Difference Equations and Boundary Value Problems

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Abstract We study multidimensional difference equations with a continual variable in the Sobolev–Slobodetskii spaces. Using ideas and methods of the theory of boundary value problems for elliptic pseudo-differential equations, we suggest to consider certain boundary value problems for such difference equations. Special boundary conditions permit to prove unique solvability for these boundary value problems in appropriate Sobolev–Slobodetskii spaces.

Keywords Difference equation • Symbol • Factorization • Index • Boundary value problem • Solvability

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1 Introduction

We consider a general difference equation of the type

$$\sum_{|k|=0}^{\infty} a_k(x)u(x+\alpha_k) = v(x), \quad x \in D, \tag{1.1}$$

where $D \subset \mathbb{R}^m$ is a canonical domain like $\mathbb{R}^m, \mathbb{R}^m_\pm = \{x \in \mathbb{R}^m : x = (x_1, \cdots, x_m), \pm x_m > 0\}, C^a_+ = \{x \in \mathbb{R}^m : x_m > a|x'|, x' = (x_1, \cdots, x_{m-1}), a > 0\}$, k is a multi-index, $|k| = k_1 + \cdots + k_m$, $\{\alpha_k\} \subset D$. Equations of a such type have a long history [4, 5, 8] and in general there is no algorithm for solving the Eq. (1.1). If so then any assertion on a solvability of such equations is very important and required. One can

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add that Eq. (1.1) appear in very distinct branches of a science like mathematical biology, technical problems, etc. Also such equations have arisen in studies of the second author [10, 11] related to boundary value problems in a plane corner. One-dimensional case for such equations was considered in [12].

Here we will start from the equation

$$\sum_{|k|=0}^{\infty} a_k u(x+\alpha_k) = v(x), \quad x \in \mathbb{R}^m_+, \tag{1.2}$$

with constant coefficients because further we will try to use a local principle [6] to obtain some results on Fredholm properties of the general Eq. (1.1). We use methods of the theory of boundary value problems for elliptic pseudo-differential equations [1, 9]. For our case of a half-space, these methods are based on the theory of one-dimensional singular integral equations and classical Riemann boundary value problem [2, 3, 7].

2 Spaces, Operators, and Symbols

2.1 Spaces

Let $S(\mathbb{R}^m)$ be the Schwartz class of infinitely differentiable rapidly decreasing at infinity functions and $S'(\mathbb{R}^m)$ be the space of distributions over the space $S(\mathbb{R}^m)$. If $u \in S(\mathbb{R}^m)$, then its Fourier transform is defined by the formula

$$\tilde{u}(\xi) = \int_{\mathbb{R}^m} e^{-ix\cdot\xi} u(x)dx.$$

Definition 2.1. A Sobolev–Slobodetskii space $H^s(\mathbb{R}^m)$, $s \in \mathbb{R}$, consists of functions (distributions) with a finite norm

$$||u||_s = \left(\int_{\mathbb{R}^m} \tilde{u}(\xi)(1+|\xi|)^{2s} d\xi \right)^{1/2}.$$

Let us note $H^0(\mathbb{R}^m) = L_2(\mathbb{R}^m)$.

The space $S(\mathbb{R}^m)$ is a dense subspace in the $H^s(\mathbb{R}^m)$ [1]. The space $H^s(\mathbb{R}^m_+)$ consists of functions from the space $H^s(\mathbb{R}^m)$ which support belongs to $\overline{\mathbb{R}^m_+}$ with induced norm. Also we need the space $H^s_0(\mathbb{R}^m_+)$ which consists of distributions from $S'(\mathbb{R}^m_+)$ admitting a continuation in the whole space $H^s(\mathbb{R}^m)$. A norm in the space $H^s_0(\mathbb{R}^m_+)$ is defined by the formula

$$||u||^+_s = \inf ||lu||_s,$$

where *infimum* is taken from all continuations l.

2.2 Operators

Here we consider difference operators with constant coefficients only of the type

$$\mathcal{D} : u(x) \longmapsto \sum_{|k|=0}^{\infty} a_k u(x + \alpha_k), \tag{2.1}$$

where $\{a_k\} and \{\alpha_k\}$ are given sequences in $\mathbb{R}^m$, and

$$\sum_{|k|=0}^{\infty} |a_k| < +\infty. \tag{2.2}$$

Definition 2.2. An operator $\mathcal{D}$ of the type (2.1) with coefficients a_k satisfying (2.2) is called difference operator with constant coefficients.

