

The question of boundary values $(I\varphi)^+(t_0) = \lim(I\varphi)(z)$ as $z \to t_0,\ z \in D$, of the function $I\varphi$ is closely related to the singular integral

$$(I^*\varphi)(t_0) = \int_\Gamma Q(t_0, t - t_0)\varphi(t) d_1 t, \quad t_0 \in \Gamma.$$

If $C^{\mu(2)}(\Gamma, \mathcal{H}_{-1})$ then by Lemma 1 (c) we can write

$$Q(t_0, t; t - t_0) = \frac{q(t_0, t)}{t - t_0}, \quad q \in C^\nu(\Gamma \times \Gamma)$$

and thus the singular integral $(I^*\varphi)(t_0)$ exists.

Let the unit tangent vector $e(t_0)$ to Γ at the point t_0 be oriented positively with respect to the domain D and $L(t_0)$ be the correspondence tangent line, which is oriented by $e(t_0)$. Let us consider the integral

$$\sigma(t_0) = \int_{L(t_0)} Q(t_0, t - z) d_1 t, \quad z \in G^+(t_0), \tag{20}$$

where the half-plane $G^+(t_0)$ is on the left from $L(t_0)$. This integral is singular with respect to ∞ and does not depend on point $z \in G^+$. It follows from the formula

$$\int_{L(t_0)} \frac{\partial Q}{\partial x_j}(t_0, t - z) d_1 t = 0, \quad z \in G^+,\ j = 1, 2,$$

which has already used in the proof of Theorem 2.

Theorem 7. *Let $\Gamma \in C^{1,\nu}$ and the generalized Cauchy kernel $Q(t; \xi) \in C^{\nu(2)}(\Gamma, \mathcal{H}_{-1})$. Then*

$$\sigma \in C^\mu(\Gamma), \quad 0 < \mu < \nu, \tag{21}$$

and for $\varphi \in C^\mu(\Gamma)$ the following formula

$$(I\varphi)^+(t_0) = \sigma(t_0)\varphi(t_0) + (I^*\varphi)(t_0), \quad t_0 \in \Gamma, \tag{22}$$

is valid. Particularly, the singular operator I^* is bounded in $C^\mu(\Gamma)$.

Proof. We can put $z = t_0 + ie(t_0) \in G^+(t_0)$ in (21). Then

$$\sigma(t_0) = \int_{L(t_0)} Q[t_0, t - t_0 - ie(t_0)]d_1t = \int_{\mathbb{R}} Q[t_0, (s - i)e(t_0)]ds =$$

$$= \int_{-1}^{1} Q[t_0, (s - i)e(t_0)]ds + \int_{-1}^{1} (Q[t_0, (1 - is)e(t_0)] - Q[t_0, e(t_0)])\frac{ds}{s}.$$

By Lemma 4 we can write

$$\sigma(t_0) = \int_{-1}^{1} q(t_0, s)\frac{|s|^{\nu-\mu}ds}{s}$$

with some function $q \in C^{\mu}(\Gamma \times [-1, 1])$, that proves the first part of the theorem.

Using notions from the proof of Theorem 2 it is easi to see that

$$\int_{\Gamma}[Q(t, t - z)\varphi(t) - Q(t_0, t - z)\varphi(t_0)]d_1t \to \int_{\Gamma}[Q(t, t - t_0)\varphi(t) - Q(t_0, t - t_0)\varphi(t_0)]d_1t$$

and

$$\left(\int_{\Gamma_\rho(t_0)} - \int_{L_\rho(t_0)}\right)Q(t_0, t - z)d_1t \to \left(\int_{\Gamma_\rho(t_0)} - \int_{L_\rho(t_0)}\right)Q(t_0, t - t_0)d_1t$$

as $z \to t_0,\ z \in S_\rho(t_0)$. So it is sufficiently to prove the equality

$$\lim_{\varepsilon\to 0} \int_{L_\rho(t_0)} Q[t_0, t - t_0 - i\varepsilon e(t_0)]d_1t = \sigma(t_0),$$

