On Discrete Solutions for Pseudo-Differential Equations

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Abstract. We study model discrete pseudo-differential equations in some canonical domains of Euclidean space. We need such consideration to obtain approximate solution for initial continuous pseudo-differential equations and related boundary value problems. We call solutions of these discrete equations by discrete solutions for pseudo-differential equations. Our main goal is finding discrete solutions and comparison between discrete and continuous solutions.

INTRODUCTION

A certain theory of pseudo-differential operators and corresponding equations was constructed in the second half of the last century [1, 2, 3], and it includes as usual boundedness theorems in different functional spaces and a certain variant of symbolic calculus. But for discrete situation there is no any variant of such a theory although there are a lot of approximate constructions for solving simplest kinds of pseudo-differential equations, for example singular integral and similar equations [4, 5, 6]. Moreover there are some recent studies for these discrete situations from algebraic or symbolic calculus point of view on the whole m-dimensional lattice $\mathbb{Z}^m$. But there are principal difficulties to transfer this approach to another discrete domains which are not $\mathbb{Z}^m$, for example a discrete half-space or a discrete cone.

DISCRETE SPACES AND OPERATORS

For studying discrete analogues of pseudo-differential equations [1, 2, 3] we will use the discrete Fourier transform for defining discrete Sobolev–Slobodetskii spaces; such discrete spaces are very convenient in the theory.

Let us denote by $u_d(\tilde{x})$ a function of a discrete variable $\tilde{x} \in h\mathbb{Z}^m, h > 0$, and by $\tilde{u}_d(\xi), \xi \in \hbar\mathbb{T}^m, \hbar = h^{-1}, \mathbb{T}^m = [-\pi, \pi]^m$, its discrete Fourier transform

$$(F_d u_d)(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^m} u_d(\tilde{x}) e^{i\tilde{x}\cdot\xi} h^m.$$

We denote $\zeta^2 = h^{-2} \sum\limits_{k=1}^{m} (e^{-ih\cdot\xi_k} - 1)^2$ and introduce the following

Definition 1. *The space $H^s(h\mathbb{Z}^m)$ of discrete functions $u_d(\tilde{x})$ with the finite norm*

$$||u_d||_s = \left(\int_{\hbar\mathbb{T}^m} (1 + |\zeta^2|)^s |\tilde{u}_d(\xi)|^2 d\xi \right)^{1/2}$$

Further, we introduce $D = \mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x', x_m), x_m > 0\}$, and $D_d = D \cap h\mathbb{Z}^m$ is a corresponding discrete domain.

Definition 2. *The space $H^s(D_d)$ consists of discrete functions from $H^s(h\mathbb{Z}^m)$ with supports in $\overline{D_d}$. A norm in the space $H^s(D_d)$ is induced by a norm of the space $H^s(h\mathbb{Z}^m)$.The space $H^s_0(D_d)$ consists of discrete functions u_d with supports in D_d and these functions must admit a continuation ℓu_d on the whole space $H^s(h\mathbb{Z}^m)$. A norm in the space $H^s_0(D_d)$ is given by the formula*

$$||u_d||^+_s = \inf ||\ell u_d||_s,$$

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where infimum is taken over all continuations ℓ.

Let $A_d(\xi)$ be a periodic function in $\mathbb{R}^m$ with basic cube of periods $\hbar\mathbb{T}^m$. Such functions we call symbols.

Definition 3. *A digital pseudo-differential operator A_d in the discrete domain D_d is called an operator of the following type*

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^m} \int\limits_{\hbar\mathbb{T}^m} A_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in D_d,$$

We will consider class E_α of symbols satisfying the following condition

$$c_1(1 + |\zeta^2|)^{\alpha/2} \leq |A_d(\xi)| \leq c_2(1 + |\zeta^2|)^{\alpha/2}$$

with constants c_1, c_2 non-depending on h.

Let us denote $\Pi_\pm = \{(\xi', \xi_m \pm i\tau), \tau > 0\}, \xi = (\xi', \xi_m) \in \mathbb{T}^m$.

SOLVABILITY

We are interested in a solvability of the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d \tag{1}$$

in the space $H^s(D_d)$ if the right hand side $v_d \in H_0^{s-\alpha}(D_d)$.

This discrete equation (1) is a discrete analogue of certain continuous equation

$$(Au)(x) = v(x), \quad x \in D \tag{2}$$

with a pseudo-differential operator A with the symbol $A(\xi)$.

To obtain solvability results on the discrete equation (1) we use the following concept [3, 7, 8, 9, 10].

