DOI 10.1007/s10958-018-3737-9
Journal of Mathematical Sciences, Vol. 230, No. 1, April, 2018

PSEUDODIFFERENTIAL EQUATIONS ON MANIFOLDS WITH COMPLICATED BOUNDARY SINGULARITIES

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UDC 517.95 + 517.983

We consider model pseudodifferential equations in canonical multidimensional domains with boundary singularities presented by the union of cones or a cone of lower dimension. We study the solvability of these equations by using the wave factorization concept. Bibliography: 22 *titles.*

By a *pseudodifferential equation* we mean an equation of the form

$$(Au)(x) = v(x), \quad x \in M, \tag{1}$$

where A is a pseudodifferential operator acting in a space of functions defined on a manifold M, u is an unknown function, and v is a given function. For a smooth compact manifold M without boundary the theory of pseudodifferential equations is well developed. Here, by the *theory* we mean a description of the Fredholm conditions of the corresponding pseudodifferential operator A. As a rule, it is also proposed a method for reducing (1) to a Fredholm equation, i.e., an equation of the form

$$(I + T)f = g,$$

where I is the identity operator and T is a compact operator in a chosen function space.

A crucial point in the study of solvability of pseudodifferential equations on manifolds with nonsmooth boundary is to find invertibility conditions for special local representatives appearing while freezing coefficients of the original pseudodifferential equation in a special canonical domain diffeomorphic to a neighborhood of a singular boundary point. Owing to the wave factorization concept, introduced by the author, it is possible to obtain a complete picture concerning the solvability of the model elliptic pseudodifferential equation in the two-dimensional case, which leads to correct statements of boundary value problems for elliptic pseudodifferential equations in domains with angular points.

In the multidimensional case, we face difficulties caused by the fact that distributions concentrated on the cone surface cannot be expressed in a general form. The author established that one can use special pseudodifferential operators in terms of which it is possible in some cases to write out a general solution and further consider boundary conditions of different type.

Translated from *Sibirskii Zhurnal Chistoi i Prikladnoi Matematiki* **16**, No. 3, 2016, pp. 3-14.

1 Simple and Complicated Singularities

Definition 1. Let $C_+^a = \{x \in \mathbb{R}^m : x = (x', x_m), x' = (x_1, \cdots, x_{m-1}), x_m > a|x'|, a > 0\}$. By a *canonical stratified singularity* in $\mathbb{R}^{n+m}$ we mean the direct product of cones $C_+^a \times C_+^b$, where $C_+^a \subset \mathbb{R}^n$ and $C_+^b \subset \mathbb{R}^m$.

1.1. Simple singularities.

Example 1. A quadrant in the plane $\mathbb{R}^2$ is the direct product of two half-axes.

Example 2. An octant in the space $\mathbb{R}^3$ is the direct product of a quadrant (a two-dimensional cone) and a half-line (a one-dimensional cone).

Example 3. The cone $C_+^a = \{x \in \mathbb{R}^3 : x_3 > a(x_1^2 + x_2^2)^{1/2},\ a > 0\} \subset \mathbb{R}^3$ is not stratified. Furthermore, in multidimensional spaces, the direct product of cones is not visualized.

Example 4. An edge of codimension k in the m-dimensional space $\{x \in \mathbb{R}^m : x = (x', x''), x'' \in \mathbb{R}^{m-k}, x' = (x_1, \cdots, x_{m-k-1}), x_{m-k-1} > a|x'''|, x''' = (x_1, \cdots, x_{m-k-2}), a > 0\}$.

Example 5. A polyhedral angle variant $P_m = \left\{x \in \mathbb{R}^m : x_m > \sum\limits_{k=1}^{m-1} a_k|x_k|, a_k > 0\right\}$.

1.2. Complicated singularities.

Example 6. A "thin" cone $T_{m-k} = \{x \in \mathbb{R}^m : x_m > a|x''|, x'' = (x_1, \cdots, x_{m-k}), x_{m-k+1} = \cdots = x_{m-1} = 0\}$.

Example 7. A "cluster" of m-dimensional cones with vertex at the origin.

Example 8. A "cluster" of cones of different dimension.

The main thesis of the author is as follows. With each canonical singularity we associate a distribution, the Fourier transform of the characteristic function of a cone. The convolution with this distribution describes the Fourier-image of the operator of restriction to the canonical singularity.

