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PSEUDO-DIFFERENTIAL EQUATIONS AND CONICAL POTENTIALS: 2-DIMENSIONAL CASE

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Abstract. We consider two-dimensional elliptic pseudo-differential equation in a plane sector. Using a special representation for an elliptic symbol and the formula for a general solution we study the Dirichlet problem for such equation. This problem was reduced to a system of linear integral equations and then after some transformations to a system of linear algebraic equations. The unique solvability for the Dirichlet problem was proved in Sobolev–Slobodetskii spaces and a priori estimate for the solution is given.

Keywords: pseudo-differential equation, wave factorization, Dirichlet problem, system of linear integral equations.

Mathematics Subject Classification: 35S15, 45A05.

1. INTRODUCTION

The theory of pseudo-differential equations and boundary value problems in domains with a smooth boundary is well-known [4], but for non-smooth boundaries there are a lot of problems. Using a local principle one can say that main difficulty is a studying invertibility of a model equation in so-called canonical domain. Such canonical domain is a half-space for a domain with a smooth boundary, and a model equation is considered in a half-space. But for a simplest domain with a non-smooth boundary the model domain is a cone. The author supposes to consider a cone like a canonical domain serving the theory of pseudo-differential equations on manifolds with non-smooth boundary [15]. Such approach is based on a special representation for an elliptic symbol. Existence of such special wave factorization for symbols of elliptic pseudo differential equations has permitted to obtain full solvability picture for model pseudo differential equations in two-dimensional case [15, 16]. Recently the author has found that there is another way to develop the theory [19–21].

The paper is devoted to verifying these construction for two-dimensional case [14] and may be the consideration in more details will allow to transfer the main results on spaces of an arbitrary dimension.

Some papers are related to this studying [1, 12] (for the Laplace equation) and [3] (for a general differential equations), there are very similar methods (the Mellin transform) and formulas.

There are other approaches for studying the problem. I wrote many times on these versions to studying a solvability for pseudo-differential equations in domains with conical points and wedges, but now I would like to speak on main difference of my papers from other authors (see, for example, [2, 7–11] and many others). Some authors consider pseudo-differential operators on compact manifolds without a boundary on which exist certain singularities, and they need a special local definition of a pseudo-differential operator in a neighborhood of the singular point, this is another problem different from my one. Other authors consider given boundary value problems on manifolds with a singular boundary, and they consider a conical point as a direct product $B \times (0, +\infty)$, where B is a base of a cone with a smooth boundary, further they use the Mellin transform and obtain a certain operator pencil. The author's approach is based on two other principles, namely the local principle and existence of the special factorization for an elliptic symbol of pseudo-differential operator at a singular point, this approach in general is described in papers [18, 22].

2. PRELIMINARIES

A pseudo differential operator A with symbol $A(\xi), \xi \in \mathbb{R}^m$, is defined by the formula

$$(Au)(x) = \int\limits_{\mathbb{R}^m} A(\xi)\tilde{u}(\xi)e^{ix\xi}d\xi, \tag{2.1}$$

where $\tilde{u}$ denotes the Fourier transform.

This is a model operator. Generally speaking one considers pseudo differential operators depending on the space variable x. An operator with symbol $A(x, \xi)$ is defined like (2.1) with the help of the formula

$$u(x) \longmapsto \int\limits_{\mathbb{R}^m} A(x, \xi)\tilde{u}(\xi)e^{ix\xi}d\xi$$

by "freezing" the space variable x.

Here we will consider the class of symbols independent of the space variable x and satisfying the following condition: there are two positive constants c_1, c_2, such that

$$c_1 \leq |A(\xi)(1 + |\xi|)^{-\alpha}| \leq c_2, \quad \forall \xi \in \mathbb{R}^m. \tag{2.2}$$

The number $\alpha \in \mathbb{R}$ we call the order of pseudo-differential operator A.

We will denote P_α the symbol class satisfying the condition (2.2).

Let us define the Sobolev–Slobodetskii functional space $H^s(\mathbb{R}^m)$ as the Hilbert space of distributions [1] with the norm

$$\|u\|_s^2 = \int\limits_{\mathbb{R}^m} |\tilde{u}(\xi)|^2 (1+|\xi|)^{2s} d\xi.$$

It is well-known that an operator from P_α is a linear bounded operator acting from $H^s(\mathbb{R}^m)$ into $H^{s-\alpha}(\mathbb{R}^m)$ [1]. Everywhere below we will use symbol $\tilde{H}^s(M)$ to denote the Fourier image of the space $H^s(M)$.

Now we will study a solvability of pseudo differential equations

$$(Au)(x) = f(x), \quad x \in C_+^a, \tag{2.3}$$

in the space $H^s(C_+^a)$, where C_+^a is m-dimensional cone

$$C_+^a = \{x \in \mathbb{R}^m : x = (x_1, \ldots, x_{m-1}, x_m), x_m > a|x'|, a > 0\},$$
$$x' = (x_1, \ldots, x_{m-1}).$$

By definition, the space $H^s(C_+^a)$ consists of distributions from $H^s(\mathbb{R}^m)$, which support belongs to C_+^a. The norm in the space $H^s(C_+^a)$ is induced by the norm from $H^s(\mathbb{R}^m)$. The right-hand side f is chosen from the space $H_0^{s-\alpha}(C_+^a)$; by definition the space $H_0^s(C_+^a)$ is a space of distributions on C_+^a, admitting a continuation to $H^s(\mathbb{R}^m)$. The norm in the space $H_0^s(C_+^a)$ is defined

$$\|f\|_s^+ = \inf \|lf\|_s,$$

where the *infimum* is taken over all continuations lf on the whole $\mathbb{R}^m$.