Lemma 2.3. *Every operator $\mathcal{D} : H^s(\mathbb{R}^m) \to H^s(\mathbb{R}^m)$ with constant coefficients is a linear bounded operator $\forall\ s \in \mathbb{R}$.*

2.3 Symbols

Definition 2.4. The function

$$\sigma_{\mathcal{D}}(\xi) = \sum_{|k|=0}^{\infty} a_k e^{-i\alpha_k \cdot \xi} \tag{2.3}$$

is called a symbol of the operator $\mathcal{D}$. The symbol $\sigma_{\mathcal{D}}(\xi)$ is called an **elliptic symbol** if $\sigma_{\mathcal{D}}(\xi) \neq 0, \forall\ \xi \in \mathbb{R}^m$.

Evidently under condition (2.2) $\sigma_{\mathcal{D}} \in L_\infty(\mathbb{R}^m)$, but everywhere below we suppose that $\sigma_{\mathcal{D}} \in C(\dot{\mathbb{R}}^m)$ taking into account that $\dot{\mathbb{R}}^m$ is a compactification of $\mathbb{R}^m$.

3 Equations and Factorization

3.1 Equations

We are interested in studying solvability of the Eq. (1.2). It can be written in the operator form

$$(\mathcal{D}u)(x) = v(x), \quad x \in \mathbb{R}^m_+, \tag{3.1}$$

assuming that v is a given function in $\mathbb{R}^m_+$, $v \in H^s_0(\mathbb{R}^m_+)$, the unknown function u is defined in $\mathbb{R}^m_+$, $u \in H^s(\mathbb{R}^m_+)$, and $\{\alpha_k\} \subset \mathbb{R}^m_+$.

By notation, $u_+(x) = u(x)$, lv is an arbitrary continuation of v on $\mathbb{R}^m_+$. Then we put

$$u_-(x) = (lv)(x) - (\mathcal{D}u_+)(x),$$

and see that $u_-(x) = 0, \forall\, x \in \mathbb{R}^m_+$, to explain this notation. Further we rewrite the last equation

$$(\mathcal{D}u_+)(x) + u_-(x) = (lv)(x)$$

and apply the Fourier transform

$$\sigma_{\mathcal{D}}(\xi)\tilde{u}_+(\xi) + \tilde{u}_-(\xi) = \widetilde{lv}(\xi). \tag{3.2}$$

To solve the Eq. (3.2) with an elliptic symbol $\sigma_{\mathcal{D}}(\xi)$, we need to introduce a concept of a factorization. Everywhere below we write $\sigma(\xi)$ instead of $\sigma_{\mathcal{D}}(\xi)$ for a brevity.

3.2 Factorization

Let us denote $\xi = (\xi', \xi_m), \xi' = (\xi_1, \cdots, \xi_{m-1})$.

Definition 3.1. Let $\sigma(\xi)$ be an elliptic symbol. Factorization of elliptic symbol $\sigma(\xi)$ is called its representation in the form

$$\sigma(\xi) = \sigma_+(\xi)\sigma_-(\xi),$$

where factors $\sigma_\pm(\xi)$ admit an analytical continuation in upper and lower complex planes $\mathbb{C}_\pm$ on the last variable ξ_m for almost all $\xi' \in \mathbb{R}^{m-1}$ and $\sigma_\pm(\xi) \in L_\infty(\mathbb{R}^m)$.

Definition 3.2. Index of factorization for the elliptic symbol $\sigma(\xi)$ is called an integer

$$æ = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\arg\sigma(\cdot, \xi_m).$$

Remark 3.3. The index æ is not really depended on ξ' because it is homotopic invariant.

Remark 3.4. It is a principal fact the index of factorization does not correlate with an order of operator. For our case the order of the operator $\mathcal{D}$ is zero in a sense of Eskin's book [1], but the index may be an arbitrary integer. It is essential the index is a **topological barrier** for a solvability.

