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PARTIAL DIFFERENTIAL EQUATIONS

# Schwarz Problem for First-Order Elliptic Systems on the Plane

**V. B. Vasil'ev[1*] and V. G. Nikolaev[2**]**

[1] *Belgorod National Research University, Belgorod, 308015 Russia*
[2] *Yaroslav-the-Wise Novgorod State University, Veliky Novgorod, 173003 Russia*
*e-mail: *vbv57@inbox.ru, **vg14@inbox.ru*

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**Abstract**—We consider the Schwarz problem for $J$-analytic functions for the case in which the Jordan basis $Q$ of the matrix $J$ contains complex conjugate vectors. Conditions on the matrix $Q$ are obtained under which there exists a unique solution of the Schwarz problem in Hölder classes.

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## 1. INTRODUCTION

The study of boundary value problems for various classes of analytic functions has a long history [1–3] and has been developed in the recent years in several directions both theoretically [4, 5] and from the viewpoint of applications to problems in the general theory of boundary value problems for (pseudo)differential equations [6–8].

The Riemann–Hilbert problem [1, p. 264; 2, p. 140] of finding an analytic function in a domain from the boundary values of its real part is one of the main boundary value problems. One possible generalization of this problem is the Schwarz problem for Douglis analytic functions, a special case of which is considered in the present paper. Note that the Schwarz problem has applications in the theory of second-order elliptic systems on the plane [9].

## 2. MAIN DEFINITIONS AND NOTATION

Let $J \in \mathbb{C}^{n\times n}$ be a matrix without real eigenvalues. A Douglis analytic function (or a $J$-analytic function with matrix $J$) in a domain $D \in \mathbb{R}^2$ is a complex $n$-vector function $\phi = \phi(z) \in C^1(D)$ satisfying the equation

$$\frac{\partial \phi}{\partial y} - J\frac{\partial \phi}{\partial x} = 0, \qquad z = x + iy \in D, \tag{1}$$

in the domain $D$ [4–7].

In the scalar case ($n = 1$), a $J$-analytic function in a domain $D$ with $J = \lambda$, $\operatorname{Im}\lambda \neq 0$, will be called a $\lambda$-holomorphic function in $D$. We adopt the convention that the subscript "$\lambda$" or "$\mu$" on a function indicates that the function belongs to the class of $\lambda$- or $\mu$-holomorphic functions, respectively; for example, $f_\lambda$, $g_\lambda$, $g_\mu$, etc. Polynomials of the form $f_\lambda(z) = (x + \lambda y)^n$, $n = 1, 2, \ldots,$ may serve as examples of $\lambda$-holomorphic functions.

**Remark 1.** Let $\lambda = a + bi$, where $a, b \in \mathbb{R}$ and $b \neq 0$. One can readily show that a holomorphic function $f(x, y)$ ($\lambda = i$) is taken to a $\lambda$-holomorphic function $f_\lambda(x', y')$ by the invertible linear transformation $x = x' + ay'$, $y = by'$. Hence the properties of $\lambda$-holomorphic functions are the same as those of ordinary holomorphic functions.

However, this is not true for $n > 1$ in the general case (see Example 1 below).

**Definition 1.** We say that a *function $\phi(z)$ corresponds to a matrix $J$* if it satisfies Eq. (1).

As was shown in the monograph [4, p. 35], the complex system (1) of first-order partial differential equations is elliptic. Consider the following *Schwarz boundary value problem* [10–12] for Eq. (1).

Let $D \subset \mathbb{R}^2$ be a finite simply connected domain bounded by a contour $\Gamma$. Find a $J$-analytic function $\phi(z) \in C(\overline{D})$ with matrix $J$ in $D$ satisfying the boundary condition

$$\operatorname{Re}\, \phi(z)|_\Gamma = \psi(t), \tag{2}$$

where $\psi(t) = (\psi_1(t), \dots, \psi_n(t))^{\mathrm{T}} \in C(\Gamma)$ is a given real vector function.

For $\psi \equiv 0$, we speak of the *homogeneous Schwarz problem*. It has the obvious solutions in the form of constant vector functions $\phi = ic$, $c \in \mathbb{R}^n$, which are usually called trivial (or constant) solutions.

It is well known that the only solutions of the homogeneous problem (2) for $n = 1$ are constants [13, p. 123]. Let us show that this is not always the case for $n = 2$.

**Example 1.** Let

$$J = \begin{pmatrix} -1+3i & 1 \\ 3+4i & 1-i \end{pmatrix}, \qquad \phi(z) = \begin{pmatrix} x^2+3y^2-1-2xy\, i \\ x^2+3y^2-1-(4x^2+2xy+4y^2)\, i \end{pmatrix}.$$

One can readily verify that the function $\phi(z)$ corresponds to the matrix $J$, which has a multiple eigenvalue $\lambda = i$. One has $\operatorname{Re}\, \phi(z)|_\Gamma = 0$ on the ellipse

$$\Gamma : x^2 + 3y^2 = 1.$$

Theorem 4 below in particular gives conditions onto the Jordan forms of $n \times n$ matrices under which such examples are impossible (that is, the homogeneous Schwarz problem only has the trivial solutions).

