S(e)MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 13, стр. 75–88 (2016) УДК 517.95;534.21
DOI 10.17377/semi.2016.13.006 MSC 35L52, 35Q35, 35Q74

SEISMIC IN COMPOSITE MEDIA: ELASTIC AND POROELASTIC COMPONENTS

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Abstract. In the present paper we consider elastic and poroelastic media having a common interface. We derive the macroscopic mathematical models for seismic wave propagation through these two different media as a homogenization of the exact mathematical model at the microscopic level. They consist of seismic equations for the each component and boundary conditions at the common interface, which separates different media. To do this we use the two-scale expansion method in the corresponding integral identities, defining the weak solution. Our results we illustrate with the numerical implementations of the inverse problem for the simplest model.

Keywords: seismic, two-scale expansion method, full wavefield inversion, numerical simulation.

1. Introduction

The present paper is devoted to a correct description of seismic wave propagation in composite media $Q \subset \mathbb{R}^3$, consisting of the elastic medium $\Omega^{(0)}$, poroelastic medium Ω, which is perforated by a periodic system of pores filled with a fluid, and common interface $S^{(0)}$ between $\Omega^{(0)}$ and Ω (see Fig.1). That is, $Q = \Omega \cup S^{(0)} \cup \Omega^{(0)}$ and $\Omega = \Omega_f \cup \Gamma \cup \Omega_s$, where Ω_s is a solid skeleton, Ω_f is a pore space (liquid domain), and Γ is a common boundary "solid skeleton-liquid domain".

Meirmanov, A.M., Mukhambetzhanov, S.T., Nurtas, M., Seismic in composite media: elastic and poroelastic components.

This research was supported by the Ministry of Education and Science of the Republic of Kazakhstan, agreement №80 from 12.02.2015. (grant 2021/GF4).

Received November, 29, 2015, published February, 16, 2016.

The structure of the heterogeneous medium Q is too complicated and makes hard a numerical simulation of seismic waves propagation in multiscale media. The main difficulty here is a presence of both components (solid and liquid) in each sufficiently small subdomain of Q. It requires to change the governing equations (from Lame's equations to the Stokes equations) at the scale of some tens microns.

There are two basic methods to describe physical processes in such media: the phenomenological method and the asymptotical one which is based on the upscaling approaches. The phenomenological approach for waves propagation through a poroelastic medium ([4], [5]) leads, in particular, to Biot model ([1]-[3]). It based on the system of axioms (relations between the parameters of the medium), which define the given physical process. But, there can be another system of axioms defining the same process. Thus, it is necessary choose the correct authenticity criterion of the mathematical description of the process. It can be, for example, the physical experiment. As a rule, each phenomenological model contains some set of phenomenological constants. Therefore, one can achieve agreement between the suggested theory and selected series of experiments changing these parameters.

The second method, suggested by R. Burridge and J. Keller [6] and E. Sanchez-Palencia [7], based on the homogenization.

It consists of:

1) an exact description of the process at the microscopic level based on the fundamental laws of continuum mechanics,

and

2) the rigorous homogenization of the obtained mathematical model.

To explain the method we consider a characteristic function $\chi_0(\boldsymbol{x})$ of the pore space Ω_f. Let L is the characteristic size of the physical domain in consideration, τ is the characteristic time of the physical process, ρ^0 is the mean density of water, and g is acceleration due gravity

In dimensionless variables

$$\boldsymbol{x} \to \frac{\boldsymbol{x}}{L},\ \boldsymbol{w} \to \alpha_\tau \frac{\boldsymbol{w}}{L},\ t \to \frac{t}{\tau},\ \boldsymbol{F} \to \frac{\boldsymbol{F}}{g},\ \rho \to \frac{\rho}{\rho^0},$$

the dynamic system for the displacements $\boldsymbol{w}$ and pressure p of the medium takes the form [6, 7, 8]:

$$\varrho \frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \nabla \cdot \mathbb{P} + \varrho \boldsymbol{F}, \tag{1}$$

$$\mathbb{P} = \chi_0\, \alpha_\mu\, \mathbb{D}(x, \frac{\partial \boldsymbol{w}}{\partial t}) + (1-\chi_0)\alpha_\lambda\, \mathbb{D}(x, \boldsymbol{w}) + \big(\chi_0 \alpha_\nu\, (\nabla \cdot \frac{\partial \boldsymbol{w}}{\partial t}) - p\big)\, \mathbb{I}, \tag{2}$$

$$p + \alpha_p \nabla \cdot \boldsymbol{w} = 0. \tag{3}$$

Equations (1)-(3) are understood in the sense of distributions as corresponding integral identities. They are equivalent to the Stokes equations

$$\varrho_f \frac{\partial \boldsymbol{v}}{\partial t} = \nabla \cdot \mathbb{P}_f + \varrho_f \boldsymbol{F},\ \frac{\partial p}{\partial t} + \alpha_{p,f} \nabla \cdot \boldsymbol{v} = 0, \tag{4}$$

$$\mathbb{P}_f = \alpha_\mu\, \mathbb{D}(x, \boldsymbol{v}) + \big(\alpha_\nu\, (\nabla \cdot \boldsymbol{v}) - p\big)\, \mathbb{I} \tag{5}$$

for the velocity $\boldsymbol{v} = \dfrac{\partial \boldsymbol{w}}{\partial t}$ and pressure p in the pore space Ω_f and the Lame equations

$$\varrho_s \frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \nabla \cdot \mathbb{P}_s + \varrho_s \boldsymbol{F}, \;\; p + \alpha_{p,s} \nabla \cdot \boldsymbol{w} = 0, \tag{6}$$

$$\mathbb{P}_s = \alpha_\lambda \, \mathbb{D}(x, \boldsymbol{w}) - p\,\mathbb{I} \tag{7}$$

for the solid displacements $\boldsymbol{w}$ and pressure p in Ω_s.

