COMPLEX VARIABLES AND ELLIPTIC EQUATIONS
2019, VOL. 64, NO. 5, 888–898
https://doi.org/10.1080/17476933.2019.1571048

On some distributions associated to boundary value problems

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ABSTRACT
We consider an elliptic pseudo-differential equation in a multidimensional cone in Sobolev–Slobodetskii space and describe its general solution for some cases. This description is based on some distributions supported on a conical surface. For certain conical surfaces we describe such distributions. It permits to use such distributions to refine a general solution.

ARTICLE HISTORY
Received 21 June 2018
Accepted 12 January 2019

COMMUNICATED BY
A. Soldatov

KEYWORDS
Pseudo-differential equation; elliptic symbol; cone; wave factorization; general solution; distribution

AMS SUBJECT CLASSIFICATIONS
Primary: 35S15; Secondary: 42B37; 46F10

1. Introduction

Earlier the author has studied model elliptic pseudo-differential equations in canonical non-smooth domains in Euclidean space $\mathbb{R}^m$, and more completed results were obtained in the plane $m=2$. This approach is based on a special factorization of an elliptic symbol and leads to full description of solvablity picture for a model elliptic pseudo-differential equation [1]. To transfer these results to multidimensional case one needs to know a general form of a distribution supported in a singular surface, this is the main difficulty. There are a lot of examples for distributions supported on different types of surfaces in m-dimensional space [2], but there is no theorem on a general form of such a distribution. The author's latest studies related to the multidimensional situations $m \geq 3$ are very dependent on such a result. Some preliminary results in this direction were obtained in papers [3–5], but here we will give more concrete calculations.

One can find other studies of pseudo-differential equations and related boundary value problems, for example, in [6,7] and subsequent publications.

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2. Preliminaries

We consider the pseudo-differential operator A:

$$u(x) \longmapsto \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} A(\xi)u(y)\,\mathrm{e}^{\mathrm{i}(y-x)\cdot\xi}\,\mathrm{d}\xi\,\mathrm{d}y, \quad x \in \mathbb{R}^m$$

in the Sobolev–Slobodetskii spaces $H^s(\mathbb{R}^m)$ with norm

$$\|u\|_s^2 = \int_{\mathbb{R}^m} |\tilde{u}(\xi)|^2(1+|\xi|)^{2s}\,\mathrm{d}\xi, \quad \tilde{u}(\xi) \equiv (Fu)(\xi) = \int_{\mathbb{R}^m} \mathrm{e}^{\mathrm{i}x\cdot\xi}u(x)\,\mathrm{d}x,$$

and introduce the following class of symbols not depending on the spatial variable x: $\exists c_1, c_2 > 0$ such that

$$c_1 \le |A(\xi)(1+|\xi|)^{-\alpha}| \le c_2, \quad \xi \in \mathbb{R}^m. \tag{1}$$

We call the number $\alpha \in \mathbb{R}$ order of the pseudo-differential operator A.

It is well known that such an operator is a linear bounded operator acting from the space $H^s(\mathbb{R}^m)$ into the space $H^{s-\alpha}(\mathbb{R}^m)$ and invertible [8]. The problem is the following: what can one say on the invertibility of the operator:

$$u(x) \longmapsto \int_D\int_{\mathbb{R}^m} A(\xi)u(y)\,\mathrm{e}^{\mathrm{i}(y-x)\cdot\xi}\,\mathrm{d}\xi\,\mathrm{d}y, \quad x \in D,$$

where $D \subset \mathbb{R}^m$ is a so-called 'canonical' domain. For the first canonical domain $D = \mathbb{R}^m_+ \equiv \{x \in \mathbb{R}^m : x = (x', x_m), x_m > 0\}$ it was studied by M.I. Vishik–G.I. Eskin [8]. Here we consider another canonical domain, namely $D = C^a_+$,

$$C^a_+ = \{x \in \mathbb{R}^m : x_m > a|x'|,\ x' = (x_1, \dots, x_{m-1}),\ a > 0\}.$$

Thus we study the solvability of the model pseudo-differential equation [1,3,4]

$$(Au_+)(x) = f(x), \quad x \in C^a_+, \tag{2}$$

in the space $H^s(C^a_+)$, where A is a pseudo-differential operator with the symbol $A(\xi)$ satisfying the condition (1). The sign '+' for the unknown function u_+ shows that this function is defined in C^a_+ only. We will recall some standard definitions.

