On a digital approximation for pseudo-differential operators

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1 Introduction
We introduce a concept of a discrete pseudo-differential operator using general ideas of the theory and would like to show correlations between continuous and discrete cases.

2 Digital pseudo-differential operators
2.1 Digital Fourier transform
Given function \( u_d \) of a discrete variable \( \tilde{x} \in h\mathbb{Z}^m, h > 0 \), we define its discrete Fourier transform by the series
\[
(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^m} e^{i\tilde{x} \cdot \xi} u_d(\tilde{x}) h^m, \quad \xi \in h\mathbb{T}^m,
\]
where \( \mathbb{T}^m = [-\pi, \pi]^m, h = (2\pi/h)^{-1} \), partial sums are taken over cubes \( Q_N = \{ \tilde{x} \in \mathbb{Z}^m : \tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_m), \max_{1 \leq k \leq m} |\tilde{x}_k| \leq N \} \).

2.2 \( h \)-operators and \( \bar{h} \)-symbols
Let \( D \subset \mathbb{R}^m \) be a domain, and \( D_d = D \cap h\mathbb{Z}^m \).

We consider the following operators
\[
(A_d u_d)(\tilde{x}) = \int_{h\mathbb{T}^m} \sum_{\tilde{y} \in hD_d} e^{i(\tilde{y} - \tilde{x}) \cdot \xi} \tilde{A}_d(\xi) \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in hD_d,
\]
and the function \( \tilde{A}_d(\xi), \xi \in h\mathbb{T}^m \) is called a symbol of the operator \( A_d \).

Also the function
\[
A_d(\tilde{x}) = \int_{h\mathbb{T}^m} e^{i\tilde{x} \cdot \xi} \tilde{A}_d(\xi) d\xi,
\]
called a kernel of the operator \( A_d \).

Definition 2.1 The symbol \( \tilde{A}_d(\xi) \) is called an elliptic symbol of the operator \( A_d \) if \( \text{ess} \inf_{\xi \in h\mathbb{T}^m} |\tilde{A}_d(\xi)| > 0 \).

Example 2.2 The digital Laplacian is the following
\[
(\Delta_{d} u_d)(\tilde{x}) = h^{-2} \sum_{k=1}^{m} (u_d(x_1, \cdots, x_k + 2h, \cdots, x_m) - 2u_d(x_1, \cdots, x_k + h, \cdots, x_m) + u_d(x_1, \cdots, x_k, \cdots, x_m)),
\]
and its symbol is
\[
\tilde{\Delta}_d(\xi) = h^{-2} \sum_{k=1}^{m} (e^{inh^k} - 1)^2.
\]

Example 2.3 The digital Calderon–Zygmund operator is defined as follows [4]
\[
(K_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in hD_d} K_d(\tilde{x} - \tilde{y}) u_d(\tilde{y}) h^m, \quad \tilde{y} \in hD_d,
\]

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3 A comparison between discrete and continual cases

3.1 An approximation rate

Let \( P_h \) be a projection \( \mathbb{R}^m \to \mathbb{Z}^m \) so that a function \( u \) defined on \( \mathbb{R}^m \) corresponds to a function \( u_d \) of a discrete variable defined on \( h\mathbb{Z}^m \), \( P_h u = u_d \). If we consider the equation

\[
(Au)(x) = v(x), \quad x \in D,
\]

where \( A \) is a classical pseudo-differential operator with the symbol \( \tilde{A}(\xi) \) [1–3] of the form

\[
(Au)(x) = \int \int_D \int_{\mathbb{R}^m} e^{i(x-y) \cdot \xi} \tilde{A}(\xi) u(y) dy d\xi,
\]

which acts in certain functional spaces \( X \to Y \), for example Sobolev–Slobodetskii spaces [3]. We say that an element \( u \in X \) is an admissible element if \( P_h u \) is defined.

**Definition 3.1** An approximation rate for operators \( A \) and \( A_d \) on an admissible element \( u \in X \) is called the following norm

\[
\mu_h(A, A_d, u) = \| (A_d P_h - P_h A) u \|_{X_h},
\]

where \( X_h \) is so-called digital realization of the space \( X \) so that the operator \( A_d : X_h \to Y_h \) is a linear bounded operator.

One of main problems is the following. How we can choose the operator \( A_d \) to obtain a good approximation rate for the operator \( A \)? We need to fix a domain \( D \) and spaces \( X, Y \).

**Theorem 3.2** Let \( D \) be a domain with a Lipschitz boundary and \( X = Y = L_2(D) \), \( X_h = Y_h = L_2(D_d) \). If \( \tilde{A}(\xi) \) is a smooth bounded function on \( \mathbb{R}^m \) and

\[
A_d(\tilde{x}) = \int_{\mathbb{R}^m} e^{i\tilde{x} \cdot \xi} \tilde{A}(\xi) d\xi
\]

then \( \mu_h(A, A_d, u) \leq c u_h \) for arbitrary smooth function \( u \in L_2(D), c_u \) is a constant.

3.2 Digital solution and comparison

**Definition 3.3** A digital solution for the equation (2) is called a solution of the equation

\[
(A_d u_d)(\tilde{x}) = (P_h v)(\tilde{x}), \quad \tilde{x} \in D_d,
\]

if it exists.

**Remark 3.4** It is not evidently that a digital solution always exists. Thus, second of main problems is obtaining a solvability for the equation (3) in the space \( X_h \) at least for small \( h \) from the solvability of the equation (2) in the space \( X \). For this purpose we need to study a solvability of discrete equations, some steps in this direction were done in [6, 7] for special conical domains \( D \) and for the whole space \( \mathbb{R}^m \) and the half-space \( \mathbb{R}^m_+ \) [4].

**Theorem 3.5** Let \( D \) be \( \mathbb{R}^m \) or \( \mathbb{R}^m_+ \), the conditions of above theorem hold. \( A \) be an elliptic invertible operator, \( u \) be a solution of the equation (2) with a smooth right-hand side \( v \), \( u_d \) be a solution of the equation (3). Then

\[
\| P_h u - u_d \|_{X_h} \leq c h.
\]

4 Conclusion

In authors’ opinion these considerations will be useful for studying certain applied problems [5] because such operators and equation are very typical for these problems.

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References