At the common boundary Γ velocities and normal tensions are continuous:

$$\frac{\partial \boldsymbol{w}}{\partial t} = \boldsymbol{v}, \;\; \mathbb{P}_s \cdot \boldsymbol{n} = \mathbb{P}_f \cdot \boldsymbol{n}. \tag{8}$$

Here $\boldsymbol{n}$ is a unit normal to Γ.

In (1)-(8) $\mathbb{D}(x, \boldsymbol{u}) = \dfrac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^*)$ is the symmetric part of $\nabla \boldsymbol{u}$, $\mathbb{I}$ is a unit tensor, $\boldsymbol{F}$ is a given vector of distributed mass forces,

$$\alpha_p = \alpha_{p,f} \chi_0 + \alpha_{p,s}(1 - \chi_0), \quad \varrho = \varrho_f \, \chi_0 + \varrho_s \, (1 - \chi_0),$$

$$\alpha_\tau = \frac{L}{g\tau^2}, \quad \alpha_\mu = \frac{2\mu}{\alpha_\tau \, \tau L g \, \rho^0}, \quad \alpha_\lambda = \frac{2\lambda}{\alpha_\tau \, L g \, \rho^0},$$

$$\alpha_\nu = \frac{2\nu}{\alpha_\tau \, \tau L g \, \rho^0}, \quad \alpha_{p,f} = \frac{\varrho_f \, c_f^2}{\alpha_\tau \, L g}, \quad \alpha_{p,s} = \frac{\varrho_s \, c_s^2}{\alpha_\tau \, L g},$$

μ is the dynamic viscosity, ν is the bulk viscosity, λ is the elastic constant, ϱ_f and ϱ_s are the respective mean dimensionless densities of the liquid in pores and the solid skeleton, correlated with the mean density of water ρ^0, and c_f and c_s are the speed of compression sound waves in the pore liquid and in the solid skeleton respectively. The mathematical model (1) – (3) can not be useful for practical needs, since the

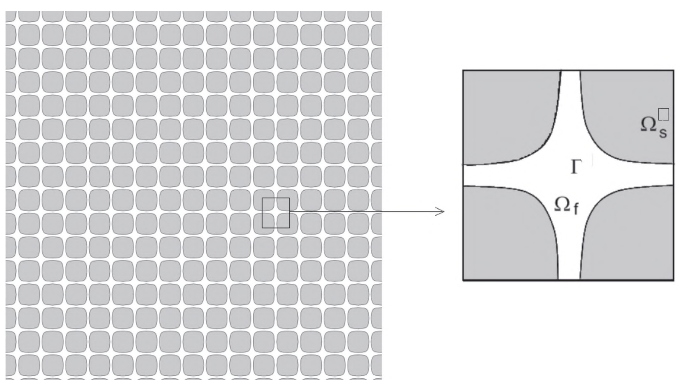

FIGURE 1. The pore structure

function χ_0 changes its value from 0 to 1 on the scale of a few microns. Fortunately, the system possesses a natural small parameter $\varepsilon = \dfrac{l}{L}$, where l is the average size of pores. Thus, the most suitable way to get a practically significant mathematical model, which approximate (1) – (3), is a homogenization or upscaling. That is, we suppose the ε-periodicity of the solid skeleton, let ε to be variable, and look for the limit in (1) – (3) as $\varepsilon \to 0$.

There are different homogenized (limiting) systems, depending on of α_μ, α_λ, ... Some of these numbers might be small and some might be large. We may represent them as a power of ε, or as functions depending on ε.

Let

$$\mu_0 = \lim_{\varepsilon\searrow 0} \alpha_\mu(\varepsilon), \quad \nu_0 = \lim_{\varepsilon\searrow 0} \alpha_\nu(\varepsilon), \quad \lambda_0 = \lim_{\varepsilon\searrow 0} \alpha_\lambda(\varepsilon),$$
$$c^2_{f,0} = \lim_{\varepsilon\searrow 0} \alpha_{p,f}(\varepsilon), \quad c^2_{s,0} = \lim_{\varepsilon\searrow 0} \alpha_{p,s}(\varepsilon),$$
$$\mu_1 = \lim_{\varepsilon\searrow 0} \frac{\alpha_\mu}{\varepsilon^2}, \quad \lambda_1 = \lim_{\varepsilon\searrow 0} \frac{\alpha_\lambda}{\varepsilon^2}.$$

It is clear that the choice of these limits depend on our willing. For example, for $\varepsilon = 10^{-2}$ and $\alpha = 2\cdot 10^{-1}$ we may state that $\alpha = 2\cdot\varepsilon^{-\frac{1}{2}}$, or $\alpha = 0.02\cdot\varepsilon^0$. It is usual procedure when we neglect some terms in differential equations with small coefficients and get more simple equations, still describing the physical process.

The detailed analyses of all possible limiting regimes has been done in [8, 9].

In order to describe the seismic in two different media (elastic and poroelastic), having a common interface we must chose one of the two methods discussed above. The first method suggests only some guesses, while the second method has a clear algorithm for the derivation of the boundary conditions. That is why we choose here the second method.

We derive new seismic equations in each component (elastic and poroelastic) and the boundary conditions on the common boundary. For these boundary conditions the very little is known and only for the liquid filtration (see for example [10]).

For three different sets of μ_0, λ_0, ... for each component we derive three different mathematical models, which describe the process with different degrees of approximation.