2 Manifolds with Nonsmooth Boundary

An m-dimensional manifold M with nonsmooth boundary is a compact topological space each point of which possesses a neighborhood diffeomorphic to some canonical set in the Euclidean space $\mathbb{R}^m$. If $x \in M$ is an interior point of a manifold, then the canonical set is the entire space $\mathbb{R}^m$. If $x \in \partial M$ is a boundary point, where the boundary is smooth, then the canonical set is the half-space $\mathbb{R}_+^m = \{x \in \mathbb{R}^m : x = (x_1, \cdots, x_m),\ x_m > 0\}$. On the boundary ∂M, we can distinguish smooth submanifolds M_k, strata of dimension k, $0 \leqslant k < (m-1)$, such that each point of M_k possesses a neighborhood diffeomorphic to an edge of codimension k (Example 4). A submanifold of type M_0 is introduced as the set of isolated singular boundary points possessing neighborhoods diffeomorphic to some canonical cone (Examples 1–5). These singularities of ∂M are simple. A complicated singularity appears if for the standard we take the cones in Examples 6–8.

3 Local Representatives and Their Invertibility

We consider Equation (1) in the Sobolev–Slobodetskii space $H^s(M)$, where M is an m-dimensional smooth compact manifold with nonsmooth boundary. In other words, the boundary can have singularities like conical points, edges of codimension k, $1 \leqslant k \leqslant m$, and so on; the unknown function u and the right-hand side v are defined on M.

We consider only local constructions of spaces since the consideration can be extended to manifolds with the help of the unity partition.

By definition, the space $H^s(C^a_+)$ consists of distributions in $H^s(\mathbb{R}^m)$ with supports in $\overline{C^a_+}$. The $H^s(C^a_+)$–norm is induced by the $H^s(\mathbb{R}^m)$-norm. The right-hand side f is taken in the space $H_0^{s-\alpha}(C^a_+)$ of distributions in $S'(C^a_+)$ admitting an extension on $H^{s-\alpha}(\mathbb{R}^m)$. The norm in $H_0^{s-\alpha}(C^a_+)$ is defined by the equality $||f||^+_{s-\alpha} = \inf ||lf||_{s-\alpha}$, where the infimum is taken over all extensions l.

We introduce the multidimensional singular integral by

$$(G_m u)(x) = \lim_{\tau\to 0+} \int\limits_{\mathbb{R}^m} \frac{u(y', y_m)dy'dy_m}{(|x'-y'|^2 - a^2(x_m - y_m + i\tau)^2)^{m/2}}$$

(we omit some constants, cf. [1]). We recall that this operator is a multidimensional counterpart of the Cauchy type integral or, more exactly, the Hilbert transform.

If $A(x,\xi)$, $(x,\xi) \in T^*M$, is the symbol of a pseudodifferential operator A (defined on the cotangent bundle of the manifold M), then in order to describe the Fredholm conditions for the operator A, we need to describe the conditions of invertibility of all its local representatives. This assertion is known as the *local principle* or the *principle of freezing coefficients*.

If M is a smooth compact manifold (without boundary), the local representative of the operator at a point is the operator of multiplication by the symbol. The (necessary and sufficient) invertibility conditions are formulated as follows: the symbol does not vanish (the symbol is usually assumed to be sufficiently smooth). Such symbols are said to be *elliptic*.

For a manifold with smooth boundary we need a new local definition of a pseudodifferential operator at the smoothness point of the boundary ∂M. While for an interior point $x \in M$ we used the local definition

$$u(x) \longmapsto \int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} A(x,\xi)u(y)e^{i(x-y)\cdot\xi}d\xi dy,$$

for $x \in \partial M$ the local structure of the operator A is defined by

$$u(x) \longmapsto \int\limits_{\mathbb{R}^m_+}\int\limits_{\mathbb{R}^m} A(x,\xi)u(y)e^{i(x-y)\cdot\xi}d\xi dy.$$

To study the invertibility of such an operator with symbol $A(\cdot,\xi)$, independent of the spatial variable x, one could use the theory of classical Riemann problem in the upper and lower half-planes [2]–[4] with parameter $\xi' = (\xi_1, \dots, \xi_{m-1})$ (cf. [5] for details). However, if the boundary ∂M has at least one conical point, this approach is not applicable.