The symbol $\overset{*}{C}{}_+^a$ denotes a conjugate cone for C_+^a:

$$\overset{*}{C}{}_+^a = \{x \in \mathbb{R}^m : x = (x', x_m),\ ax_m > |x'|\},$$

$C_-^a \equiv -C_+^a,\ T(C_+^a)$ denotes the radial tube domain over the cone C_+^a, i.e. the domain in a complex space $\mathbb{C}^m$ of type $\mathbb{R}^m + iC_+^a$.

Further, let us define a special multi-dimensional singular integral by the formula

$$(G_m u)(x) = \lim_{\tau \to 0+} \int\limits_{\mathbb{R}^m} \frac{u(y', y_m) dy' dy_m}{(|x'-y'|^2 - a^2(x_m - y_m + i\tau)^2)^{m/2}}.$$

To describe the solvability picture for the equation (2.3) we will introduce the following definition.

Definition 2.1. By wave factorization for the symbol $A(\xi)$, satisfying the condition (2.2), we mean its representation in the form

$$A(\xi) = A_{\neq}(\xi) A_{=}(\xi),$$

where the factors $A_{\neq}(\xi), A_{=}(\xi)$ satisfy the following conditions:

1) $A_{\neq}(\xi), A_{=}(\xi)$ are defined for all admissible values $\xi \in \mathbb{R}^m$, without may be, the points $\{\xi \in \mathbb{R}^m : |\xi'|^2 = a^2\xi_m^2\}$;
2) $A_{\neq}(\xi), A_{=}(\xi)$ admit an analytical continuation into radial tube domains $T(\overset{*}{C}{}^a_+)$, $T(\overset{*}{C}{}^a_-)$, respectively, with estimates

$$|A_{\neq}^{\pm 1}(\xi + i\tau)| \leq c_1(1 + |\xi| + |\tau|)^{\pm æ},$$
$$|A_{=}^{\pm 1}(\xi - i\tau)| \leq c_2(1 + |\xi| + |\tau|)^{\pm(\alpha - æ)}, \quad \forall \tau \in \overset{*}{C}{}^a_+ .$$

The number $æ \in \mathbb{R}$ is called index of wave factorization.

Everywhere below we will suppose that the wave factorization mentioned exists.

3. STUDYING TRANSMUTATION OPERATORS

Let us denote T_a the transformation $\mathbb{R}^m \longrightarrow \mathbb{R}^m$ of the following type

$$\begin{cases} t_1 = x_1, \\ \dots\dots\dots\dots \\ t_{m-1} = x_{m-1}, \\ t_m = x_m - a|x'|. \end{cases}$$

(obviously, it one-to-one transforms ∂C^a_+ into hyperplane $x_m = 0$).

Explicit calculations give simple answer:

$$FT_a u = V_a \tilde{u},$$

where F is the Fourier transform

$$\tilde{u}(\xi) = \int\limits_{\mathbb{R}^m} e^{ix\cdot\xi} dx,$$

and V_a is a special operator (roughly speaking it is a pseudo-differential operator with symbol $e^{-ia|\xi'|\xi_m}$), and further one can construct the general solution for our pseudo differential equation (2.3).

Indeed, it follows from the relations

$$(FT_au)(\xi) = \int\limits_{\mathbb{R}^m} e^{ix\cdot\xi} u(x_1, \dots, x_{m-1}, x_m - a|x'|)dx$$
$$= \int\limits_{\mathbb{R}^m} e^{iy'\xi'} e^{i(y_m + a|y'|)\xi_m} u(y_1, \dots, y_{m-1}, y_m)dy$$
$$= \int\limits_{\mathbb{R}^{m-1}} e^{ia|y'|\xi_m} e^{iy'\xi'} \hat{u}(y_1, \dots, y_{m-1}, \xi_m)dy',$$

where $\hat{u}$ denotes the Fourier transform on the last variable, and the Jacobian of T_a is equal to 1 everywhere without origin and bounded. According to the properties of Fourier transform the product of two functions becomes their convolution. Roughly speaking the operator V_a is a convolution for $m-1$ variables, and a multiplier for the last variable.

Moreover, the following relations are valid [19].

Lemma 3.1. *Operators T_a and V_a have the following properties:*

1. $V_a = FT_aF^{-1}$,
2. $T_a^{-1} = T_{-a}$,
3. $V_a^{-1} = V_{-a}$.