We start with the integral identities, defining the weak solution $\boldsymbol{w}^\varepsilon$ and p^ε, and use the two-scale expansion method [11, 12], when we look for the solution in the form

$$\boldsymbol{w}^\varepsilon(\boldsymbol{x},t) = \boldsymbol{w}(\boldsymbol{x},t) + \boldsymbol{W}_0(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}) + \varepsilon\,\boldsymbol{W}_1(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}) + o(\varepsilon),$$
$$p^\varepsilon(\boldsymbol{x},t) = p(\boldsymbol{x},t) + P_0(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}) + \varepsilon\,P_1(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}) + o(\varepsilon)$$

with 1-periodic in the variable $\boldsymbol{y}$ functions $\boldsymbol{W}_i(\boldsymbol{x},t,\boldsymbol{y})$, $P_i(\boldsymbol{x},t,\boldsymbol{y})$, $i = 0,1,...$

This method is rather heuristic and may lead to the wrong answer. But our guesses are based upon the strong theory, suggested by G. Nguetseng [13, 14]. For the rigorous derivation of seismic equations in poroelastic media, which dictate the correct two-scale expansion, see [8].

Finally, to calculate limits as $\varepsilon\to 0$ in corresponding integral identities, we apply the well-known result

$$\lim_{\varepsilon\to 0}\int_\Omega F(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon},t)dxdt = \int_\Omega \big(\int_Y F(\boldsymbol{x},\boldsymbol{y},t)dy\big)dxdt \tag{9}$$

for any smooth 1-periodic in the variable $\boldsymbol{y}\in Y$ function $F(\boldsymbol{x},\boldsymbol{y},t)$.

2. The statement of the problem

For the sake of simplicity we suppose that $Q = \{\boldsymbol{x} = (x_1,x_2,x_3)\in\mathbb{R}^3 : x_3 > 0\}$, $\Omega^{(0)} = \{\boldsymbol{x} = (x_1,x_2,x_3)\in\mathbb{R}^3 : 0 < x_3 < H\}$, $\Omega = \{\boldsymbol{x} = (x_1,x_2,x_3)\in\mathbb{R}^3 : x_3 > H\}$, $\boldsymbol{F} = 0$, and

$$\alpha_{p,f} = \bar{c}^2_f,\ \alpha_{p,s} = \bar{c}^2_s.$$

Let Y be a unit cube in $\mathbb{R}^3$, $Y = Y_f \bigcup \gamma \bigcup Y_s$. We assume that pore space Ω_f^ε is a periodic repetition in Ω of the elementary cell εY_f, the solid skeleton Ω_s^ε is a periodic repetition in Ω of the elementary cell εY_s, and the boundary Γ^ε between a pore space and a solid skeleton is a periodic repetition in Ω of the boundary $\varepsilon\gamma$.

The detailed description of the sets Y_f and Y_s is done in [8].

Due to these suppositions

$$\chi_0(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x}) = \chi(\frac{\boldsymbol{x}}{\varepsilon}),$$

where $\chi(\boldsymbol{y})$ is a 1-periodic function such that $\chi(\boldsymbol{y}) = 1$ for $\boldsymbol{y} \in Y_f$ and $\chi(\boldsymbol{y}) = 0$ for $\boldsymbol{y} \in Y_s$.

For a fixed $\varepsilon > 0$ the displacement vector $\boldsymbol{w}^\varepsilon$ and pressure p^ε satisfy Lame's system

$$\varrho_s^{(0)} \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2} = \nabla \cdot \mathbb{P}_s^{(0)}, \ p^\varepsilon + \bar{c}_{s,0}^2 \nabla \cdot \boldsymbol{w}^\varepsilon = 0, \tag{10}$$

$$\mathbb{P}_s^{(0)} = \alpha_\lambda^{(0)} \, \mathbb{D}(x, \boldsymbol{w}^\varepsilon) - p^\varepsilon \, \mathbb{I} \tag{11}$$

in the domain $\Omega^{(0)}$ for $t > 0$, and the system (1)-(3) with $\chi_0 = \chi^\varepsilon(\boldsymbol{x})$, $\varrho = \varrho^\varepsilon = \varrho_f \, \chi^\varepsilon + \varrho_s \, (1 - \chi^\varepsilon)$, and $\alpha_p = \alpha_p^\varepsilon = \alpha_{p,f}\chi^\varepsilon + \alpha_{p,s}(1 - \chi^\varepsilon)$ in the domain Ω for $t > 0$.

On the common boundary $S^{(0)} = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = H\}$ the displacement vector and normal tensions are continuous:

$$\lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}^0 \\ \boldsymbol{x} \in \Omega^{(0)}}} \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t) = \lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}^0 \\ \boldsymbol{x} \in \Omega}} \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t), \ \boldsymbol{x}^0 \in S^{(0)}, \tag{12}$$

$$\lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}^0 \\ \boldsymbol{x} \in \Omega^{(0)}}} \mathbb{P}^{(0)}(\boldsymbol{x}, t) \cdot \boldsymbol{e}_3 = \lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}^0 \\ \boldsymbol{x} \in \Omega}} \mathbb{P}(\boldsymbol{x}, t) \cdot \boldsymbol{e}_3, \ \boldsymbol{x}^0 \in S^{(0)}, \tag{13}$$

where $\boldsymbol{e}_3 = (0, 0, 1)$.