where we take into account that

$$\int_{L_\rho(t_0)} Q(t_0, t - t_0)d_1t = 0.$$

Since

$$\int_{L_\rho(t_0)} Q(t_0, t - z)d_1t = \int_{|s|\le\rho/\varepsilon} Q[t_0, (1 - i)se(t_0)]ds,$$

this equality is obvious. □

Let two generalized Cauchy kernels $Q_j(t, \xi),\ j = 1, 2$, are given. The expression

$$Q(t; \xi, \eta) = Q_1(t, \xi)\eta_1 + Q_2(t, \xi)\eta_2, \quad \eta = \eta_1 + i\eta_2 \in \mathbb{C},$$

is called the Cauchy kernel if the function $Q(t; \xi, \xi)$ does not depend on ξ. For example, this condition is satisfied for the case of the classical Cauchy kernel

$$Q(\xi, \eta) = \frac{\eta}{2\pi i\xi}.$$

Let us consider the Cauchy type integral

$$(I\varphi)(z) = \int_{\Gamma} Q(t; t - z, dt)\varphi(t), \quad z \in D, \tag{23}$$

where $dt = dt_1 + idt_2$ and contour Γ is oriented.

We prove the following result which consists with the famous theorem (see monograph by N.I. Muskhelishvili) for the classical Cauchy kernel.

Theorem 8. *Let $\Gamma = \partial D$ be a smooth contour oriented positively with respect to D and the Cauchy kernel $Q(t;\xi,\eta) \in C^{\mu(1)}(\Gamma, \mathcal{H}_{-1})$. Then the operator I defined by (23) is bounded $C^\mu(\Gamma) \to C^\mu(\overline{D})$ with a corresponding norm estimate. Nevertheless the formula (22) for boundary values holds with the coefficient*

$$\sigma(t_0) = \frac{1}{2}\int_{\mathbb{T}} Q(t_0;\xi,d\xi), \tag{24}$$

where $\mathbb{T}$ denotes the unit circumference, oriented counterclockwise.

Particularly the singular operator I^* is bounded in $C^\mu(\Gamma)$.

Proof. For fixed $t \in \Gamma$ the differential form $Q(t;\xi,d\xi) = Q_1(\xi)d\xi_1 + Q_2(\xi)d\xi_2$ is closed i.e.

$$\frac{\partial Q_2}{\partial \xi_1} = \frac{\partial Q_1}{\partial \xi_2}. \tag{25}$$

Indeed by definition we have equalities

$$Q_j(\xi) + \frac{\partial Q_1}{\partial \xi_j}\xi_1 + \frac{\partial Q_2}{\partial \xi_j}\xi_2 = 0,\ j = 1,2,$$

and the Euler identity for homogeneous functions.

$$Q_j(\xi) = \frac{\partial Q_j}{\partial \xi_1}\xi_1 + \frac{\partial Q_j}{\partial \xi_2}\xi_2,\ j = 1,2.$$

It implies (25) from these equalities at once.

Let $z_0 \in D$ and $\varepsilon > 0$ such that $\{|z - z_0| \le \varepsilon\} \subseteq D$. By virtue of (24) and (25) we can write

$$\int_\Gamma Q(t_0, t - z_0, dt) = \int_{|t-z_0|=\varepsilon} Q(t_0, t - z_0, dt) = 2\sigma(t_0). \tag{26}$$

It is established analogously the following relation for the singular integral

$$\int_\Gamma Q(t_0, t - t_0, dt) = \sigma(t_0). \tag{27}$$

From (26) it follows that

$$\int_\Gamma \frac{\partial Q}{\partial x_j}(t_0, t - z, dt) = 0,\ z \in D,$$

and particularly the partial derivatives of $\phi = I\varphi$ we can be represented in the form

$$\frac{\partial \phi}{\partial x_j}(z) = \int_\Gamma \left[\frac{\partial Q}{\partial \xi_j}(t, t - z, dt)\varphi(t) - \frac{\partial Q}{\partial \xi_j}(t_0, t - z, dt)\varphi(t_0)\right],\ j = 1,2.$$

So analogously to the proof of Theorem 2 we obtain the estimate (13) and hence the operator I is bounded in C^μ.