3.1. 2-DIMENSIONAL CASE

Let us consider the case $m = 2$ in details. So we have

$$(FT_au)(\xi) = \int\limits_{-\infty}^{+\infty} e^{ia|y_1|\xi_2} e^{iy_1\xi_1} \hat{u}(y_1, \xi_2)dy_1$$
$$= \int\limits_{-\infty}^{+\infty} \chi_+(y_1)e^{iay_1\xi_2} e^{iy_1\xi_1} \hat{u}(y_1, \xi_2)dy_1 + \int\limits_{-\infty}^{+\infty} \chi_-(y_1)e^{-iay_1\xi_2} e^{iy_1\xi_1} \hat{u}(y_1, \xi_2)dy_1$$
$$= \int\limits_{-\infty}^{+\infty} \chi_+(y_1)e^{iy_1(a\xi_2+\xi_1)} \hat{u}(y_1, \xi_2)dy_1 + \int\limits_{-\infty}^{+\infty} \chi_-(y_1)e^{iy_1(-a\xi_2+\xi_1)} \hat{u}(y_1, \xi_2)dy_1.$$

The last two summands are the Fourier transforms of functions

$$\chi_+(y_1)e^{iy_1(a\xi_2+\xi_1)}\hat{u}(y_1, \xi_2), \quad \chi_-(y_1)e^{iy_1(-a\xi_2+\xi_1)}\hat{u}(y_1, \xi_2)$$

on the first variable y_1 respectively, so we can use the following properties [4] (these are Sokhotskii formulas [5, 6] and we write them for a one variable)

$$\int\limits_{-\infty}^{+\infty} \chi_+(x)e^{ix\xi}u(x)dx = \frac{1}{2}\tilde{u}(\xi) + v.p.\frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{\tilde{u}(\eta)d\eta}{\xi - \eta},$$

$$\int_{-\infty}^{+\infty}\chi_-(x)e^{ix\xi}u(x)dx = \frac{1}{2}\tilde{u}(\xi) - v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{u}(\eta)d\eta}{\xi-\eta}.$$

Taking into account these properties we have

$$(FT_au)(\xi) = \frac{\tilde{u}(a\xi_2+\xi_1,\xi_2)+\tilde{u}(-a\xi_2+\xi_1,\xi_2)}{2}$$
$$+ v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{u}(\eta,\xi_2)d\eta}{a\xi_2+\xi_1-\eta} - v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{u}(\eta,\xi_2)d\eta}{-a\xi_2+\xi_1-\eta} \equiv (V_a\tilde{u})(\xi).$$

4. A GENERAL SOLUTION

Here we will consider the equation (2.3) for the case $æ - s = n+\delta, n\in\mathbb{N}, |\delta|<1/2$, only. The following result is valid [19].

Theorem 4.1. *General solution of the equation* (2.3) *in Fourier image is given by the formula*

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)Q_n(\xi)G_mQ_n^{-1}(\xi)A_{=}^{-1}(\xi)\widetilde{lf}(\xi)$$
$$+ A_{\neq}^{-1}(\xi)V_{-a}F\left(\sum_{k=0}^{n-1}c_k(x')\delta^{(k)}(x_m)\right),$$

where $c_k(x')\in H^{s_k}(\mathbb{R}^{m-1})$ *are arbitrary functions,* $s_k = s - æ + k + 1/2$, $k=0,1,2,\dots,n-1$, lf *is an arbitrary continuation of* f *on* $H^{s-\alpha}(\mathbb{R}^m)$, δ *is the Dirac mass-function,* $Q_n(\xi)$ *is an arbitrary polynomial satisfying the condition* (2.2) *for* $\alpha = n$.

Using this representation one can suggest different statements of boundary value problems for the equation (2.3).

5. BOUNDARY VALUE PROBLEMS

Let us consider very simple case, when $f\equiv 0,\ a=1,\ n=1$. Then the formula above takes the form

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi_1).$$

The main problem for this situation is the following: What kind of conditions do we need for $\tilde{u}(\xi)$ to determine uniquely $\tilde{c}_0(\xi_1)$? According to above calculations we have

$$\tilde{u}(\xi) = \frac{\tilde{c}_0(\xi_2+\xi_1)+\tilde{c}_0(-\xi_2+\xi_1)}{2A_{\neq}(\xi_1,\xi_2)}$$
$$+ A_{\neq}^{-1}(\xi_1,\xi_2)\left(v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{c}_0(\eta)d\eta}{\xi_2+\xi_1-\eta} - v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{c}_0(\eta)d\eta}{-\xi_2+\xi_1-\eta}\right).$$

Let us make the change of variables

$$\begin{cases} t_1 = \xi_2 + \xi_1, \\ t_2 = -\xi_2 + \xi_1, \end{cases}$$

and denote

$$a_{\neq}(t_1, t_2) \equiv A_{\neq}\left(\frac{t_1 + t_2}{2}, \frac{t_1 - t_2}{2}\right).$$

Then we can rewrite

$$\tilde{U}(t_1, t_2) = \frac{\tilde{c}_0(t_1) + \tilde{c}_0(t_2)}{2a_{\neq}(t_1, t_2)} + a_{\neq}^{-1}(t_1, t_2)\left(v.p.\frac{i}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\tilde{c}_0(\eta)d\eta}{t_1 - \eta} - v.p.\frac{i}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\tilde{c}_0(\eta)d\eta}{t_2 - \eta}\right).$$

Let us introduce

$$v.p.\frac{i}{\pi}\int\limits_{-\infty}^{+\infty}\frac{\tilde{c}_0(\eta)d\eta}{t_1 - \eta} \equiv \tilde{d}_0(t_1), \quad v.p.\frac{i}{\pi}\int\limits_{-\infty}^{+\infty}\frac{\tilde{c}_0(\eta)d\eta}{t_2 - \eta} \equiv \tilde{d}_0(t_2).$$

Then we have

$$\tilde{U}(t_1, t_2) = \frac{\tilde{c}_0(t_1) + \tilde{c}_0(t_2) + \tilde{d}_0(t_1) - \tilde{d}_0(t_2)}{2a_{\neq}(t_1, t_2)} \equiv \frac{\tilde{c}(t_1) + \tilde{d}(t_2)}{2a_{\neq}(t_1, t_2)},$$

where we put $\tilde{c}(t_1) \equiv \tilde{c}_0(t_1) + \tilde{d}_0(t_1), \tilde{d}(t_2) \equiv \tilde{c}_0(t_2) - \tilde{d}_0(t_2)$.