The problem is completed with the boundary condition

$$\mathbb{P}_s^{(0)} \cdot \boldsymbol{e}_3 = -p^0(\boldsymbol{x}', t)\boldsymbol{e}_3, \ \boldsymbol{x}' = (x_1, x_2) \tag{14}$$

on the outer boundary $S = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$ for $t > 0$ and homogeneous initial conditions

$$\boldsymbol{w}^\varepsilon(\boldsymbol{x}, 0) = \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}(\boldsymbol{x}, 0) = 0. \tag{15}$$

Let $\varsigma(\boldsymbol{x})$ be the characteristic function of the domain Ω and

$$\tilde{\varrho}^\varepsilon = (1 - \varsigma)\varrho_s^{(0)} + \varsigma \, \varrho^\varepsilon, \ \tilde{\alpha}_p^\varepsilon = (1 - \varsigma)\bar{c}_{s,0}^2 + \varsigma \, \alpha_p^\varepsilon.$$

Then the above formulated problem takes the form

$$\tilde{\varrho}^\varepsilon \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2} = \nabla \cdot \big((1 - \varsigma)\mathbb{P}_s^{(0)} + \varsigma\mathbb{P}\big), \tag{16}$$

$$p^\varepsilon + \tilde{\alpha}_p^\varepsilon \nabla \cdot \boldsymbol{w}^\varepsilon = 0, \tag{17}$$

$$\mathbb{P} = \chi^\varepsilon \, \alpha_\mu \, \mathbb{D}(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}) + (1 - \chi^\varepsilon)\alpha_\lambda \, \mathbb{D}(x, \boldsymbol{w}^\varepsilon) - \big(\chi^\varepsilon \frac{\alpha_\nu}{\bar{c}_f^2} \frac{\partial p^\varepsilon}{\partial t} + p^\varepsilon\big) \, \mathbb{I}, \tag{18}$$

where in (18) we have used the consequence of (17) in the form

$$\varsigma\,\chi^{\varepsilon}\alpha_{\nu}\,(\nabla\cdot\frac{\partial \boldsymbol{w}^{\varepsilon}}{\partial t})=-\varsigma\,\chi^{\varepsilon}\frac{\alpha_{\nu}}{\bar{c}_f^2}\frac{\partial p^{\varepsilon}}{\partial t}.$$

Equation (16) is understood in the sense of distributions. That is, for any smooth functions $\boldsymbol{\varphi}$ with a compact support in $\overline{Q}$ the following integral identity

$$\int_{Q_T}\Big(\tilde{\varrho}^{\,\varepsilon}\frac{\partial^2\boldsymbol{w}^{\varepsilon}}{\partial t^2}\cdot\boldsymbol{\varphi}+\big((1-\varsigma)\mathbb{P}_s^{(0)}+\varsigma\mathbb{P}\big):\mathbb{D}(x,\boldsymbol{\varphi})+\nabla\cdot(p^0\boldsymbol{\varphi})\Big)dxdt=0 \tag{19}$$

holds true. We call such solution the *weak solution*.

In (19) $Q_T=Q\times(0,T)$ and the convolution $\mathbb{A}:\mathbb{B}$ of two tensors $\mathbb{A}=(A_{ij})$ and $\mathbb{B}=(B_{ij})$ is defined as $\mathbb{A}:\mathbb{B}=\mathrm{tr}(\mathbb{A}\cdot\mathbb{B})=\sum\limits_{i,j=1}^{3}A_{ij}B_{ji}$.

Using standard methods one can prove that for any positive $\varepsilon>0$ and given smooth function p^0 there exists a unique weak solution of the problem (16)-(18) which makes sense to the integral identity (19).

We look for the limit of the weak solutions for the following case:

(**I**) $\mu_0=\lambda_0=\lambda_0^{(0)}=0,\ \mu_1=\lambda_1=\infty,\ 0\leqslant\nu_0<\infty,\ \lambda_0^{(0)}=\lim\limits_{\varepsilon\to 0}\alpha_{\lambda}^{(0)}$.

3. Homogenization: case (I)

According to [9], the two-scale expansion for the weak solution of the problem (16)-(18) under conditions (**I**) has the form

$$\boldsymbol{w}^{\varepsilon}(\boldsymbol{x},t)=\boldsymbol{w}(\boldsymbol{x},t)+o(\varepsilon),\ p^{\varepsilon}(\boldsymbol{x},t)=p(\boldsymbol{x},t)+o(\varepsilon), \tag{20}$$

where $\lim\limits_{\varepsilon\to 0}o(\varepsilon)=0$.

The substitution (20) into (19) results in the integral identity

$$\int_{\Omega_T}\Big(\big(\chi(\frac{\boldsymbol{x}}{\varepsilon})\varrho_f+\big(1-\chi(\frac{\boldsymbol{x}}{\varepsilon})\big)\varrho_s\big)\frac{\partial^2\boldsymbol{w}}{\partial t^2}\cdot\boldsymbol{\varphi}-(\chi(\frac{\boldsymbol{x}}{\varepsilon})\frac{\alpha_{\nu}}{\bar{c}_f^2}\frac{\partial p}{\partial t}+p)\nabla\cdot\,\boldsymbol{\varphi}\Big)dxdt$$
$$+\int_{Q_T}\nabla\cdot(p^0\boldsymbol{\varphi})dxdt+\int_{\Omega_T^{(0)}}\big(\varrho_s^{(0)}\frac{\partial^2\boldsymbol{w}}{\partial t^2}\cdot\boldsymbol{\varphi}-p\,(\nabla\cdot\,\boldsymbol{\varphi})\big)dxdt=o(\varepsilon). \tag{21}$$

Now we use (9) and after the limit in (21) as $\varepsilon\to 0$ arrive at the following integral identity

$$\int_{\Omega_T}\big(\hat{\varrho}\frac{\partial^2\boldsymbol{w}}{\partial t^2}\cdot\boldsymbol{\varphi}-(m\frac{\nu_0}{\bar{c}_f^2}\frac{\partial p}{\partial t}+p)\nabla\cdot\,\boldsymbol{\varphi}\big)dxdt+\int_{Q_T}\nabla\cdot(p^0\boldsymbol{\varphi})dxdt$$
$$+\int_{\Omega_T^{(0)}}\big(\varrho_s^{(0)}\frac{\partial^2\boldsymbol{w}}{\partial t^2}\cdot\boldsymbol{\varphi}-p\,(\nabla\cdot\,\boldsymbol{\varphi})\big)dxdt=0, \tag{22}$$

where $\hat{\varrho}=m\varrho_f+(1-m)\varrho_s$ and $m=\int_Y\chi(\boldsymbol{y})dy$.