Proposition 3.5. *If* $æ = 0$*, then for any elliptic symbol* $\sigma(\xi)$*, a factorization*

$$\sigma(\xi) = \sigma_+(\xi)\sigma_-(\xi)$$

exists, and it is unique up to a constant.

This is classical result, see details in [1–3, 7].

4 Solvability and Boundary Value Problems

4.1 Solvability

Everywhere below we will denote $\tilde{H}(D)$ the Fourier image of the space $H(D)$.

Theorem 4.1. *If* $|s| < 1/2$*,* $æ = 0$*, then the Eq. (3.1) has a unique solution* $u \in H^s(\mathbb{R}^m_+)$ *for arbitrary right-hand side* $v \in H^s_0(\mathbb{R}^m_+)$*.*

Proof is a very simple. It is based on properties of the Hilbert transform

$$(H_{\xi'}u)(\xi', \xi_m) = \frac{1}{\pi i} v.p. \int_{-\infty}^{+\infty} \frac{u(\xi', \eta_m)d\eta_m}{\xi_m - \eta_m}$$

which is a linear bounded operator $H^s(\mathbb{R}^m) \to H^s(\mathbb{R}^m)$ for $|s| < 1/2$ [1]. This operator generates two projectors on some spaces consisting of boundary values of analytical functions in $\mathbb{C}_\pm$ on the last variable ξ_m [1–3, 7]

$$\Pi_\pm = 1/2(I \pm H_{\xi'}),$$

so that the representation

$$f = f_+ + f_- \equiv \Pi_+ f + \Pi_- f$$

is unique for arbitrary $f \in H^s(\mathbb{R}^m)$, $|s| < 1/2$. Further after factorization we write the equality (3.2) in the form

$$\sigma_+(\xi)\tilde{u}_+(\xi) + \sigma_-^{-1}(\xi)\tilde{u}_-(\xi) = \sigma_-^{-1}(\xi)\widetilde{lv}(\xi),$$

and else

$$\sigma_+(\xi)\tilde{u}_+(\xi) - (\Pi_+(\sigma_-^{-1} \cdot \widetilde{lv}))(\xi) = (\Pi_-(\sigma_-^{-1} \cdot \widetilde{lv}))(\xi) - \sigma_-^{-1}(\xi)\tilde{u}_-(\xi).$$

So the left-hand side belongs to the space $\widetilde{H}^s(\mathbb{R}^m_+)$ and the left-hand side belongs to the space $\widetilde{H}^s(\mathbb{R}^m_-)$, and these should be zero. Hence

$$\tilde{u}_+(\xi) = \sigma_+^{-1}(\xi)(\Pi_+(\sigma_-^{-1} \cdot \widetilde{lv}))(\xi).$$

It completes the proof. △

4.2 General Solution

Let æ $\in \mathbb{Z}$. First we introduce a function

$$\omega(\xi', \xi_m) = \left(\frac{\xi_m - i|\xi'| - i}{\xi_m + i|\xi'| + i}\right)^{æ},$$

which belongs to $C(\dot{\mathbb{R}}^m)$.

Evidently the functions $z \pm i|\xi'|$ for fixed $\xi' \in \mathbb{R}^{m-1}$ are analytical functions in complex half planes $\mathbb{C}_\pm$. Moreover

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} d \arg \frac{\xi_m - i|\xi'| - i}{\xi_m + i|\xi'| + i} = 1.$$

According to the index property [1–3, 7], a function

$$\omega^{-1}(\xi', \xi_m)\sigma(\xi', \xi_m)$$

has a vanishing index, and it can be factorized

$$\omega^{-1}(\xi', \xi_m)\sigma(\xi', \xi_m) = \sigma_+(\xi', \xi_m)\sigma_-(\xi', \xi_m),$$

so we have

$$\sigma(\xi', \xi_m) = \omega(\xi', \xi_m)\sigma_+(\xi', \xi_m)\sigma_-(\xi', \xi_m),$$

where

$$\sigma_\pm(\xi', \xi_m) = \exp(\Psi^\pm(\xi', \xi_m)), \quad \Psi^\pm(\xi', \xi_m) = \frac{1}{2\pi i} \lim_{\tau \to 0+} \int_{-\infty}^{+\infty} \frac{\ln(\omega^{-1}\sigma)(\xi, \eta_m)d\eta_m}{\xi_m \pm i\tau - \eta_m}.$$