Assuming that we know the following two integrals

$$\int\limits_{-\infty}^{+\infty}\tilde{U}(t_1, t_2)dt_1 = \tilde{g}_1(t_2), \qquad \int\limits_{-\infty}^{+\infty}\tilde{U}(t_1, t_2)dt_2 = \tilde{g}_2(t_1)$$

and integrating the last quality first on t_1 and then on t_2 we can obtain the following 2×2-system of linear integral equations with respect to two unknown functions $\tilde{c}, \tilde{d}$:

$$\begin{cases} \int\limits_{-\infty}^{+\infty} K(t_1, t_2)\tilde{c}(t_1)dt_1 + b_1(t_2)\tilde{d}(t_2) = \tilde{g}_1(t_2), \\ b_2(t_1)\tilde{c}(t_1) + \int\limits_{-\infty}^{+\infty} K(t_1, t_2)\tilde{d}(t_2)dt_2 = \tilde{g}_2(t_1), \end{cases} \tag{5.1}$$

where we use the following notations

$$K(t_1, t_2) = (2a_{\neq}(t_1, t_2))^{-1}, \; b_1(t_2) = \int\limits_{-\infty}^{+\infty} K(t_1, t_2)dt_1, \; b_2(t_1) = \int\limits_{-\infty}^{+\infty} K(t_1, t_2)dt_2.$$

5.1. THE DIRICHLET PROBLEM

We consider here the following problem in the 2-dimensional sector

$$C^1_+ = \{x \in \mathbb{R}^2 : x = (x_1, x_2), x_2 > |x_1|\}.$$

Definition 5.1. Let $\overset{\circ}{E}_\alpha$ be a class of symbols which are homogeneous of order α and satisfy the condition

$$c_1|\xi|^\alpha \leq |A(\xi)| \leq c_2|\xi|^\alpha.$$

Let A be a pseudo-differential operator with the symbol $A(\xi_1, \xi_2) \in \overset{\circ}{E}_\alpha$ and set the following problem: find the function $u \in H^s(C^1_+)$ such that

$$(Au)(x) = 0, \quad x \in C^1_+, \tag{5.2}$$

$$u_{|x_1+x_2=0} = v_1, \quad u_{|-x_1+x_2=0} = v_2, \tag{5.3}$$

where v_1, v_2 are given functions on angle sides from $H^{s-1/2}(\mathbb{R}_+)$.

Theorem 5.2. *Let* $æ - s = 1 + \delta, |\delta| < 1/2$. *Then the problem* (5.2), (5.3) *has a unique solution* $u \in H^s(C^1_+)$ *iff the system* (5.1) *of linear integral equations is uniquely solvable.*

5.2. THE HOMOGENEOUS WAVE FACTORIZATION

For this case we need a special type of the wave factorization, i.e. the so-called homogeneous wave factorization [15]. Let us note that this case is more pleasant for consideration because we can reduce the system (5.1) to a system of linear algebraic equations by the Mellin transform [15].

Definition 5.3. We say that $A(\xi) \in \overset{\circ}{E}_\alpha$ admits a homogeneous wave factorization with respect to the cone C^a_+ if it can be represented in the form

$$A(\xi) = A_{\neq}(\xi) \cdot A_{=}(\xi),$$

where the factors $A_{\neq}(\xi), A_{=}(\xi)$ satisfy the following conditions:

(1) $A_{\neq}(\xi), A_{=}(\xi)$ are defined for all admissible values $\xi \in \mathbb{R}^m$, without may be, the points $\{\xi \in \mathbb{R}^m : |\xi'|^2 = a^2\xi_m^2\}$;
(2) $A_{\neq}(\xi), A_{=}(\xi)$ admit an analytical continuation into radial tube domains $T(\overset{*}{C}{}^a_+), T(\overset{*}{C}{}^a_-)$ and are homogeneous of order æ and $\alpha -$ æ, respectively.

The number æ $\in \mathbb{R}$ is called index of homogeneous wave factorization.

Using Definition 5.3 we will try to simplify our conclusions and to obtain more convenient solvability conditions.

Everywhere below we assume that the factor $A_{\neq}(\xi_1, \xi_2)$ is homogeneous function of order æ.