Next we rewrite (17) as

$$\Big(\frac{(1-\varsigma)}{\bar{c}_{s,0}^2}+\frac{\varsigma\,\chi(\frac{\boldsymbol{x}}{\varepsilon})}{\bar{c}_f^2}+\frac{\varsigma\,\big(1-\chi(\frac{\boldsymbol{x}}{\varepsilon})\big)}{\bar{c}_s^2}\Big)p^{\varepsilon}+\nabla\cdot\,\boldsymbol{w}^{\varepsilon}=0, \tag{23}$$

multiply the result by a smooth function $\psi(\boldsymbol{x},t)$ with a compact support in Q, and integrate by part over domain Q_T:

$$\int_{Q_T}\Big(\psi\,\big(\frac{(1-\varsigma)}{\bar{c}^2_{s,0}}+\frac{\varsigma\,\chi(\frac{\boldsymbol{x}}{\varepsilon})}{\bar{c}^2_f}+\frac{\varsigma\,\big(1-\chi(\frac{\boldsymbol{x}}{\varepsilon})\big)}{\bar{c}^2_s}\big)p^{\,\varepsilon}-\nabla\,\psi\cdot\boldsymbol{w}^{\varepsilon}\Big)dxdt=0. \tag{24}$$

As above, we substitute (20) into (24) and pass to the limit as $\varepsilon\to 0$:

$$\int_{Q_T}\Big(\psi\,\big(\frac{(1-\varsigma)}{\bar{c}^2_{s,0}}+\frac{\varsigma\,m}{\bar{c}^2_f}+\frac{\varsigma\,(1-m)}{\bar{c}^2_s}\big)p-\nabla\,\psi\cdot\boldsymbol{w}\Big)dxdt=0. \tag{25}$$

Integral identities (22) and (25), completed with initial conditions

$$\boldsymbol{w}(\boldsymbol{x},0)=\frac{\partial\boldsymbol{w}}{\partial t}(\boldsymbol{x},0)=0, \tag{26}$$

form **mathematical model (I)** of seismic in composite media.

In fact, these identities contain the differential equations in Ω and $\Omega^{(0)}$ and the boundary conditions on S and $S^{(0)}$.

Let $\boldsymbol{\varphi}$ be a smooth function with a compact support in $\Omega^{(0)}$. Rewriting (22) as

$$\int_{\Omega^{(0)}_T}(\varrho^{(0)}_s\frac{\partial^2\boldsymbol{w}}{\partial t^2}+\nabla\,p)\cdot\boldsymbol{\varphi}dxdt=0$$

and using the arbitrary choice of $\boldsymbol{\varphi}$ we conclude that

$$\varrho^{(0)}_s\frac{\partial^2\boldsymbol{w}}{\partial t^2}=-\nabla\,p \tag{27}$$

in the domain $\Omega^{(0)}$ for $t>0$.

For functions $\boldsymbol{\varphi}$ with a compact support in Ω (22) implies

$$\hat{\varrho}\frac{\partial^2\boldsymbol{w}}{\partial t^2}=-\nabla(p+m\frac{\nu_0}{\bar{c}^2_f}\frac{\partial p}{\partial t}),\quad\hat{\varrho}=m\varrho_f+(1-m)\varrho_s \tag{28}$$

in the domain Ω for $t>0$.

Now, if we choose $\boldsymbol{\varphi}=(\varphi_1,\varphi_2,\varphi_3)$ with a compact support in Q and $\varphi_3(\boldsymbol{x},t)\neq 0$ for $\boldsymbol{x}\in S^{(0)}$, then the integration by parts in (22) together with (27) and (28) result in

$$\int_{S^{(0)}_T}\big(p^- -(p^+ +m\frac{\nu_0}{\bar{c}^2_f}\frac{\partial p^+}{\partial t})\big)\varphi_3dxdt=0,$$

where

$$p^-(x_1,x_2,t)=p(x_1,x_2,H-0,t),\quad p^+(x_1,x_2,t)=p(x_1,x_2,H+0,t).$$

Therefore

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega^{(0)}}}p(\boldsymbol{x},t)=\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega}}\big(p(\boldsymbol{x},t)+m\frac{\nu_0}{\bar{c}^2_f}\frac{\partial p}{\partial t}(\boldsymbol{x},t)\big),\quad\boldsymbol{x}^0\in S^{(0)}. \tag{29}$$

Finally, for functions $\boldsymbol{\varphi}$ with a compact support in $\Omega^{(0)}$ and $\varphi_3(\boldsymbol{x},t)\neq 0$ for $\boldsymbol{x}\in S$ the integration by parts in (22) together with (27) result in

$$\int_{S_T}(p-p^0)\varphi_3dxdt=0,$$

or

$$p(\boldsymbol{x},t)=p^0(\boldsymbol{x},t),\quad\boldsymbol{x}\in S. \tag{30}$$

In the same way as above, it can be shown that (25) implies continuity equations

$$\frac{1}{\bar{c}^2_{s,0}}\,p + \nabla\cdot\boldsymbol{w} = 0 \tag{31}$$

and

$$\Big(\frac{m}{\bar{c}^2_f} + \frac{(1-m)}{\bar{c}^2_s}\Big)\,p + \nabla\cdot\boldsymbol{w} = 0 \tag{32}$$

in the domains $\Omega^{(0)}$ and Ω respectively, and the boundary condition

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega^{(0)}}} \boldsymbol{e}_3\cdot\boldsymbol{w}(\boldsymbol{x},t) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega}} \boldsymbol{e}_3\cdot\boldsymbol{w}(\boldsymbol{x},t),\ \ \boldsymbol{x}^0\in S^{(0)} \tag{33}$$

on the common boundary $S^{(0)}$.