Now the Eq. (3.2), we rewrite in the form

$$(\xi_m + i|\xi'| + i)^{-æ}\sigma_+(\xi)\tilde{u}_+(\xi) + (\xi_m - i|\xi'| - i)^{-æ}\sigma_-^{-1}(\xi)\tilde{u}_-(\xi)$$
$$= (\xi_m - i|\xi'| - i)^{-æ}\sigma_-^{-1}(\xi)\widetilde{lv}(\xi). \tag{4.1}$$

Let us note the right-hand side of the Eq. (4.1) belongs to the space $\widetilde{H}^{s+æ}(\mathbb{R}^m)$. If $|æ + s| < 1/2$, we go to Sect. 4.1.

4.2.1 Positive Case

If $s + æ > 1/2$, we choose a minimal $n \in \mathbb{N}$ so that $0 < s + æ - n < 1/2$. Further we use a decomposition formula for operators $\Pi_\pm$ [1] for $\tilde{f} \in \widetilde{H}^{s+æ}(\mathbb{R}^m)$

$$\Pi_\pm \tilde{f} = \sum_{k=1}^{n} \frac{\Pi' \Lambda_\pm^{k-1}\tilde{f}}{\Lambda_\pm^k} + \frac{1}{\Lambda_\pm^n}\Pi_\pm \Lambda_\pm^n \tilde{f}, \tag{4.2}$$

where

$$\Lambda_\pm(\xi', \xi_m) = \xi_m \pm |\xi'| \pm i, \quad (\Pi'\tilde{f})(\xi') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{f}(\xi', \xi_m)d\xi_m.$$

We rewrite the Eq. (4.1)

$$\sigma_+(\xi)\tilde{w}_+(\xi) + \sigma_-^{-1}(\xi)\tilde{w}_-(\xi) = \tilde{h}(\xi),$$

where $\tilde{w}_\pm(\xi) = (\xi_m \pm i|\xi'| \pm i)^{-æ}\tilde{u}_\pm(\xi), \tilde{h}(\xi) = (\xi_m - i|\xi'| - i)^{-æ}\sigma_-^{-1}(\xi)\widetilde{lv}(\xi)$.

Obviously $\tilde{w}_\pm \in \widetilde{H}^{s+æ}(\mathbb{R}^m_\pm), \tilde{h} \in \widetilde{H}^{s+æ}(\mathbb{R}^m)$. We set $s+æ-n = \alpha, 0 < \alpha < 1/2$. Since $s + æ = n + \alpha > \alpha$ then $\tilde{h} \in \widetilde{H}^{s+æ}(\mathbb{R}^m) \Longrightarrow h \in \widetilde{H}^{\alpha}(\mathbb{R}^m)$. According to Theorem 4.1, we have a solution of the last equation $\tilde{w}_+ \in \widetilde{H}^{\alpha}(\mathbb{R}^m_+)$ in the form

$$\tilde{w}_+(\xi) = \sigma_+^{-1}(\xi)(\Pi_+\tilde{h})(\xi).$$

Thus

$$\tilde{u}_+(\xi) = (\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi)(\Pi_+\tilde{h})(\xi),$$

so that $\tilde{u}_+ \in \widetilde{H}^{\alpha-æ}(\mathbb{R}^m_+)$. Now we apply the formula (4.2) to the expression $\Pi_+\tilde{h}$ and obtain the following representation

$$\tilde{u}_+(\xi) = \sum_{k=1}^{n} \frac{\tilde{c}_k(\xi')}{\sigma_+(\xi)\Lambda_+^{k-æ}(\xi', \xi_m)} + \frac{1}{\sigma_+(\xi)\Lambda_+^{n-æ}(\xi', \xi_m)}(\Pi_\pm\Lambda_+^n\tilde{h})(\xi', \xi_m), \tag{4.3}$$

where $\tilde{c}_k = (\Pi'\Lambda_+^{k-1})\tilde{h}$. It is not hard concluding $\tilde{c}_k \in \widetilde{H}^{s_k}(\mathbb{R}^{m-1})$, $s_k = s + æ - k + 1/2$. So we have the following

Proposition 4.2. *If $s + æ > 1/2$, then for the solution of the Eq. (3.1), the representation (4.3) is valid.*

Note. One can prove that the functions $\tilde{c}_k \in \widetilde{H}^{s_k}(\mathbb{R}^{m-1}) and s_k = s + æ - k + 1/2$ are defined uniquely.