By a *conical boundary point* we mean a point possessing a neighborhood diffeomorphic to the cone $C^a_+ = \{x \in \mathbb{R}^m : x_m > a|x'|, x' = (x_1, \dots, x_{m-1}), a > 0\}$.. Consequently, it is necessary to modify the local definition of a pseudodifferential operator in a neighborhood of a conical point: the local operator has the form

$$u(x) \longmapsto \int\limits_{C^a_+}\int\limits_{\mathbb{R}^m} A(x,\xi)u(y)e^{i(x-y)\cdot\xi}d\xi dy. \quad (2)$$

To study the invertibility of the operator (2), the author proposed [1, 6] the concept of wave factorization of an elliptic symbol at a singular boundary point and, based on this concept, described the Fredholm conditions for Equation (1).

There are many other approaches to the theory of boundary value problems on nonsmooth manifolds (cf., for example, [7]–[9]).

4 Wave Factorization

Definition 2. A symbol $A(\xi)$ is *elliptic* if there exist $c_1, c_2 > 0$ such that

$$c_1 \leqslant |A(\xi)(1+|\xi|)^{-\alpha}| \leqslant c_2.$$

The number $\alpha \in \mathbb{R}$ is called the *order* of the operator A.

We set $\overset{*}{C}{}^a_+ = \{\xi \in \mathbb{R}^m : a\xi_m > |\xi'|\}$ and $\overset{*}{C}{}^a_- = -\overset{*}{C}{}^a_+$. A subset $T(\overset{*}{C}{}^a_+)$ of $\mathbb{C}^m$ of the form $\mathbb{R}^m + i\,\overset{*}{C}{}^a_+$ is referred to as a *radial tubular domain* over the cone $\overset{*}{C}{}^a_+$ (cf. [10]–[12]).

Definition 3. By the *wave factorization* of an elliptic symbol $A(\xi)$ we mean the representation $A(\xi) = A_{\neq}(\xi)A_{=}(\xi)$, where $A_{\neq}(\xi)$ and $A_{=}(\xi)$ satisfy the following conditions:

1) $A_{\neq}(\xi)$ and $A_{=}(\xi)$ are defined for all $\xi \in \mathbb{R}^m$ except for, possibly, the points $\{\xi \in \mathbb{R}^m : |\xi'|^2 = a^2\xi_m^2\}$,

2) $A_{\neq}(\xi)$ and $A_{=}(\xi)$ admit analytic extensions to the radial tubular domains $T(\overset{*}{C}{}^a_+)$ and $T(\overset{*}{C}{}^a_-)$ respectively, and following estimates hold:

$$|A_{\neq}^{\pm 1}(\xi + i\tau)| \leqslant c_1(1+|\xi|+|\tau|)^{\pm æ},$$

$$|A_{=}^{\pm 1}(\xi - i\tau)| \leqslant c_2(1+|\xi|+|\tau|)^{\pm(\alpha - æ)} \quad \forall \tau \in \overset{*}{C}{}^a_+ .$$

The number $æ \in \mathbb{R}$ is called the *wave factorization index.*

For classes of symbols admitting the wave factorization we refer to [6, 15]. Classes of analytic functions in radial tubular domains over cones are described in [10]–[12].

5 Multidimensional Riemann Problem

In the literature, there are different generalizations of the classical Riemann problem (cf., for example [13, 14]). One of variants was proposed by the author [1, 6, 15] for describing the Noetherian conditions for multidimensional singular integral equations in nonsmooth domains in terms of the wave factorization of the symbol of an elliptic operator. This approach turns out to be very convenient for studying the solvability of pseudodifferential equations and boundary value problems in domains with nonsmooth boundary (cf. [16]–[22]). We formulate this variant in the simplest form. We assume that C^m is a convex acute cone in $\mathbb{R}^m$ ($m \geqslant 2$) and $A(\mathbb{R}^m)$ is the subspace of $L_2(\mathbb{R}^m)$ of square Lebesgue integrable functions $u(x)$ admitting analytic extensions to the radial tubular domain $T(\overset{*}{C}{}^m)$ over the conjugate cone $\overset{*}{C}{}^m = \{x \in \mathbb{R}^m : (x, y) > 0 \ \forall y \in C^m\}$ and satisfying the condition

$$\sup_{y \in \overset{*}{C}{}^m} \int_{\mathbb{R}^m} |u(x+iy)|^2\, dx \leqslant \text{const}.$$