Let us consider formulas (22), (24). According to the proof of Theorem 7 it is sufficient to prove this formula for $Q(t_0, \xi, \eta)$ and $\varphi = 1$. In this case it follows from (26), (27) immediately.

Notice that (24) coincides with the corresponding formula (20) defined by

$$\sigma(t_0) = \int_{L(t_0)} Q(t_0, t - z, dt), \quad z \in G^+(t_0). \tag{28}$$

It is sufficient to apply the form $Q(t_0, t - z, dt)$ in the domain $G_n = \{|z - t_0| < n\} \cap G^-(t_0)$, where $n = 1, 2, \dots$ and $G^-(t_0)$ is the half-plane on the left f $L(t_0)$. Then

$$\int_{\partial G_n} Q(t_0, t - z, dt) = \left(\int_{L_n} - \int_{\Gamma_n}\right) Q(t_0, t - z, dt) = 0,$$

where $L_n = \{|z - t_0| < n\} \cap L(t_0)$ and Γ_n is the correspondence semi-circumference. It remains to note that the integral

$$\int_{\Gamma_n} Q(t_0, t - z, dt)$$

coincides with (28). □

It is easy to prove the following differentiation formula of the function $\phi = I\varphi$, defined by (23).

Lemma 9. *Let $\Gamma \in C^{1,\mu}$, $\varphi \in C^1(\Gamma)$, the Cauchy kernel Q belong to $C^{1,\mu(1)}(\Gamma, \mathcal{H}_{-1})$ and $Q_0(t;\xi,\eta) = Q_t'(t;\xi,\eta)$, where prime denotes differentiation with respect to arc length parameter.*

Then for function $\phi = I\varphi$ with density $\varphi \in C^1(\Gamma)$ the following differentiation formula holds:

$$\left(\eta_1 \frac{\partial \phi}{\partial x_1} + \eta_2 \frac{\partial \phi}{\partial x_2}\right)(z) = \int_\Gamma Q_0(t, t - z, \eta)\varphi(t) d_1 t + \int_\Gamma Q(t, t - z, \eta)\varphi'(t) d_1 t.$$

Obviously, the function Q_0 in this lemma is in fact the generalized Cauchy kernel. Therefore together with Theorems 2 and 8 we can obtain the following result.

Theorem 10. *Let a smooth contour $\Gamma \in C^{1,\mu}$ be oriented positively with respect to D and the Cauchy kernel $Q(t;\xi,\eta) \in C^{1,\mu(2)}(\Gamma, \mathcal{H}_{-1})$. Then the operator I is bounded $C^{1,\mu}(\Gamma) \to C^{1,\mu}(\overline{D})$ with a corresponding norm estimate.*

Let us apply these results to the singular Cauchy integral

$$(I^*\varphi)(t_0) = \int_\Gamma Q(t; t - t_0, dt)\varphi(t), \quad t_0 \in \Gamma.$$

Corollary 11. *Under the conditions of Theorem 10 the singular operator I^* is bounded in $C^{1,\mu}(\Gamma)$, with the corresponding norm estimate. Wherein the derivative of function $\psi = I^*\varphi$ is given by the formula*

$$\psi'(t_0) = \int_\Gamma Q_0[t, t - t_0, e(t_0)]\varphi(t) d_1 t + \int_\Gamma Q[t, t - t_0, e(t_0)]\varphi'(t) d_1 t,$$

where $Q_0 = Q_t'$.

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