By definition the space $H^s(C^a_+)$ consists of functions from the space $H^s(\mathbb{R}^m)$ with supports in $\overline{C^a_+}$. The norm in the space $H^s(C^a_+)$ is induced by the norm of the space $H^s(\mathbb{R}^m)$.

Let us denote by $S(\mathbb{R}^m)$ the Schwartz class of infinitely differentiable at infinity rapidly decreasing functions, by $S'(\mathbb{R}^m)$ the space of distributions over $S(\mathbb{R}^m)$, and by $S'(C^a_+)$ the space of distributions from $S'(\mathbb{R}^m)$ with supports in $\overline{C^a_+}$. The right-hand side f in Equation (2) is taken from the space $H_0^{s-\alpha}(C^a_+)$ consisting of distributions from $S'(C^a_+)$ which admit a continuation into the whole space $H^{s-\alpha}(\mathbb{R}^m)$. A norm in the space $H_0^{s-\alpha}(C^a_+)$ is defined by the formula

$$\|f\|^+_{s-\alpha} = \inf \|lf\|_{s-\alpha},$$

where *infimum* is taken over all continuations lf.

Furthermore, we introduce the multidimensional singular integral operator:

$$(G_m u)(x) = \lim_{\tau\to 0+} \int_{\mathbb{R}^m} \frac{u(y', y_m)\, dy'\, dy_m}{\left(|x'-y'|^2 - a^2(x_m - y_m + i\tau)^2\right)^{m/2}}$$

(we omit certain constants, see [1]). We recall that this integral is a multidimensional analogue of the Cauchy type integral; more exactly it is a variant of multidimensional Hilbert transform.

To introduce the next definition we add a new notation.

Let $\overset{*}{C^a_+}$ be the conjugate cone for the cone C^a_+:

$$\overset{*}{C^a_+} = \{x \in \mathbb{R}^m : x = (x', x_m), ax_m > |x'|\},$$

$C^a_- \equiv -C^a_+$, $T(C^a_+)$ be the radial tube domain over the cone C^a_+, i.e. a domain of the multidimensional complex space $\mathbb{C}^m$ of the type $\mathbb{R}^m + iC^a_+$. The sign $\sim$ is used for the Fourier transform, so $\tilde{u}_+$ is the Fourier transform of the function u_+, and $\tilde{H}$ is the Fourier image of the space H.

Definition 2.1: The wave factorization of an elliptic symbol $A(\xi)$ is called its representation in the form

$$A(\xi) = A_{\neq}(\xi) A_{=}(\xi),$$

where the factors $A_{\neq}(\xi), A_{=}(\xi)$ must satisfy the conditions:

(1) $A_{\neq}(\xi)$, $A_{=}(\xi)$ are defined for all $\xi \in \mathbb{R}^m$ may be except $\{\xi \in \mathbb{R}^m : |\xi'|^2 = a^2\xi_m^2\}$;
(2) $A_{\neq}(\xi), A_{=}(\xi)$ admits an analytic continuation into the radial tube domain $T(\overset{*}{C^a_+})$, $T(\overset{*}{C^a_-})$, respectively with estimates

$$|A_{\neq}^{\pm 1}(\xi + i\tau)| \le c_1(1 + |\xi| + |\tau|)^{\pm æ},$$

$$|A_{=}^{\pm 1}(\xi - i\tau)| \le c_2(1 + |\xi| + |\tau|)^{\pm(\alpha - æ)}, \quad \forall \tau \in \overset{*}{C^a_+}.$$

The number $æ \in \mathbb{R}$ is called the index of the wave factorization.

Here we will consider Equation (2) for the case $æ - s = n + \delta, n \in \mathbf{N}, |\delta| < 1/2$, only. Let us note that for the case $æ - s = \delta$ the solution of Equation (2) is unique, and it can be reconstructed by the operator G_m [1]. The general solution for our case can be constructed using the following scheme. We continue f on the entire $\mathbb{R}^m$ and denote this continuation by lf. We put

$$u_-(x) = lf(x) - (Au_+)(x),$$

and rewrite Equation (2) as

$$(Au_+)(x) + u_-(x) = lf(x).$$

Now applying the Fourier transform and the wave factorization we can write the initial equation in the form

$$A_{\neq}(\xi)\tilde{u}_+(\xi) + A_{=}^{-1}(\xi)\tilde{u}_-(\xi) = A_{=}^{-1}(\xi)\tilde{lf}(\xi). \tag{3}$$

Let us note that the function $A_{=}^{-1}(\xi)\tilde{lf}(\xi)$ belongs to the space $\tilde{H}^{s-æ}(\mathbb{R}^m)$. So, if we take an arbitrary polynomial $Q(\xi)$ satisfying the condition

$$|Q(\xi)| \sim (1+|\xi|)^n,$$

then the function $Q^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{lf}(\xi)$ will belong to the space $\tilde{H}^{-\delta}(\mathbb{R}^m)$.