Lemma 5.4. *The functions $b_1(t_2), b_2(t_1)$ are homogeneous of order $-æ+1$.*

Indeed, let us verify $b_1(t_2)$. Let $\lambda \in \mathbb{R}$ be a positive number, then we have

$$b_1(\lambda t_2) = \int\limits_{-\infty}^{+\infty} K(t_1, \lambda t_2)dt_1,$$

and after the change $t_1 = \lambda t$ we obtain

$$b_1(\lambda t_2) = \int\limits_{-\infty}^{+\infty} K(\lambda t, \lambda t_2)\lambda dt = \int\limits_{-\infty}^{+\infty} \lambda^{-æ} K(t, t_2)\lambda dt$$

$$= \lambda^{-æ+1} \int\limits_{-\infty}^{+\infty} K(t, t_2)dt = \lambda^{-æ+1} b_1(t_2);$$

thus suppose additionally that for all $t_1 \neq 0, t_2 \neq 0$

$$b_1(t_2) \neq 0, \quad b_2(t_1) \neq 0. \tag{5.4}$$

Further, we rewrite the system (5.1) in the following form

$$\begin{cases} \int_{-\infty}^{+\infty} M(t_1, t_2)\tilde{c}(t_1)dt_1 + \tilde{d}(t_2) = \tilde{f}_1(t_2), \\ \tilde{c}(t_1) + \int_{-\infty}^{+\infty} N(t_1, t_2)\tilde{d}(t_2)dt_2 = \tilde{f}_2(t_1), \end{cases} \tag{5.5}$$

where

$$b_1^{-1}(t_2)K(t_1, t_2) = M(t_1, t_2),$$
$$b_2^{-1}(t_1)K(t_1, t_2) = N(t_1, t_2), b_1^{-1}(t_2)\tilde{g}_1(t_2) = \tilde{f}_1(t_2), b_2^{-1}(t_1)\tilde{g}_2(t_1) = \tilde{f}_2(t_1).$$

Lemma 5.5. *The kernels $M(t_1, t_2), N(t_1, t_2)$ of the system* (5.5) *are homogeneous of order -1.*

It implies that the Mellin transform [13] can be very useful in this situation, and we use the technique developed in [17]. For completeness we give here these calculations. We rewrite the system (5.5)

$$\begin{cases} \int_{-\infty}^{0} M(t_1, t_2)\tilde{c}(t_1)dt_1 + \int_{0}^{+\infty} M(t_1, t_2)\tilde{c}(t_1)dt_1 + \tilde{d}(t_2) = \tilde{f}_1(t_2), \\ \tilde{c}(t_1) + \int_{-\infty}^{0} N(t_1, t_2)\tilde{d}(t_2)dt_2 + \int_{0}^{+\infty} N(t_1, t_2)\tilde{d}(t_2)dt_2 = \tilde{f}_2(t_1), \end{cases}$$

then once again with change of variables

$$\begin{cases} \int_{0}^{+\infty} M(-t_1, t_2)\tilde{c}(-t_1)dt_1 + \int_{0}^{+\infty} M(t_1, t_2)\tilde{c}(t_1)dt_1 + \tilde{d}(t_2) = \tilde{f}_1(t_2), \\ \tilde{c}(t_1) + \int_{0}^{+\infty} N(t_1, -t_2)\tilde{d}(-t_2)dt_2 + \int_{0}^{+\infty} N(t_1, t_2)\tilde{d}(t_2)dt_2 = \tilde{f}_2(t_1). \end{cases}$$

Now we introduce new unknowns $\tilde{c}_0(t), \tilde{c}_1(t), \tilde{d}_0(t), \tilde{d}_1(t)$ defined on $\mathbb{R}_+$ instead of unknowns $c(t), d(t)$ defined on $\mathbb{R}$. We put

$$\tilde{c}_0(t) = \tilde{c}(t), \quad \tilde{c}_1(t) = \tilde{c}(-t),\, t > 0, \quad \tilde{d}_0(t) = \tilde{d}(t), \quad \tilde{d}_1(t) = \tilde{d}(-t),\, t > 0,$$

the same for right-hand sides

$$\tilde{f}_{10}(t) = \tilde{f}_1(t), \quad \tilde{f}_{11}(t) = \tilde{f}_1(-t), t > 0, \quad \tilde{f}_{20}(t) = \tilde{f}_2(t), \quad \tilde{f}_{21}(t) = \tilde{f}_2(-t), t > 0,$$

for the kernel $M(t_1, t_2)$

$$M_{11}(t_1, t_2) = M(t_1, t_2), \quad M_{12}(t_1, t_2) = M(-t_1, t_2), \quad M_{21}(t_1, t_2) = M(t_1, -t_2),$$
$$M_{22}(t_1, t_2) = M(-t_1, -t_2), \quad t_1, t_2 > 0,$$

and analogously for $N(t_1, t_2)$.

Hence, we can obtain 4×4-system of linear integral equations on a half-axis with respect to 4 unknowns c_0, c_1, d_0, d_1

$$\begin{cases} \int\limits_0^{+\infty} M_{11}(t_1, t_2)\tilde{c}_0(t_1)dt_1 + \int\limits_0^{+\infty} M_{12}(t_1, t_2)\tilde{c}_1(t_1)dt_1 + \tilde{d}_0(t_2) = \tilde{f}_{10}(t_2), \\ \int\limits_0^{+\infty} M_{21}(t_1, t_2)\tilde{c}_0(t_1)dt_1 + \int\limits_0^{+\infty} M_{22}(t_1, t_2)\tilde{c}_1(t_1)dt_1 + \tilde{d}_1(t_2) = \tilde{f}_{11}(t_2), \\ \tilde{c}_0(t_1) + \int\limits_0^{+\infty} N_{11}(t_1, t_2)\tilde{d}_0(t_2)dt_2 + \int\limits_0^{+\infty} N_{21}(t_1, t_2)\tilde{d}_1(t_2)dt_2 = \tilde{f}_{20}(t_1), \\ \tilde{c}_1(t_1) + \int\limits_0^{+\infty} N_{12}(t_1, t_2)\tilde{d}_0(t_2)dt_2 + \int\limits_0^{+\infty} N_{22}(t_1, t_2)\tilde{d}_1(t_2)dt_2 = \tilde{f}_{22}(t_1). \end{cases} \tag{5.6}$$