Let $B(\mathbb{R}^m)$ be the direct complement of $A(\mathbb{R}^m)$ in $L_2(\mathbb{R}^m) = A(\mathbb{R}^m) \oplus B(\mathbb{R}^m)$. It is required to find a pair of functions $\Phi^+(x) \in A(\mathbb{R}^m)$, $\Phi^-(x) \in B(\mathbb{R}^m)$ satisfying the linear relation

$$\Phi^+(x) = W(x)\Phi^-(x) + w(x) \tag{3}$$

everywhere in $\mathbb{R}^m$. The problem (3) appears as a result of localization of a multidimensional singular integral (pseudodifferential) equation at a conical boundary point and is solved with the help of the Bochner integral [10, 11].

We note that our statement of the Riemann problem differs from that given in [14].

6 Structure of Solution and Solvability Conditions

Let $C = \bigcup_{j=1}^{n} C_j$, where C_j are convex acute cones with common vertex at the origin, $C_j \bigcap C_k = \varnothing$, $k \neq j$. We consider Equation (1) with $M = C$. For every C_j, $j = 1, \dots, n$, we introduce the multidimensional singular integral with the Bochner kernel [10, 11] by

$$(B_j u)(x) = \lim_{\tau \to 0+} \int_{\mathbb{R}^m} B_j(x' - y', x_m - y_m + i\tau) u(y', y_m) dy' dy_m.$$

This singular integral appears as the Fourier transform of the product of the characteristic function of a cone and some integrable function. It is closely connected with the multidimensional Riemann problem in its simplest variant and presents one of possible multidimensional generalizations of Cauchy type integrals and, respectively, the Hilbert transform.

The Fourier-image of a space H is denoted by $\widetilde{H}$.

We formulate the main result for Equation (1) (we refer to [19]–[22] for details in the case of a single convex cone). The general solution can be constructed as follows. We denote by $Q_n(\xi)$ a polynomial of degree n such that $|Q_n(\xi)| \sim (1 + |\xi|)^n$. We denote by $E_a(\xi', \xi_m)$ the $(m-1)$-dimensional Fourier transform ($y' \to \xi'$ in the sense of the theory of distributions) of the function $e^{-ia|y'|\xi_m}$ and introduce the operator

$$(V_a \widetilde{u})(\xi') = (E_a * \widetilde{u})(\xi) \equiv \int_{\mathbb{R}^{m-1}} E_a(\xi' - \eta', \xi_m) \widetilde{u}(\eta', \xi_m) d\eta'.$$

Denote by $\mathscr{T}_k$ the rotation of $\mathbb{R}^m$ sending the cone C_k to the cone $C_+^{a_k}$. We consider n auxiliary multidimensional Riemann problems of type (3) under the assumption that $W(\xi) \neq 0$, $\xi \in \mathbb{R}^m$:

$$\widetilde{U}_k(\xi) = W(\xi)\widetilde{V}_k(\xi) + \widetilde{w}_k(\xi), k = 1, \dots, n, W(\xi) = -\frac{1}{n}\left(A(\xi) - \frac{n-1}{n}\right)^{-1}$$

with arbitrary $\widetilde{w}_k(\xi) \in \widetilde{H}^s(\mathbb{R}^m)$. We assume that $W(\xi)$ admits the wave factorization with respect to C_k with index æ$_k$, æ$_k - s = n_k + \delta$, $n_k \in \mathbb{N}$, $|\delta| < 1/2$, and denote by $W_{k,\neq}(\xi)$ and $W_{k,=}(\xi)$ elements of the wave factorization [6] of the symbol $W(\xi)$ with respect to the cone C_k.

Theorem 1. *The general solution to Equation* (1) *in the Fourier images is expressed by*

$$\widetilde{u}_+(\xi) = \sum_{k=1}^{n} W_{k,\neq}^{-1}(\xi) Q_{n_k}(\xi) B_k Q_{n_k}^{-1}(\xi) W_{k,=}^{-1}(\xi) \widetilde{w}_k(\xi)$$
$$+ \sum_{k=1}^{n} W_{k,\neq}^{-1}(\xi) \mathscr{T}_k^{-1} V_{-a_k} F\Big(\sum_{j=1}^{n_k} c_j(x') \delta^{(j-1)}(x_m)\Big),$$

where $c_j(x') \in H^{s_j}(\mathbb{R}^{m-1})$ *are arbitrary,* $s_j = s - æ_k + j - 1/2$*,* $j = 1, 2, \ldots, n_k$*,* $k = 1, \ldots, n$*,* lf *is an arbitrary extension of* f *to* $H^{s-\alpha}(\mathbb{R}^m)$*.*