Furthermore, according to the theory of the multidimensional Riemann problem [1] we can represent the latter function as a sum of two summands, this is a so-called jump problem which can be solved by the operator G_m:

$$Q^{-1}A_{=}^{-1}\tilde{lf} = f_+ + f_-,$$

where $f_+ \in \tilde{H}^{-\delta}(C_+^a)$, $f_- \in \tilde{H}^{-\delta}(\mathbb{R}^m \setminus C_+^a)$,

$$f_+ = G_m(A_{=}^{-1}\widetilde{lf}), \quad f_- = (I - G_m)(A_{=}^{-1}\widetilde{lf}).$$

Multiplying the equality (3) by $Q^{-1}(\xi)$ we rewrite it in the form

$$Q^{-1}A_{\neq}\tilde{u}_+ + Q^{-1}A_{=}^{-1}\tilde{u}_- = f_+ + f_-,$$

or

$$Q^{-1}A_{\neq}\tilde{u}_+ - f_+ = f_- - Q^{-1}A_{=}^{-1}\tilde{u}_-.$$

In other words

$$A_{\neq}\tilde{u}_+ - Qf_+ = Qf_- - A_{=}^{-1}\tilde{u}_-. \tag{4}$$

Now we can use the following result [1,5].

Lemma 2.2: *If $\tilde{u}_- \in \tilde{H}^s(\mathbb{R}^m \setminus C_+^a)$, $A_{=}^{-1}$ is a factor of the wave factorization then $A_{=}^{-1}\tilde{u}_- \in \tilde{H}^{s-\alpha+æ}(\mathbb{R}^m \setminus C_+^a)$.*

The left-hand side of the equality (4) belongs to the space $\tilde{H}^{-n-\delta}(C_+^a)$, but the right-hand side belongs to the space $\tilde{H}^{-n-\delta}(\mathbb{R}^m \setminus C_+^a)$. Therefore, we have

$$F^{-1}(A_{\neq}\tilde{u}_+ - Qf_+) = F^{-1}(Qf_- - A_{=}^{-1}\tilde{u}_-),$$

where the left-hand side belongs to the space $H^{-n-\delta}(C_+^a)$, but right-hand side belongs to the space $H^{-n-\delta}(\mathbf{R}^m \setminus C_+^a)$, from which we conclude immediately that this is a distribution supported on the surface ∂C_+^a.

It is left to determine the form for such a distribution.

3. Distributions and change of variables

3.1. Test functions

Let C be a convex cone in the space $\mathbb{R}^m$, not including any entire straight line. This is important because we use the theory of analytic functions of several complex variables [9–11]. Moreover, we suppose that the surface of this cone is given by the equation $x_m = \varphi(x'), x' = (x_1, \dots, x_{m-1})$, where $\varphi : \mathbb{R}^{m-1} \to \mathbb{R}$ is a smooth function in $\mathbb{R}^{m-1} \setminus \{0\}$, and $\varphi(0) = 0$.

Let us introduce the change of variables:

$$
\begin{aligned}
t_1 &= x_1\\
t_2 &= x_2\\
&\vdots\\
t_{m-1} &= x_{m-1}\\
t_m &= x_m - \varphi(x')
\end{aligned}
$$

and denote this operator by $T_\varphi : \mathbb{R}^m \to \mathbb{R}^m$.

Obviously, this is a smooth transformation excluding the origin. We would like to define a special change of variables for distributions, and we need a special class of test functions. As such a space we take the Lizorkin space $\Phi(\mathbb{R}^m)$ [12] which is a subspace of the Schwartz space $S(\mathbb{R}^m)$. Such functions vanish at the origin with all their derivatives. If we denote by $\Phi'(\mathbb{R}^m), S'(\mathbb{R}^m)$ the corresponding spaces of distributions then $\Phi'(\mathbb{R}^m) \supset S'(\mathbb{R}^m)$, and all operations with distributions from $\Phi'(\mathbb{R}^m)$ will be valid for distributions from $S'(\mathbb{R}^m)$.