Applying the Mellin transform to the system (5.6) we obtain the following system of linear algebraic equations:

$$\begin{cases} \hat{M}_{11}(\lambda)\hat{c}_0(\lambda) + \hat{M}_{12}(\lambda)\hat{c}_1(\lambda) + \hat{d}_0(\lambda) = \hat{f}_{10}(\lambda), \\ \hat{M}_{21}(\lambda)\hat{c}_0(\lambda) + \hat{M}_{22}(\lambda)\hat{c}_1(\lambda) + \hat{d}_1(\lambda) = \hat{f}_{11}(\lambda), \\ \hat{c}_0(\lambda) + \hat{N}_{11}(\lambda)\hat{d}_0(\lambda) + \hat{N}_{21}(\lambda)\hat{d}_1(\lambda) = \hat{f}_{20}(\lambda), \\ \hat{c}_1(\lambda) + \hat{N}_{12}(\lambda)\hat{d}_0(\lambda) + \hat{N}_{22}(\lambda)\hat{d}_1(\lambda) = \hat{f}_{21}(\lambda), \end{cases} \tag{5.7}$$

with the matrix

$$P_4(\lambda) = \begin{pmatrix} \hat{M}_{11}(\lambda) & \hat{M}_{12}(\lambda) & 1 & 0 \\ \hat{M}_{12}(\lambda) & \hat{M}_{221}(\lambda) & 0 & 1 \\ 1 & 0 & \hat{N}_{11}(\lambda) & \hat{N}_{21}(\lambda) \\ 0 & 1 & \hat{N}_{12}(\lambda) & \hat{N}_{22}(\lambda) \end{pmatrix},$$

where we used the notations: $\hat{M}_{ij}(\lambda)$ denotes Mellin transform for the function $M(1, t)$, and $\hat{N}_{ij}(\lambda)$ denotes Mellin transform for the function $N(t, 1), i, j = 1, 2$.

5.3. WEIGHTED H^s-SPACES AND SOLVABILITY

In case when $\det P_4(\lambda) \neq 0$ ($\Re e \lambda = 1/2$) the system (5.6) has a unique solution, and we can define $c(x)$, $d(x)$ knowing this solution. Substituting them in general solution we will define $\tilde{U}(x)$, but this solution generally speaking will not belong to class $H^s(C_+^a)$ because in case æ > 0 the pseudo-differential operator $a_{\neq}^{-1}$ is not bounded in space scale H^s, i.e., the symbol has singularity of order æ at origin. In connection with this fact we will introduce weighted H^s-spaces which "annihilate" such singularity and give the possibility to obtain a priori estimate of the solution there.

Let us denote by $H^{s,æ}(\mathbb{R}^m)$ the space of distributions $u(x)$ for which their Fourier transforms are locally integrable in the Lebesgue sense function $\tilde{u}(\xi)$ such that

$$\|u\|_{s,æ}^2 = \int\limits_{\mathbb{R}^m} |\tilde{u}(\xi)|^2 |\xi|^{2æ} (1+|\xi|)^{2(s-æ)} d\xi < +\infty.$$

It is natural to call $H^{s,æ}(\mathbb{R}^m)$ weighted H^s-spaces with weight

$$\left(\frac{|\xi|}{1+|\xi|}\right)^{æ}.$$

In space scale $H^{s,æ}(\mathbb{R}^m)$, pseudo-differential operators with homogeneous symbols have series of properties needed. We will collect below some of them. We denote $S(\mathbb{R}^m)$ the Schwartz class of infinitely differentiable rapidly decreasing at infinity functions.

Theorem 5.6. *Let A be a pseudo-differential operator with symbol which is belonging to $C^\infty(\mathbb{R}^m \setminus \{0\})$ and homogeneous of order æ. Then*

$$\|Au\|_{s-æ} \leq c_{s,æ} \|u\|_{s,æ}, \quad \forall u \in S(\mathbb{R}^m).$$

Proof. We have

$$\|Au\|_{s-æ}^2 = \int\limits_{\mathbb{R}^m} |A(\xi)u(\xi)|^2 (1+|\xi|)^{2(s-æ)} d\xi,$$

and since $A(\xi)$ is homogeneous of order æ and $A(\xi) \in C^\infty(\mathbb{R}^m \setminus \{0\})$, then

$$c_1 \leq \left| A(\xi)|\xi|^{-æ} \right| \leq c_2.$$

Hence,

$$\|Au\|_{s-æ}^2 \leq c_{s,æ} \int\limits_{\mathbb{R}^m} |\tilde{u}(\xi)|^2 |\xi|^{2æ} (1+|\xi|)^{2(s-æ)} d\xi = c_{s,æ} \|u\|_{s,æ}^2. \qquad \square$$

If we will denote by $[v]_{s,æ}$ the norm in space $H^{s,æ}(\mathbb{R}^{m-1})$

$$[v]_{s,æ}^2 = \int\limits_{\mathbb{R}^m} |\tilde{v}(\xi')|^2 |\xi'|^{2æ} (1+|\xi'|)^{2(s-æ)} d\xi', \qquad \xi' = (\xi_1, \ldots, \xi_{m-1}),$$

then a proof of the following lemma is the same as in [4].