Differential equations (27), (28), (31), and (32), boundary conditions (29), (30), and (33), and initial conditions (26) constitute the **mathematical model (I)** in its differential form.

4. One dimensional model for the case (I): numerical implementations

4.1. **Direct problem.** For the sake of simplicity we consider the space, which consists of the following subdomains: $\Omega_1 = \{x\in\mathbb{R} : 0 < x < H_1\}$, $\Omega_2 = \{x\in\mathbb{R} : H_1 < x < H_2\}$, and $\Omega_3 = \{x\in\mathbb{R} : x > H_2\}$. Differential equations (27), (28), (31), and (32) result in

$$\frac{1}{\hat{c}^2(x)}\frac{\partial^2 p}{\partial t^2} = div\Big(\frac{1}{\hat{\rho}(x)}\nabla\big(p + m\frac{\nu_0}{\bar{c}^2_f}\frac{\partial p}{\partial t}\big)\Big)$$

where $\dfrac{1}{\hat{c}^2} = \dfrac{m}{\bar{c}^2_f} + \dfrac{(1-m)}{\bar{c}^2_s}$ and $\hat{\rho} = m\rho_f + (1-m)\rho_s$ are respectively average wave propagation velocity and average density of the medium.

Applying now the Fourier transformation we arrive at

$$\frac{d^2\hat{P}}{dX^2} + \frac{\hat{\rho}\omega^2}{(1-\frac{m\nu_0}{c_f^2}i\omega)\hat{c}^2}\hat{P} = 0 \tag{34}$$

where $\hat{P}(x,\omega)$-the pressure obtained after Fourier transform.

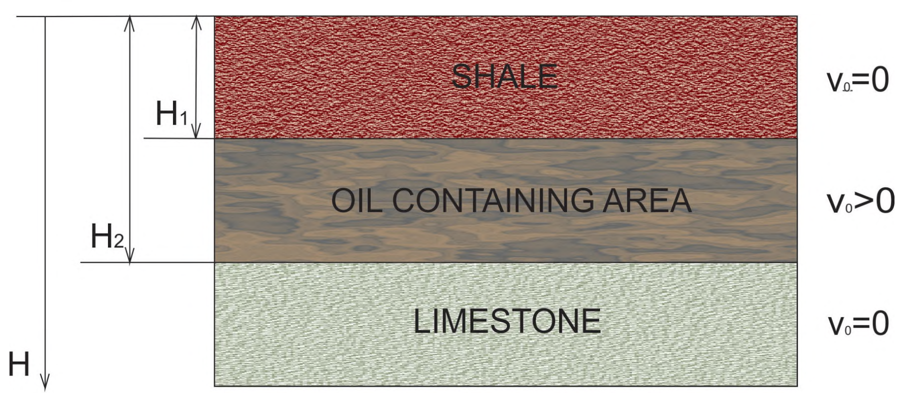

Figure 2. Scheme of arrangement of layers

Depending on the exact physical properties, the sedimentary rock zone is divided into three subdomains. The value of the geometry of pores, viscosity of fluid, density of rock, and velocity of seismic wave considered in each layers to be different. In the experiment in order to get numerical solution, it's assumed that the first medium is a shale, the second medium is oil saturated sandstone, and the third medium is a limestone (see Fig.2).

Parameter medium	Density (kg/m^3)	Elastic wave velocity (m/s)
Oil	850	1330
Shale	1600	1500
Sandstone	2250	2500
Limestone	2700	3000

TABLE 1. The average values of density and sound velocity.

Let us suppose that there is a plane wave which propagates from ∞. Then the general solution of equation (34) for $-\infty < X \leq H_1$ in the case $\nu_0 = 0$ is written down as:

$$\hat{P}_1 = \exp\left\{\frac{i\omega\sqrt{\hat{\rho}_1}}{\hat{c}_1}x\right\} + A_2\exp\left\{\frac{-i\omega\sqrt{\hat{\rho}_1}}{\hat{c}_1}x\right\}. \tag{35}$$

The general solution of equation (34) for $H_1 \leq x < H_2$ in the case $\nu_0 > 0$ is represented as:

$$\hat{P}_2 = B_1\exp\left\{\frac{i\omega\sqrt{\hat{\rho}_2}}{\hat{c}_2\sqrt{1-\frac{m\nu_0}{c_f^2}i\omega}}x\right\} + B_2\exp\left\{\frac{-i\omega\sqrt{\hat{\rho}_2}}{\hat{c}_2\sqrt{1-\frac{m\nu_0}{c_f^2}i\omega}}x\right\}. \tag{36}$$

Finally the general solution for $x \geq H_2$ in the case $\nu_0 = 0$ will be the following:

$$\hat{P}_3 = D_1\exp\left\{i\omega\frac{\sqrt{\hat{\rho}_3}}{\hat{c}_3}x\right\}. \tag{37}$$

Continuity condition in contact media will be:

$$[\hat{P}_1 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_1]_{H_{1-0}} = [\hat{P}_2 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_2]_{H_{1+0}} \tag{38}$$

$$\hat{c}_1^2\frac{d}{dx}[\hat{P}_1 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_1]_{H_{1-0}} = \hat{c}_2^2\frac{d}{dx}[\hat{P}_2 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_2]_{H_{1+0}} \tag{39}$$

$$[\hat{P}_2 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_2]_{H_{2-0}} = [\hat{P}_3 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_3]_{H_{2+0}} \tag{40}$$

$$\hat{c}_2^2\frac{d}{dx}[\hat{P}_2 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_2]_{H_{2-0}} = \hat{c}_3^2\frac{d}{dx}[\hat{P}_3 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_3]_{H_{2+0}} \tag{41}$$

These relations are nothing else but the system of linear algebraic equations for the coefficients A_2, B_1, B_2, D_1 which can be easily resolved by any direct method. These coefficients are used in order to construct the solution in time frequency domain and after inverse Fourier transform in time the solution in the time domain can be easily recovered (see Fig.3).