Definition 4. *A periodic factorization of an elliptic symbol $A_d(\xi) \in E_\alpha$ is called its representation in the form*

$$A_d(\xi) = A_{d,+}(\xi) A_{d,-}(\xi),$$

where the factors $A_{d,\pm}(\xi)$ admit an analytic continuation into half-strips $\hbar\Pi_\pm$ on the last variable ξ_m for almost all fixed $\xi' \in \hbar\mathbb{T}^{m-1}$ and satisfy the estimates

$$|A_{d,+}^{\pm 1}(\xi)| \leq c_1(1 + |\hat{\zeta}^2|)^{\pm\frac{\text{æ}}{2}}, \quad |A_{d,-}^{\pm 1}(\xi)| \leq c_2(1 + |\hat{\zeta}^2|)^{\pm\frac{\alpha-\text{æ}}{2}},$$

with constants c_1, c_2 non-depending on h,

$$\hat{\zeta}^2 \equiv \hbar^2 \left(\sum_{k=1}^{m-1} (e^{-ih\xi_k} - 1)^2 + (e^{-ih(\xi_m + i\tau)} - 1)^2 \right), \quad \xi_m + i\tau \in \hbar\Pi_\pm.$$

The number æ $\in \mathbb{R}$ is called an index of periodic factorization.

Remark 1. *Such periodic factorization exists at least for smooth symbols $A_d(\xi)$.*

Let us denote by ℓv_d an arbitrary continuation of v_d into a whole $h\mathbb{Z}^m$.

Theorem 1. *If the elliptic symbol $A_d(\xi) \in E_\alpha$ admits periodic factorization with index æ so that $|\text{æ} - s| < 1/2$ then the the equation (1) has unique solution in the space $H^s(D_d)$*

$$\tilde{u}_d(\xi) = A_{d,+}^{-1}(\xi) P_{\xi'}^{per}(A_{d,-}^{-1}(\xi)\widetilde{\ell v_d}(\xi)).$$

$$(P_{\xi'}^{per}\tilde{u}_d)(\xi) \equiv \frac{1}{2}\left(\tilde{u}_d(\xi) + \frac{h}{2\pi i} v.p. \int\limits_{-\hbar\pi}^{\hbar\pi} \tilde{u}_d(\xi', \eta_m) \cot\frac{h(\xi_m - \eta_m)}{2} d\eta_m \right),$$

for arbitrary right-hand side $v_d \in H_0^{s-\alpha}(D_d)$,

Theorem 2. *Let $æ - s = n+\delta, n \in \mathbf{N}, |\delta| < 1/2$. Then a general solution of the equation* (1) *in Fourier images has the following form*

$$\tilde{u}_d(\xi) = A^{-1}_{d,+}(\xi)X_n(\xi)P^{per}_{\xi'}(X^{-1}_n(\xi)A^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi)) + A^{-1}_{d,+}(\xi)\sum_{k=0}^{n-1} c_k(\xi')\hat{\zeta}^k_m,$$

where $X_n(\xi)$ is an arbitrary polynomial of order n of variables $\hat{\zeta}_k = \hbar(e^{-ih\xi_k} - 1), k = 1, \cdots, m$, satisfying the condition (2), *$c_k(\xi'), j = 0, 1, \cdots, n-1$, are arbitrary functions from $H_{s_k}(h\mathbf{T}^{m-1})$, $s_k = s - æ + k - 1/2$.*

Theorem 3. *Let $æ - s = -n+\delta, n \in \mathbb{N}, |\delta| < 1/2$. Then the equation (1) has a solution in the space $H^s(D_d)$ iff*

$$c_j(\xi') = 0, \ \forall a.a.\ \xi' \in \hbar\mathbb{T}^{m-1}, \ j = 0, 1. \cdots, n.$$

where

$$c_j(\xi') = \int\limits_{-\hbar\pi}^{\hbar\pi} (e^{ih\xi_m} - 1)^j A^{-1}_{d,-}(\xi', \xi_m)(\widetilde{\ell v_d})(\xi', \xi_m)d\xi_m, \quad j = 0, 1, \cdots, n,$$

Remark 2. *One can easily verify that all assertions of theorems 1–3 do not depend on the continuation ℓv_d.*

COMPARISON

Let $A(\xi)$ be a locally integrable function in $\mathbb{R}^m$ satisfying the condition

$$c_1(1 + |\xi|)^\alpha \le |A(\xi)| \le c_2(1 + |\xi|)^\alpha,$$

where c_1, c_2 are constants. We will consider the equation (2) where the operator A has such symbol $A(\xi)$. Then we construct the periodic symbol $A_d(\xi)$ in the following way. We take a restriction of $A(\xi)$ on the cube $\hbar\mathbb{T}^m$ and periodically extend it onto a whole $\mathbb{R}^m$. We consider such discrete operator A_d as an approximate operator for A.