Lemma 5.7. *Let $s > 1/2$. Then any function $u(x', x_m) \in H^{s,æ}(\mathbb{R}^m)$ is continuous on $x_m \in \mathbb{R}$ with its values in $H^{s-1/2,æ}(\mathbb{R}^{m-1})$. The estimate holds:*

$$\max_{x_m \in \mathbb{R}} [u(x', x_m)]_{s-1/2,æ} \leq c\|u\|_{s,æ}, \quad \forall u \in H^{s,æ}(\mathbb{R}^m).$$

Lemma 5.8. *Let $c(x_1) \in H^{s-\alpha+1/2}(\mathbb{R}_+)$, A is a pseudo-differential operator with symbol $A_{\neq}(\xi) \in C^\infty\left(\mathbb{R}^2 \setminus \{0\}\right)$ which is homogeneous of order α. Then*

$$\|c(x_1)a_{\neq}^{-1}(x)\|_{s,\alpha}^2 \leq c'[c]_{s-\alpha+1/2} \quad (-1 < s - \alpha + 1/2 < 0).$$

Proof. We have

$$\begin{aligned}
\|c(x_1)a_{\neq}^{-1}(x)\|_{s,\alpha}^2 &= \int\limits_{\mathbb{R}^2} \frac{|\tilde{c}(x_1)|^2}{|a_{\neq}(x)|^2}|x|^{2\alpha}(1+|x|)^{2(s-\alpha)}dx \\
&\leq \int\limits_{\mathbb{R}^2} |\tilde{c}(x_1)|^2(1+|x|)^{2(s-\alpha)}dx_1dx_2 \\
&\leq c' \int\limits_{-\infty}^{+\infty} |\tilde{c}(x_1)|^2(1+|x_1|)^{2(s-\alpha+1)}dx_1 \int\limits_{-\infty}^{+\infty}(1+|x|)^{-2}dx_2 \\
&= c' \int\limits_{-\infty}^{+\infty} |\tilde{c}(x_1)|^2(1+|x_1|)^{2(s-\alpha+1/2)}dx_1 = c'[c]_{s-\alpha+1/2}.
\end{aligned}$$

□

Lemma 5.9. *Let $b(x) \in C^\infty(\mathbb{R} \setminus \{0\})$, $b(x)$ be homogeneous of order $1 - æ$, and represent symbol of pseudo-differential operator b. Then*

$$\|b^{-1}v\|_{s-æ+1/2} \leq c\|v\|_{s-1/2,æ-1}, \quad \forall v \in H^{s-1/2,æ-1}(\mathbb{R}).$$

Proof.

$$\begin{aligned}
\|b^{-1}v\|_{s-æ+1/2}^2 &= \int\limits_{-\infty}^{+\infty} |b^{-1}(x)|^2|\tilde{v}(x)|^2(1+|x|)^{2(s-æ+1/2)}dx \\
&\leq c\int\limits_{-\infty}^{+\infty} |x|^{-2(1-æ)}|\tilde{v}(x)|^2(1+|x|)^{2(s-æ+1/2)}dx \\
&= c\int\limits_{-\infty}^{+\infty} \left(\frac{|x|}{1+|x|}\right)^{2(æ-1)} |\tilde{v}(x)|(1+|x|)^{2(s-1/2)}dx \\
&= c\|v\|_{s-1/2,æ-1}.
\end{aligned}$$

□

Remark 5.10. Since $s - æ + 1/2 = s - 1/2 - (æ - 1)$ we can reformulate Lemma 5.9 by the following way: if $b(x) \in C^\infty(\mathbb{R} \setminus \{0\})$ and $b(x)$ is homogeneous of order α then

$$\|bv\|_{s-\alpha} \leq c\|v\|_{s,\alpha},$$

and it follows from Theorem 5.6.

Let us note also that

$$\|u\|_{s,æ} \leq c\|u\|_{s,æ-1}. \tag{5.8}$$

In fact,

$$\|u\|^2_{s,æ} = \int\limits_{\mathbb{R}^m} |\tilde{u}(\xi)|^2 \left(\frac{|\xi|}{1+|\xi|}\right)^{2æ} (1+|\xi|)^{2s} d\xi$$

$$\leq c \int\limits_{\mathbb{R}^m} |\tilde{u}(\xi)|^2 \left(\frac{|\xi|}{1+|\xi|}\right)^{2(æ-1)} (1+|\xi|)^{2s} d\xi = c\|u\|_{s,æ-1},$$

because $\left(\frac{|\xi|}{1+|\xi|}\right)^{-1} \geq 1$.

Now we are ready to formulate the result on solvability of the Dirichlet problem.