Let f be a local integrable function which generates a distribution defined by the formula

$$(f, \psi) = \int_{\mathbb{R}^m} f(x)\psi(x)\,\mathrm{d}x.$$

We define a functional $T_\varphi f$ by the formula

$$(T_\varphi f, \psi) = (f, T_\varphi^{-1}\psi),$$

because

$$
\begin{aligned}
(T_\varphi f, \psi) &= \int_{\mathbb{R}^m} (T_\varphi f)(x)\psi(x)\,\mathrm{d}x \equiv \int_{\mathbb{R}^m} f(T_\varphi x)\psi(x)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^m} f(x)\psi(T_\varphi^{-1}x)\,\mathrm{d}x = (f, T_\varphi^{-1}\psi).
\end{aligned}
$$

One can easily verify that the Jacobian of the transformation T_φ equals to 1 everywhere except 0. Since we choose such test functions the point 0 should be ignored.

We will use a result which follows from Schwartz theorem on one-dimensional distributions from $S'(\mathbb{R})$ supported at the origin 0 [2,10].

Lemma 3.1: *If a distribution $f \in S'(\mathbb{R}^m)$ is supported in the hyper-plane $x_m = 0$ then it has the form*

$$f(x) = \sum_{k=0}^{n} c_k(x') \otimes \delta^{(k)}(x_m), \quad x = (x', x_m), \tag{5}$$

where $c_k \in S'(\mathbb{R}^{m-1})$, $k = 0, 1, \ldots, n$, are arbitrary distributions.

3.2. Change of variables

Here we define a change of variables for distributions.

Definition 3.2: The operator T_φ is defined by the formula

$$(T_\varphi f, \psi) = (f, T_\varphi^{-1}\psi), \quad \forall \psi \in S(\mathbb{R}^m).$$

The next two assertions are very simple.

Lemma 3.3: *If a distribution $f \in S'(\mathbb{R}^m)$ is supported on ∂C then $T_\varphi f$ is supported on $\mathbb{R}^{m-1}$.*

Theorem 3.4: *An arbitrary distribution $f \in S'(\mathbb{R}^m)$ supported on the conical surface ∂C can be written in the form*

$$f(x) = T_\varphi^{-1}\left(\sum_{k=0}^{n} c_k(y') \otimes \delta^{(k)}(y_m)\right), \tag{6}$$

where $c_k \in S'(\mathbb{R}^{m-1})$, $k = 0, 1, \ldots, n$, are arbitrary distributions.

4. The Fourier transform

It is a natural way to ask what is the Fourier transform of a distribution supported on a conical surface. So, for example, the Fourier transform of the distribution (6) is a polynomial of order n with respect to the variable ξ_m with coefficients $\tilde{c}_k(\xi')$, $k = 0, 1, \ldots, n$.

For functions $u(x)$ from $S(\mathbb{R}^m)$ the Fourier transform is defined by the formula

$$(Fu)(\xi) \equiv \tilde{u}(\xi) = \int_{\mathbb{R}^m} e^{ix\cdot\xi} u(x)\, dx.$$

The Fourier transform for distributions is defined as

$$(Ff, \psi) = (f, F\psi),$$

therefore

$$(FT_\varphi f, \psi) = (f, T_\varphi^{-1} F\psi).$$

These properties imply that in order to define correlations between T_φ and F one can operate with test functions only.

Let $f \in S'(\mathbb{R}^m)$ be a distribution supported on ∂C. According to Theorem 3.4 it has the special form (6). We have the relations

$$
\begin{aligned}
(Ff, \psi) &= \left(FT_\varphi^{-1}\left(\sum_{k=0}^{n} c_k(y') \otimes \delta^{(k)}(y_m)\right), \psi \right) \\
&= \left(FT_\varphi^{-1}F^{-1}F\left(\sum_{k=0}^{n} c_k(y') \otimes \delta^{(k)}(y_m)\right), \psi \right) \\
&= \left(V_\varphi\left(\sum_{k=0}^{n} \tilde{c}_k(\xi')\xi_m^k\right), \psi \right),
\end{aligned}
$$

where we use the new notation $FT_\varphi^{-1}F^{-1} \equiv V_\varphi$; this operator V_φ is defined on $S(\mathbb{R}^m)$; let us note that this operator is invertible and $V_\varphi^{-1} = FT_\varphi F^{-1}$, $T_\varphi^{-1} = T_{-\varphi}$. According to the latter observations we can conclude that