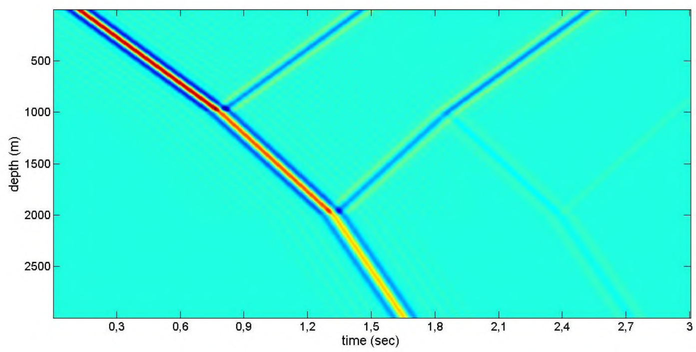

FIGURE 3. Propagation of seismic waves in different layers

4.2. **Inverse problem.** In inverse problem [15] except $\hat{P}(x,\omega)$ the values H_1, H_2, $\hat{c}_2$, ν_0, m are unknown as well. To determine these values one needs some additional information about solution of the direct problem - data of inverse problem. Usually they are given as function $\overline{P}(\omega)$ at $X = 0$. The most widespread way is to search for these values by minimization of the data misfit functional being L_2 norm of the difference of given functions and computed for some current values of unknown parameters. During the numerical simulations, we use the following list of information. For these data, the characteristic dimension of depth $L \sim 3000m$, characteristic time $\tau \sim 0{-}3sec$, $\alpha_\tau \sim 33$, the porosity of the sandstone $m \sim 0.007$, average size of pores $l \sim 30micron$, then $\varepsilon \sim 10^{-10}$, $\mu \sim 5 * 10^{-3}Pa.s$, $\nu_0 \sim 3.8cCt = 3.8 * 10^{-6}m^2/s$, $\lambda \sim 1 * 10^9 Pa$. Therefore

Coefficients	corresponding values(dimensionless)	approximate values(by ε)
The first layer is shale		
$\alpha_{p,s}$	3.63	$\varepsilon^{1/10}$
The second layer is sandstone(oil containing area)		
$\alpha_{p,s}$	14.21	$\varepsilon^{2/10}$
$\alpha_{p,f}$	1.52	$\varepsilon^{1/10}$
α_ν	$3.01 * 10^{-12}$	$\varepsilon^{6/5}$
α_μ	$3.37 * 10^{-12}$	$\varepsilon^{6/5}$
μ_1	$3.37 * 10^{8}$	$\varepsilon^{-4/5}$
α_λ	0.22	$\varepsilon^{1/5}$
λ_1	$0.22 * 10^{20}$	ε^{-2}
The third layer is limestone		
$\alpha_{p,s}$	24.54	$\varepsilon^{2/10}$

TABLE 2. The values of the coefficients used in practice.

$$F_i(H_1^i, H_2^i) = \int_{\omega_1}^{\omega_n} |\hat{P}_i(\omega, H_1^i, H_2^i) - \overline{P}(\omega, H_1, H_2)|^2 d\omega \to 0 \tag{42}$$

$$F_i(H_1^i, \hat{c}_2^i) = \int_{\omega_1}^{\omega_n} |\hat{P}_i(\omega, H_1^i, \hat{c}_2^i) - \overline{P}(\omega, H_1, \hat{c}_2)|^2 d\omega \to 0 \tag{43}$$

$$F_i(H_2^i, \hat{c}_2^i) = \int_{\omega_1}^{\omega_n} |\hat{P}_i(\omega, H_2^i, \hat{c}_2^i) - \overline{P}(\omega, H_2, \hat{c}_2)|^2 d\omega \to 0 \tag{44}$$

Here $\overline{P}(\omega, \ldots\ldots)$ is the given wave fields at $X = 0$, while $\hat{P}(\omega, \ldots\ldots)$ are wave fields computed for some current values of the desired parameters. In our numerical experiments the minimum is searched by the Nelder-Mead technique [17].

4.2.1. *Recovery of H_1 and H_2.* Behavior of the data misfit functional for this statement is represented in Fig.4 and Fig.5. As one can see this functional is convex and has the unique minimum point. Therefore this inverse problem is well resolved.

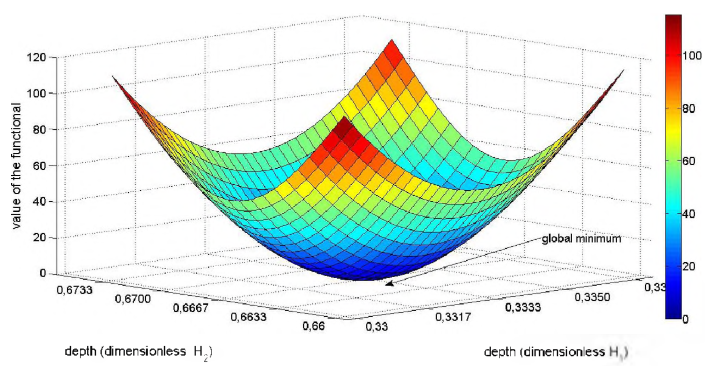

FIGURE 4. Minimization of the functional $F(H_1, H_2)$.