The proof is based on the results of [21] by reducing to the case of half-spaces with the help of the following transformations. Denote by T_a the bijection $\mathbb{R}^m \longrightarrow \mathbb{R}^m$ sending ∂C_+^a to the hyperplane $x_m = 0$ and defined by $t_k = x_k$, $k = 1, \ldots, m-1$, $t_m = x_m - a|x'|$.

Lemma 1. $FT_a = V_a F$.

Proof. The connection between the Fourier transform and the operator T_a is established by the direct computation:

$$(FT_a u)(\xi) = \int\limits_{\mathbb{R}^m} e^{-ix\cdot\xi} u(x_1, \ldots, x_{m-1}, x_m - a|x'|)dx$$
$$= \int\limits_{\mathbb{R}^m} e^{-iy'\xi'} e^{-i(y_m + a|y'|)\xi_m} u(y_1, \ldots, y_{m-1}, y_m)dy$$
$$= \int\limits_{\mathbb{R}^{m-1}} e^{-ia|y'|\xi_m} e^{-iy'\xi'} \widehat{u}(y_1, \ldots, y_{m-1}, \xi_m)dy',$$

where $\widehat{u}$ denotes the Fourier transform with respect to the last variable. If the pseudodifferential operator is defined by

$$(Au)(x) = \int\limits_{\mathbb{R}^m} e^{ix\xi} A(\xi)\widetilde{u}(\xi)d\xi$$

and the direct Fourier transform is expressed by

$$\widetilde{u}(\xi) = \int\limits_{\mathbb{R}^m} e^{-ix\xi} u(x)dx,$$

then we have (at least, formally) the following connection:

$$(FT_a u)(\xi) = \int\limits_{\mathbb{R}^{m-1}} e^{-ia|y'|\xi_m} e^{-iy'\xi'} \widehat{u}(y_1, \ldots, y_{m-1}, \xi_m)dy.$$

In other words, if $E_a(\xi', \xi_m)$ denotes the $(m-1)$-dimensional Fourier transform ($y' \to \xi'$ in the sense of the theory of distributions) of the function $e^{-ia|y'|\xi_m}$, then the last formula is written as

$$(FT_a u)(\xi) = (E_a * \widetilde{u})(\xi),$$

where the symbol $*$ denotes the convolution with respect to the first $m-1$ variables and the multiplication with respect to the last variable ξ_m. Thus, V_a is the combination of convolution and multiplier with kernel $E_a(\xi', \xi_m)$. □

Thus, to describe the construction of a solution to Equation (1), we need to solve the corresponding auxiliary problem for each cone C_k, $k = 1, \ldots, n$, separately; moreover, for a special "composed" symbol we need to use the wave factorization with respect to each cone C_k with index $æ_k$. We note that the structure of the symbol $W(\xi)$ looks like that of the original symbol $A(\xi)$. Moreover, owing to the computations performed in [17], it is hopeful to obtain informative results on the solvability of Equation (1) in the case of more complicated singularities with cones of less dimension than the dimension of the space.

7 Asymptotic Expansions for Small Cone Opening

We consider the model two-dimensional equation (1) in the canonical cone $C_+^a = \{x \in \mathbb{R}^2 : x_2 > a|x_1|, a>0\}$. In the case of the wave factorization of the symbol $A(\xi)$, we can write out the solution with the help of the singular integral operator [1]:

$$(K_a u)(x) = \frac{a}{2\pi^2} \lim_{\tau\to 0+} \int\limits_{\mathbb{R}^2} \frac{u(y)dy}{(x_1-y_1)^2 - a^2(x_2-y_2+i\tau)^2}.$$

This operator plays a role of a local representative of the operator A in a neighborhood of an angular point of the manifold. This operator is represented by convolution, and the parameter a is the angular opening α, $x_2 > a|x_1|, a = \operatorname{ctg}\alpha$.