$$
Ff = V_\varphi\left(\sum_{k=0}^{n} \tilde{c}_k(\xi')\xi_m^k\right).
$$

4.1. *The operator* V_φ

We start from the equality

$$
FT_\varphi^{-1} = V_\varphi F
$$

and conclude that this operator acts in Fourier images. It is more convenient to start from the left-hand side. We fix $u \in S(\mathbb{R}^m)$ and calculate:

$$
\begin{aligned}
(FT_\varphi^{-1}u)(\xi) &= \int_{\mathbb{R}^m} e^{iy\cdot\xi}(T_\varphi^{-1}u)(y)\,dy = \int_{\mathbb{R}^m} e^{iy\cdot\xi}u(y', y_m + \varphi(y'))\,dy \\
&= \int_{\mathbb{R}^m} e^{ix\cdot\xi}\,e^{-i\xi_m\varphi(x')}u(x', x_m)\,dx'\,dx_m = \int_{\mathbb{R}^{m-1}} e^{ix'\cdot\xi'}\,e^{-i\xi_m\varphi(x')}\hat{u}(x', \xi_m)\,dx',
\end{aligned}
$$

where $\hat{u}(x', \xi_m)$ denotes the Fourier transform of the function $u(x', x_m)$ with respect to the variable x_m. Taking into account properties of the Fourier transform one can make the following conclusion. Let us denote

$$
F_{x'\to\xi'}(e^{-i\xi_m\varphi(x')}) \equiv K_\varphi(\xi', \xi_m),
$$

and after this we obtain an integral representation for the operator V_φ:

$$
(FT_\varphi^{-1}u)(\xi) = \int_{\mathbb{R}^m} K_\varphi(\xi' - \eta', \xi_m)\tilde{u}(\eta', \xi_m)\,d\eta'.
$$

4.2. A first example: m = 2

The case $m=2$ is very good, there is only one mentioned cone. We write it as

$$C_+^a = \{x \in \mathbb{R}^2 : x = (x_1, x_2), x_2 > a|x_1|,\ a > 0\},$$

and further evaluate:

$$\begin{aligned}(FT_\varphi u)(\xi) &= \int_{-\infty}^{+\infty} e^{ia|y_1|\xi_2}\, e^{iy_1\xi_1}\, \hat{u}(y_1,\xi_2)\, dy_1,\\ &= \int_{-\infty}^{+\infty} \chi_+(y_1)\, e^{iay_1\xi_2}\, e^{iy_1\xi_1}\, \hat{u}(y_1,\xi_2)\, dy_1\\ &\quad + \int_{-\infty}^{+\infty} \chi_-(y_1)\, e^{-iay_1\xi_2}\, e^{iy_1\xi_1}\, \hat{u}(y_1,\xi_2)\, dy_1\\ &= \int_{-\infty}^{+\infty} \chi_+(y_1)\, e^{iy_1(a\xi_2+\xi_1)}\, \hat{u}(y_1,\xi_2)\, dy_1\\ &\quad + \int_{-\infty}^{+\infty} \chi_-(y_1)\, e^{-iy_1(a\xi_2-\xi_1)}\, \hat{u}(y_1,\xi_2)\, dy_1,\end{aligned}$$

where $\hat{u}(y_1,\xi_2)$ denotes the one-dimensional Fourier transform with respect to the second variable, $\chi_\pm$ are indicators of $\mathbb{R}_\pm$.

The latter two summands are Fourier transforms of functions

$$\chi_+(y_1)\hat{u}(y_1,\xi_2), \qquad \chi_-(y_1)\hat{u}(y_1,\xi_2)$$

with respect to the variable y_1 so that we can use the following properties [8]; these properties are called Sokhotskii formulas [13,14]:

$$\begin{aligned}\int_{-\infty}^{+\infty} \chi_+(x)\, e^{ix\xi}\, u(x)\, dx &= \frac{1}{2}\tilde{u}(\xi) + v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta)\, d\eta}{\xi-\eta},\\ \int_{-\infty}^{+\infty} \chi_-(x)\, e^{ix\xi}\, u(x)\, dx &= \frac{1}{2}\tilde{u}(\xi) - v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{\tilde{u}(\eta)\, d\eta}{\xi-\eta}.\end{aligned}$$

Taking these properties into account we conclude

$$\begin{aligned}(FT_\varphi^{-1}u)(\xi) &= \frac{\tilde{u}(\xi_1+a\xi_2,\xi_2)+\tilde{u}(\xi_1-a\xi_2,\xi_2)}{2}\\ &\quad + v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{u}(\eta,\xi_2)\, d\eta}{\xi_1+a\xi_2-\eta} - v.p.\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{\tilde{u}(\eta,\xi_2)\, d\eta}{\xi_1-a\xi_2-\eta} \equiv (V_\varphi\tilde{u})(\xi).\end{aligned}$$