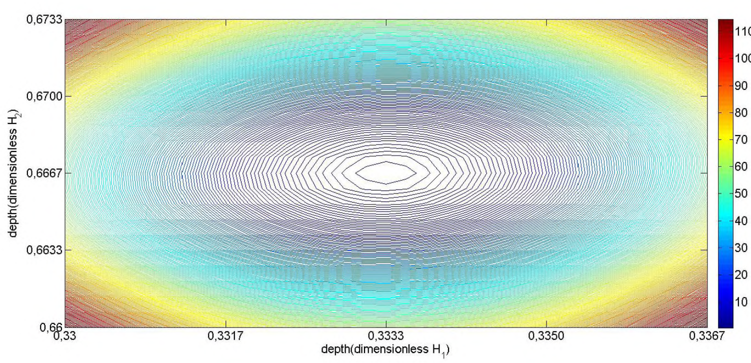

FIGURE 5. Level line of the functional $F(H_1, H_2)$.

4.2.2. *Recovery of H_1 and c_2.* Now we come to the non convex functional and therefore inverse problem may have few solutions (see Fig.6, Fig.7).

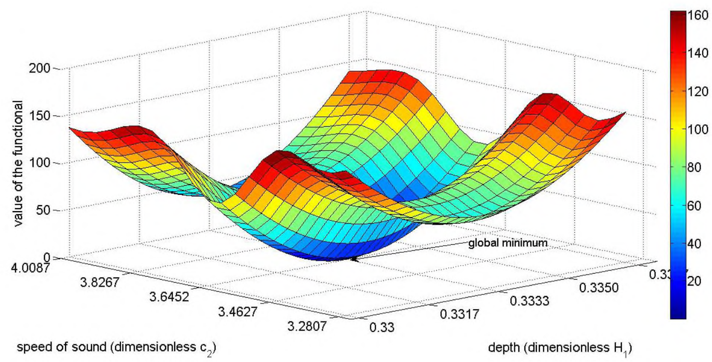

FIGURE 6. Minimization of the functional $F(H_1, \hat{c}_2)$.

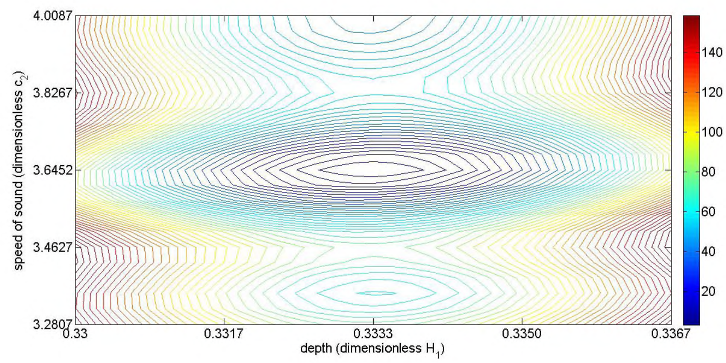

FIGURE 7. Level line functional $F(H_1, \hat{c}_2)$.

4.2.3. *Recovery of H_2 and c_2.* This statement also generates non convex functional, but now it has excellent resolution with respect to H_2 (see Fig.8, Fig.9).

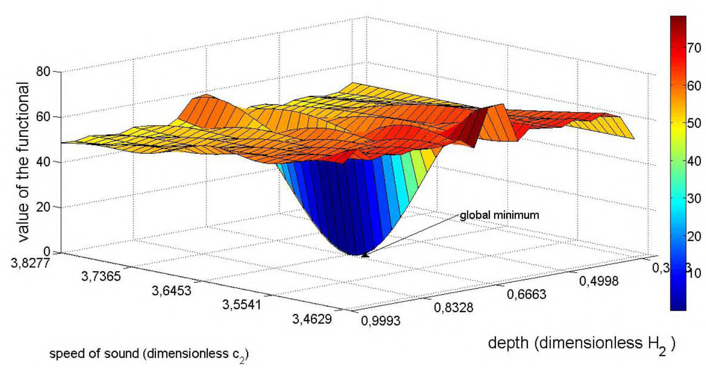

FIGURE 8. Minimization of the functional $F(H_2, \hat{c}_2)$.

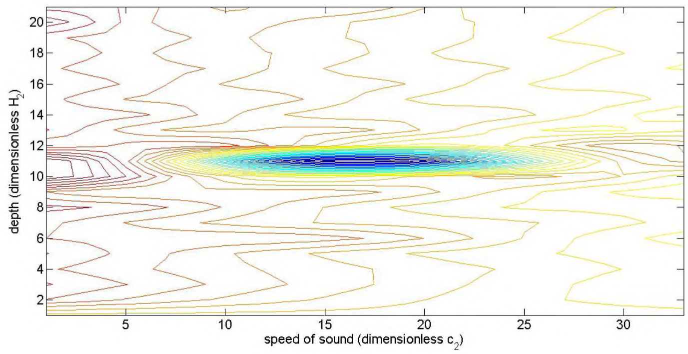

FIGURE 9. Level line functional $F(H_2, \hat{c}_2)$.

Conclusions. In this publication we have shown how to derive mathematical models for composite media using its microstructure. As a rule, there is some set of models depending on given criteria μ_0, $\lambda_0, \ldots$ of the physical process in consideration. For a fixed set of criteria the corresponding model describes some of the main features of the process.

In the paper the simplest inverse problem was dealt with - recovery of elastic parameters of the layer by Nelder-Mead algorithm. In the future we are planning to establish connection upscaling procedure and scattered waves and apply on this base recent developments of true-amplitude imaging on the base of Gaussian beams for both reflected and scattered waves ([18], [19].

Conflict of Interests. The authors declare that there is no conflict of interests regarding the publication of this manuscript.

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