Theorem 5.11. *Let $v_1, v_2 \in H^{s-1/2,æ-1}(\mathbb{R}_+)$, A be an elliptic pseudo-differential operator with symbol $A(\xi) \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ which is homogeneous of order α, æ be an index of homogeneous wave factorization for $A(\xi)$ with respect to C^1_+. Let the conditions* (5.4) *be fulfilled, and*

$$\inf |\det P_4(\lambda)| \neq 0, \quad \Re e\lambda = 1/2.$$

Then there exists a unique solution of the Dirichlet problem (5.2), (5.3) *in space $H_{s,æ}(C^1_+)$, $æ - s = 1 + \delta$, $|\delta| < 1/2$. The following a priori estimate holds*

$$\|u\|_{s,æ} \leq c \left([v_1]_{s-1/2,\, æ-1} + [v_2]_{s-1/2,\, æ-1}\right).$$

Proof. Using previous considerations we need to prove the a priori estimate only.

By Lemma 5.8 we obtain

$$\|u\|_{s,æ} = \|U\|_{s,æ} = \|\tilde{U}\|_{s,æ} \leq c_1 \left([c]_{s-æ+1/2} + [d]_{s-æ+1/2}\right),$$

and it is left to estimate $[c]_{s-æ+1/2}$, $[d]_{s-æ+1/2}$ by norms of functions v_1, v_2.

Before let's note the following. First, $s - æ + 1/2 = -1 - \delta + 1/2 = -1/2 - \delta$, and hence, $-1 < s - æ + 1/2 < 0$, and it gives

$$[c]_{s-æ+1/2} \leq [c]_0 \text{ (analogously for } d). \tag{5.9}$$

Second, if $u \in H^{s,æ}(C^1_+)$ then v_1, v_2 as traces of this function belong to space $H^{s-1/2,æ}(\mathbb{R}_+)$ in Lemma 5.7, but according to statement of the theorem they are taken from class $H^{s-1/2,æ-1}(\mathbb{R}_+)$. This "contradiction" is eliminated by inequality (5.8).

After corresponding transformations we obtain the system (5.7). We have denoted the matrix of this system by $\hat{P}_4(\lambda)$, and its inverse matrix will be denoted by $\hat{P}_4^{-1}(\lambda)$, if $\det \hat{P}_4(\lambda) \neq 0$, $\Re e\lambda = 1/2$. Vector with components $\hat{c}_0(\lambda)$, $\hat{c}_1(\lambda)$, $\hat{d}_0(\lambda)$, $\hat{d}_1(\lambda)$ we will denote $\hat{C}(\lambda)$, and vector with components $\hat{f}_{10}(\lambda)$, $\hat{f}_{11}(\lambda)$, $\hat{f}_{20}(\lambda)$, $\hat{f}_{21}(\lambda)$ we will denote $\hat{F}(\lambda)$.

Then

$$\hat{C}(\lambda) = \hat{P}_4^{-1}(\lambda)\hat{F}(\lambda),$$

and since components of matrix $\hat{P}_4^{-1}(\lambda)$ are bounded (by virtue of Mellin transform properties [13]), then

$$||\hat{C}(\lambda)||_0 \leq c'||\hat{F}(\lambda)||_0,$$

where

$$||\hat{C}(\lambda)||_0^2 = \frac{1}{2\pi i}\int\limits_{1/2-i\infty}^{1/2+i\infty}|\hat{C}(\lambda)|^2 d\lambda.$$

Now by virtue of Parseval's equality

$$\int\limits_0^{+\infty}|C(x)|^2dx = \frac{1}{2\pi i}\int\limits_{1/2-i\infty}^{1/2+i\infty}|\hat{C}(\lambda)|^2 d\lambda$$

we obtain

$$\int\limits_0^{+\infty}|C(x)|^2dx \leq c'\int\limits_0^{+\infty}|F(x)|^2dx,$$

or returning to $c(x)$, $d(x)$,

$$[c]_0 \leq c_1\left(\left[b_1^{-1}v_1\right]_0 + \left[b_2^{-1}v_2\right]_0\right),$$

$$[d]_0 \leq c_2\left(\left[b_1^{-1}v_1\right]_0 + \left[b_2^{-1}v_2\right]_0\right).$$

By Lemma 5.9, we have

$$\left[b_1^{-1}v_1\right]_0 \leq c_1\left[v_1\right]_{s-1/2,æ-1},$$

$$\left[b_2^{-1}v_2\right]_0 \leq c_2\left[v_2\right]_{s-1/2,æ-1},$$

whence taking into account the inequality (5.9), we obtain

$$[c]_{s-æ+1/2} \leq c_1\left([v_1]_{s-1/2,æ-1} + [v_2]_{s-1/2,æ-1}\right),$$

$$[d]_{s-æ+1/2} \leq c_2\left([v_1]_{s-1/2,æ-1} + [v_2]_{s-1/2,æ-1}\right),$$

and then

$$||u_+||_{s,æ} \leq c'\left([v_1]_{s-1/2,æ-1} + [v_2]_{s-1/2,æ-1}\right). \qquad \square$$

Let us note that earlier for two-dimensional case the author obtained certain integral equations for determining unknown functions, and study these equations by Mellin transform reducing them to a system of linear difference equations [17].

6. CONCLUSION

We have shown in the paper that method of transmutation operators leads to the same results which were obtained by the author earlier, but these methods can be useful for multidimensional case also. We will develop these multidimensional considerations in forthcoming papers.

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