4.3. An example: m ≥ 3

As it was shown the kernel K_φ is computable for concrete function $\varphi(x')$. Let $\varphi(x') = a|x'|$, $a > 0$. If we will look at the formulas from [11] (see also [15] in which a real analogue of these formulas is given as the Poisson kernel) we will find

$$K_\varphi(\xi',\xi_m) = \frac{a2^{m-1}\pi^{(m-2)/2}\Gamma(m/2)}{\left(\xi_1^2+\xi_2^2+\cdots\xi_{m-1}^2-a^2\xi_m^2\right)^{m/2}}.$$

Therefore for such a multidimensional cone the operator V_φ looks like

$$(V_\varphi\tilde{u})(\xi) = \int_{\mathbb{R}^{m-1}} \frac{a2^{m-1}\pi^{(m-2)/2}\Gamma(m/2)\tilde{u}(\eta',\xi_m)\,d\eta'}{\left((\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2+\cdots(\xi_{m-1}-\eta_{m-1})^2-a^2\xi_m^2\right)^{m/2}}. \quad (7)$$

In our opinion we could call it *a conical potential.*

Remark 4.1: Of course the formula (7) should be treated in the distributional sense. Below we give such a definition for the operator V_φ in the space $S'(\mathbb{R}^m)$.

Definition 4.1: For a distribution $f \in S'(\mathbb{R}^m)$ the transform V_φ is given by the formula

$$(V_\varphi\tilde{f},\psi) \equiv (\tilde{f}, V_{-\varphi}\psi), \quad \forall\psi \in S(\mathbb{R}^m).$$

5. The operator V_φ and a general solution

Let C be a convex cone not including an entire straight line. Let us introduce the Bochner kernel [9–11]

$$B_m(z) = \int_C e^{ix\cdot z}\,dx, \quad z = \xi + i\tau,$$

and the related integral operator

$$(B_m u)(x) = \lim_{\tau\to 0+}\int_{\mathbb{R}^m} B_m(x-y+i\tau)u(y)\,dy, \quad x \in \mathbb{R}^m.$$

The reader can easily adapt Definition 2.1 for an arbitrary cone $C = \{x \in \mathbb{R}^m : x_m > \varphi(x')\}$ and he can obtain the following result for Equation (2) in a cone C.

Theorem 5.1: *If the symbol $A(\xi)$ admits the wave factorization with the index* æ, æ $- s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$, *then the general solution of Equation (2) in Fourier images is given by the formula*

$$\tilde{u}_+(\xi) = A_{\neq}^{-1}(\xi)Q(\xi)B_mQ^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{lf}(\xi) + A_{\neq}^{-1}(\xi)V_\varphi^{-1}F\left(\sum_{k=1}^{n}c_k(x')\delta^{(k-1)}(x_m)\right),$$

where $c_k(x') \in H^{s_k}(\mathbb{R}^{m-1})$ are arbitrary functions, $s_k = s -$ æ $+ k - 1/2, k = 1,2,\ldots,n$, lf is an arbitrary continuation of f onto $H^{s-\alpha}(\mathbb{R}^m)$.

The corresponding result on a general solution of Equation (2) in the cone C_+^a was proved in [5].

Remark 5.1: It is very simple to verify that the result does not depend on the continuation lf [1,8].

Some special cases are very interesting, for example if $C = C^a_+$.

Corollary 5.2: *If $f \equiv 0, n = 1$, then we have the following form for the general solution in the space $H^s(C^a_+)$:*

$$\tilde{u}_+(\xi) = A^{-1}_{\neq}(\xi) \int_{\mathbb{R}^{m-1}} \frac{a 2^{m-1} \pi^{(m-2)/2} \Gamma(m/2) \tilde{c}(\eta')\, \mathrm{d}\eta'}{\left((\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + \cdots (\xi_{m-1} - \eta_{m-1})^2 - a^2 \xi_m^2\right)^{m/2}},$$

where $c(x') \in H^{s-æ+1/2}(\mathbb{R}^{m-1})$ is an arbitrary function.

Using these results one needs to add some additional conditions to determine uniquely the unknown functions c_k. We will consider this problem in forthcoming papers.

Acknowledgments

The author is very grateful to referees for their timely and helpful comments.

Disclosure statement

No potential conflict of interest was reported by the author.

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