4.2.2 Negative Case

If $s+æ < -1/2$, we choose a polynomial $Q_n(\xi)$ without real zeroes so that $-1/2 < s + æ + n < 0$, and use the equality

$$\sigma_+(\xi)\tilde{w}_+(\xi) + \sigma_-^{-1}(\xi)\tilde{w}_-(\xi) = \tilde{h}(\xi)$$

from Sect. 4.2.1 once again. Since $\tilde{h} \in \widetilde{H}^{s+æ}(\mathbb{R}^m)$, we represent

$$\tilde{h} = Q\Pi_+(Q^{-1}\tilde{h}) + Q\Pi_-(Q^{-1}\tilde{h})$$

because $Q^{-1}\tilde{h} \in \widetilde{H}^{s+æ+n}(\mathbb{R}^m)$. Further we work with the equality

$$\sigma_+(\xi)\tilde{w}_+(\xi) + \sigma_-^{-1}(\xi)\tilde{w}_-(\xi) = Q\Pi_+(Q^{-1}\tilde{h}) + Q\Pi_-(Q^{-1}\tilde{h})$$

or in other words

$$\sigma_+(\xi)\tilde{w}_+(\xi) - Q\Pi_+(Q^{-1}\tilde{h}) = Q\Pi_-(Q^{-1}\tilde{h}) - \sigma_-^{-1}(\xi)\tilde{w}_-(\xi)$$

So the left-hand side belongs to the space $\widetilde{H}^{s+æ}(\mathbb{R}^m_+)$, and the left-hand side belongs to the space $\widetilde{H}^{s+æ}(\mathbb{R}^m_-)$ so it is distribution supported on $\mathbb{R}^{m-1}$. Its general form in Fourier images is [1]

$$\sum_{j=1}^{n} \tilde{c}_j(\xi')\xi_m^{j-1}.$$

Thus we have the formula ($\tilde{g}_+ = \Pi_+(Q^{-1}\tilde{h})$

$$(\xi_m + i|\xi'| + i)^{-æ}\sigma_+(\xi)\tilde{u}_+(\xi) - Q_n(\xi)g_+(\xi) = \sum_{j=1}^{n} \tilde{c}_j(\xi')\xi_m^{j-1}.$$

and a lot of solutions

$$\tilde{u}_+(\xi) = (\xi_m+i|\xi'|+i)^{æ}\sigma_+^{-1}(\xi)Q_n(\xi)g_+(\xi)+(\xi_m+i|\xi'|+i)^{æ}\sigma_+^{-1}(\xi)\sum_{j=1}^{n} \tilde{c}_j(\xi')\xi_m^{j-1}.$$

It is left to verify that functions $\tilde{C}_j(\xi) = (\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi)\tilde{c}_j(\xi')\xi_m^j$ belong to $\widetilde{H}^s(\mathbb{R}^m)$. We have

$$||C_j||_s^2 = \int_{\mathbb{R}^m} |\tilde{c}_j(\xi')|^2|\xi_m + i|\xi'| + i|^{2æ}|\sigma_+^{-2}(\xi)||\xi_m|^{2j}(1 + |\xi|)^{2s}d\xi,$$

and passing to repeated integral, we first calculate

$$\int_{-\infty}^{+\infty} |\xi_m + i|\xi'| + i|^{2æ}|\xi_m|^{2j}(1 + |\xi|)^{2s}d\xi_m,$$

which exists only if $æ + j + s < -1/2$. Hence we obtain after integration that $C_j \in H^{æ+j+s+1/2}(\mathbb{R}^{m-1})$.

Thus we have proved the following

Theorem 4.3. *If $s + æ < -1/2$, then the Eq. (3.1) has many solutions in the space $H^s(\mathbb{R}_+^m)$, and the formula for a general solution in Fourier image*

$$\tilde{u}_+(\xi', \xi_m) = (\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi)Q_n(\xi)\tilde{g}_+(\xi', \xi_m)$$
$$+(\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi', \xi_m)\sum_{k=0}^{æ-1} c_k(\xi')\xi_m^k$$

holds, where $c_k \in H^{s_k}(\mathbb{R}^{m-1}), s_k = -æ + k + 1/2, k = 0, \cdots, æ - 1$ are arbitrary functions.