A distribution such that the convolution with this distribution is the operator K_a has the form

$$K_a(\xi_1,\xi_2) \equiv \frac{a}{2\pi^2}\frac{1}{\xi_1^2 - a^2\xi_2^2}.$$

To have an impression what happens if the angular opening becomes small, it is desirable to clarify the behavior of the operator K_a (the distribution $K_a(\xi_1,\xi_2)$) as $a^{-1} \to 0$.

On functions φ in the Schwarz class $S(\mathbb{R}^2)$ of infinitely differentiable functions rapidly decreasing with all their derivatives, this distribution is defined by the formula

$$(K_a,\varphi) = \frac{a}{2\pi^2}\int\limits_{\mathbb{R}^2} \frac{\varphi(\xi_1,\xi_2)d\xi}{\xi_1^2 - a^2\xi_2^2}.$$

7.1. Zeroth approximation. The limit distribution

$$\lim_{a\to\infty} \frac{a}{2\pi^2}\frac{1}{\xi_1^2 - a^2\xi_2^2} = \frac{i}{2\pi}\mathscr{P}\frac{1}{\xi_1} \otimes \delta(\xi_2)$$

was obtained in [17] as $a \to +\infty$, where the notation $\mathscr{P}$ is taken from [12] and $\otimes$ denotes the direct product of distributions. Here, δ is the Dirack δ-function acting on $\varphi \in S(\mathbb{R})$ by the rule $(\delta,\varphi) = \varphi(0)$ and the distribution $\mathscr{P}\frac{1}{x}$ is defined by

$$\left(\mathscr{P}\frac{1}{x},\varphi\right) = v.p.\int\limits_{-\infty}^{+\infty} \frac{\varphi(x)dx}{x} \equiv \lim_{\varepsilon\to 0+}\Bigl(\int\limits_{-\infty}^{-\varepsilon} + \int\limits_{\varepsilon}^{+\infty}\Bigr)\frac{\varphi(x)dx}{x}.$$

7.2. Complete expansion. It turns out that the distribution $K_a(\xi_1,\xi_2)$ admits an asymptotic expansion in powers of a^{-1}.

Lemma 2. *If a distribution a acts on a test function $\varphi \in S(\mathbb{R})$ by the formula*

$$(a,\varphi) = \int\limits_{-\infty}^{+\infty} \xi^k\varphi(\xi)d\xi,$$

then the distribution a has the form $a(\xi) = \widetilde{\delta^{(k)}}(\xi)$, where the symbol $\sim$ means the inverse Fourier transform F^{-1}.

Proof. Indeed, we have $F\delta = \mathbf{1}$, where $\mathbf{1}$ is identically equal to 1 in the sense of the theory of distributions, so that $F^{-1}\mathbf{1} = \delta$. Since $(F(\varphi^{(k)}))(\xi) = (-1)^k \xi^k \widetilde{\varphi}(\xi)$, we can write

$$
\begin{aligned}
(a, \varphi) = (a, F\psi) &= \int\limits_{-\infty}^{+\infty} \xi^k \widetilde{\psi}(\xi) d\xi = (\mathbf{1}, \xi^k \widetilde{\psi}(\xi)) = (\mathbf{1}, FF^{-1}(\xi_k \widetilde{\psi}(\xi))) \\
&= (F\mathbf{1}, F^{-1}(\xi_k \widetilde{\psi}(\xi))) = (F\mathbf{1}, (-1)^k \psi^{(k)}(x)) = (\delta, (-1)^k \psi^{(k)}(x)) \\
&= (\delta^{(k)}, \psi) = (\delta^{(k)}, F^{-1}\varphi) = (F^{-1}\delta^{(k)}, \varphi),
\end{aligned}
$$

where $\psi = F^{-1}\varphi$. Thus, we obtain the required equality. □

Using this lemma and the expansion of a test function into the Maclaurin series, it is possible to describe the asymptotic behavior of $K_a(\xi_1, \xi_2)$.

Theorem 2. *The following formula holds in the sense of the theory of distributions*:

$$
K_a(\xi_1, \xi_2) = \frac{i}{2\pi} \mathscr{P} \frac{1}{\xi_1} \otimes \delta(\xi_2) + \sum_{m,n} c_{m,n}(a) \widetilde{\delta^{(m)}}(\xi_1) \otimes \delta^{(n)}(\xi_2),
$$

where $c_{m,n}(a) \to 0$ *as* $a \to +\infty$.

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Submitted on December 27, 2015