A construction for the restriction operator Q_h for functions $u \in S(\mathbb{R}^m)$ is the following. We take the Fourier transform $\tilde{u}(\xi)$, then its restriction on $\hbar\mathbb{T}^m$ and periodically continue it onto a whole $\mathbb{R}^m$. Further we apply the inverse discrete Fourier transform

$$(F^{-1}_d \tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^m}\int\limits_{\hbar\mathbb{T}^m} e^{-i\tilde{x}\cdot\xi}\tilde{u}_d(\xi)d\xi, \quad \tilde{x} \in h\mathbb{Z}^m$$

and obtain a discrete function which is denoted by $(Q_h u)(\tilde{x}), \tilde{x} \in h\mathbb{Z}^m$.

Further, we introduce under consideration the following discrete equation

$$(A_d u_d)(\tilde{x}) = (Q_h(\ell v))(\tilde{x}), \quad \tilde{x} \in D_d, \tag{3}$$

Definition 5. *A discrete solution for the equation (2) is called a solution of the equation (3) if it exists.*

Let $S(\mathbb{R}^m)$ be the Schwartz class of infinitely differentiable rapidly decreasing at infinity functions; this class is dense in every space $H^s(\mathbb{R}^m)$ [3]. If we will put some conditions on the symbol $A(\xi)$ then we can compare discrete and continuous solutions. Moreover, we suppose that $|æ - s| < 1/2$ and the equation (2) is uniquely solvable in the space $H^s(D)$ for an arbitrary right hand side $v \in H^{s-\alpha}_0(D)$, and $A_\pm(\xi)$ are elements of factorization in Eskin's sense [3]. Then we have the following result.

Theorem 4. *If the symbol $A(\xi)$ satisfies the condition and is infinitely differentiable in $\mathbb{R}^m$ with the factors $A_\pm(\xi)$, u is a solution of the equation (2), u_d is a solution of the equation (3) then for $v \in S(\mathbb{R}^m)$ we have the following error estimate*

$$|u(\tilde{x}) - u_d(\tilde{x})| \le Ch^\beta, \quad \forall \tilde{x} \in D_d,$$

for arbitrary $\beta > 0$.

CONCLUSION

Of course these studies are initial results in this direction. More complicated situations related to boundary value problems [9] as well to a conical singularity[11, 12] require separate studying.

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REFERENCES

[1] M. Taylor, *Pseudodifferential Operators* (Princeton University Press, Princeton, 1981).
[2] F. Treves, *Introduction to Pseudodifferential Operators and Fourier Integral Operators* (Springer, New York, 1980).
[3] G. Eskin, *Boundary Value Problems for Elliptic Pseudodifferential Equations* (AMS, Providence, 1981).
[4] A. V. Vasil'ev and V. B. Vasil'ev, Differ. Equ. **51**, 652–660 (2015).
[5] A. Vasil'ev and V. Vasil'ev, Doklady Math. **92**, 585–589 (2015).
[6] A. Vasilyev and V. Vasilyev, "Discrete singular integrals in a half-space," in *Current Trends in Analysis and Its Applications*, Trends in Mathematics. Research Perspectives, edited by V. Mityushev and M. Ruzhansky (Birkhauser, Basel, 2015), pp. 663–670.
[7] A. V. Vasil'ev and V. B. Vasil'ev, Math. Model. Anal. **23**, 492–506 (2018).
[8] V. B. Vasilyev, "Discreteness, periodicity, holomorphy, and factorization," in *Integral Methods in Science and Engineering. V.1. Theoretical Technique*, edited by C. Constanda, M. D. Riva, P. Lamberti, and P. Musolino (Birkhauser/Springer, Cham, 2017), pp. 315–324.
[9] V. Vasilyev, "On discrete boundary value problems," in *International Conference Functional Analysis in Interdisciplinary Applications (FAIA2017)*, AIP Conf. Proc., Vol. 1880, edited by T. Kal'menov and M. Sadybekov (AIP Publishing, Melville, 2017) p. 050010.
[10] A. Vasilyev and V. Vasilyev, "On some discrete boundary value problems in canonical domains," in *Differential and Difference Equations with Applications. ICDDEA, Amadora, Portugal, June 2017*, Springer Proceedings in Mathematics & Statistics, Vol. 230, edited by S. Penelas, T. Caraballo, P. Kloeden, and J. Graef (Birkhauser/Springer, Cham, 2018), pp. 569–579.
[11] V. B. Vasil'ev, *Wave Factorization of Elliptic Symbols: Theory andApplications. Introduction to the Theory of Boundary Value Problems in Non-Smooth Domains* (Kluwer Academic Publishers, Dordrecht–Boston–London, 2000).
[12] V. Vasilyev, "The periodic cauchy kernel, the periodic bochner kernel, discrete pseudo-differential operators," in *Proceedings of the International Conference on Numerical Analysis and Applications (ICNAAM-2016)*, AIP Conf. Proc., Vol. 1863, edited by T. Simos and C. Tsitouras (AIP Publishing, Melville, 2017) p. 140014.