Corollary 4.4. *If under assumptions of the Theorem 4.3 $v \equiv 0$, then a general solution of the equation*

$$(\mathcal{D}u)(x) = 0, \quad x \in \mathbb{R}_+^m \tag{4.4}$$

has the form

$$\tilde{u}_+(\xi) = (\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi', \xi_m)\sum_{k=1}^{n} \tilde{c}_k(\xi')\xi_m^{k-1}. \tag{4.5}$$

4.3 Boundary Conditions

For a brevity we consider a homogeneous equation using the Corollary 4.4. We need some additional conditions to uniquely determine the functions $\tilde{c}_k, k = 1, \cdots, n$. It is an interesting fact that we cannot use the same conditions for positive and negative æ. Moreover the boundary operators in a certain sense are determined by the formula for a general solution. We consider below very simple boundary operators. Usually such operators are traces of some pseudo-differential operators on the hyperplane $x_m = 0$. But it is possible not for all cases.

4.3.1 Positive Case

Let us assume we know the values of $\tilde{u}_+$ in n distinct hyperplanes from $\mathbb{R}^m$ of type $\xi_m = p_j$. We denote $\tilde{u}_+(\xi', p_j) \equiv \tilde{r}_j(\xi')$ and obtain from the formula (4.5) the following system of linear algebraic equations

$$\sum_{k=1}^{n} \tilde{c}_k(\xi')p_j^{k-1} = \tilde{r}_j(\xi')(p_j + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi', p_j), \qquad j = 1, \cdots, n.$$

Obviously the system is uniquely solvable because its matrix has the Vandermonde determinant. To formulate a corresponding boundary value problem, we need some preliminaries.

We take the following boundary conditions

$$\int_{-\infty}^{+\infty} u_+(x', x_m)e^{-ip_jx_m}dx_m = r_j(x'), \quad j = 1, \cdots, n. \tag{4.6}$$

It will mean $\tilde{u}_+(\xi', p_j) = \tilde{r}_j(\xi')$. If $u_+ \in H^s(\mathbb{R}^m_+)$ then $r_j \in H^{s-1/2}(\mathbb{R}^m_+)$ [1]. So we have the following

Theorem 4.5. *Let $r_j \in H^{s-1/2}(\mathbb{R}^{m-1}), j = 1, \cdots, n$. Then the boundary value problem (4.4), (4.6) has a unique solution in the space $H^s(\mathbb{R}^m_+)$.*

Note. One can consider a linear combination of the conditions (4.6) and require nonvanishing the associated determinant.

4.3.2 Negative Case

This case admits integration for the right-hand side of the formula (4.5); thus, we take boundary conditions in the standard form

$$(A_ju_+)(x)_{|x_m=0} = r_j(x'), \quad j = 1, \cdots, n, \tag{4.7}$$

where A_j are pseudo-differential operators with symbols $A_j(\xi', \xi_m)$ satisfying the condition

$$|A_j(\xi', \xi_m) \sim (1 + |\xi'| + |\xi_m|)^{\gamma_j}.$$

Let us denote

$$a_{jk}(\xi') = \int_{-\infty}^{+\infty} A_j(\xi', \xi_m)(\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi', \xi_m)\xi_m^{k-1}d\xi_m.$$

Theorem 4.6. *Let* $\gamma_j + æ + k < -1, r_j \in H^{s_j}(\mathbb{R}^{m-1}), s_j = s - \gamma_j - 1/2, \ \forall j,$ $k = 1, \cdots, n$, *and the*

$$\inf_{\xi' \in \mathbb{R}^{m-1}} |\det(a_{jk}(\xi'))_{j,k=1}^n| > 0.$$

Then the boundary value problem (4.4),(4.7) *has a unique solution in the space* $H^s(\mathbb{R}_+^m)$.

5 Conclusion

There are a lot of possibilities to state distinct problems for the Eq. (3.1) adding some additional conditions. Also it seems to be interesting to transfer this approach and results to a discrete case, i.e., for spaces of a discrete variable. This will be discussed elsewhere.

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