## 3. SOME AUXILIARY ASSERTIONS

Let $D \subset \mathbb{R}^2$ be a finite simply connected domain whose boundary $\partial D = \Gamma$ is a Lyapunov curve (contour). Let $\overline{D}$ be the closure of $D$; i.e., $\overline{D} = D \sqcup \Gamma$. By $C^{1,\sigma}(\Gamma)$, $\sigma \in (0,1)$, we denote the class of functions whose first derivatives are Hölder continuous on $\Gamma$. Accordingly, $C^{1,\sigma}(\overline{D}\,)$ is the class of functions whose first partial derivatives are Hölder continuous in the closed domain $\overline{D}$.

We will use the following three theorems.

**Theorem 1.** *Assume that $\Gamma = \partial D$ is a Lyapunov contour and the function $\varphi(t)$ belongs to the class $C^{1,\sigma}(\Gamma)$. Then there exists a unique (up to a constant) $\lambda$-holomorphic function $f(z) \in C^{1,\sigma}(\overline{D}\,)$ in $D$ satisfying the boundary condition* $\operatorname{Re}\, f(z)|_\Gamma = \varphi(t)$.

Theorem 1 follows from the well-known similar result [2, p. 155] for holomorphic functions ($\lambda = i$) and from Remark 1.

**Theorem 2** [12]. *Assume that $\Gamma = \partial D$ is a Lyapunov contour and* $(\operatorname{Im}\lambda)(\operatorname{Im}\mu) < 0$. *Then the problem*

$$(f_\lambda + g_\mu)|_\Gamma = \varphi(t), \qquad f_\lambda,\, g_\mu \in H^\sigma(\overline{D}) \tag{3}$$

*is solvable up to a constant for any boundary function $\varphi(t)$ in the Hölder class $H^\sigma(\Gamma)$, $0 < \sigma < 1$.*

**Theorem 3** [10]. *Assume that $\Gamma = \partial D$ is a Lyapunov contour and all eigenvalues of the matrix $J \in \mathbb{C}^{n\times n}$ lie above or below the real axis simultaneously. Then the null space of the Schwarz problem is finite-dimensional in each of the classes $H^\sigma(\overline{D}\,)$, $0 < \sigma < 1$. Here the functions $\phi(z)$ with the property* $\operatorname{Re}\, \phi(z) \equiv 0$ *are taken into account as well.*

Further, let us prove three lemmas that will be used in the proof of the main Theorem 4.

Theorem 2 was proved in [12] for the special case of $\mu = \bar{\lambda}$ with the use of boundary properties of Cauchy type integrals. Let us present an alternative proof for this case, which permits considering a slightly wider function class. The following assertion holds.

**Lemma 1.** *Assume that* $\Gamma = \partial D$ *is a Lyapunov contour and the boundary function* $\varphi(t)$ *belongs to the class* $C^{1,\sigma}(\Gamma)$. *If* $\mu = \bar{\lambda}$, *then there exists a unique solution (up to a constant) of problem* (3) *in the function classes* $f_\lambda, g_\mu \in C^{1,\sigma}(\overline{D})$.

**Proof.** Set $\varphi = \varphi_1 + i\varphi_2$, where $\varphi_1, \varphi_2 \in C^{1,\sigma}(\Gamma)$ are real functions. According to Theorem 1, we construct a $\lambda$-holomorphic function $p(z) \in C^{1,\sigma}(\overline{D})$ from the condition $\operatorname{Re} p(z)|_\Gamma = \varphi_1(t)$ and a $\bar{\lambda}$-holomorphic function $h(z) \in C^{1,\sigma}(\overline{D})$ from the condition $\operatorname{Re} h(z)|_\Gamma = \varphi_2(t)$. Then the functions $\bar{p}$ and $\bar{h}$ belong to the class $C^{1,\sigma}(\overline{D})$ as well.

Set $\operatorname{Im} p(z)|_\Gamma = \varphi_1^*$ and $\operatorname{Im} h(z)|_\Gamma = \varphi_2^*$. We have

$$(p(z) + i\bar{h}(z) + \bar{p}(z) + ih(z))|_\Gamma = \varphi_1 + i\varphi_1^* + i(\varphi_2 - i\varphi_2^*) \\ + \varphi_1 - i\varphi_1^* + i(\varphi_2 + i\varphi_2^*) = 2(\varphi_1 + i\varphi_2). \tag{4}$$

We introduce the following notation for the functions on the left-hand side in Eq. (4):

$$f_\lambda(z) = \frac{1}{2}[p(z) + i\bar{h}(z)], \qquad g_\mu(z) = g_{\bar{\lambda}}(z) = \frac{1}{2}[\bar{p}(z) + ih(z)]. \tag{5}$$

The functions (5) give the desired solution $f_\lambda$, $g_\mu$ of problem (3) for $\mu = \bar{\lambda}$ in the class $C^{1,\sigma}(\overline{D})$. The proof of the lemma is complete.

**Lemma 2.** *Assume that* $\Gamma = \partial D$ *is a Lyapunov contour,* $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, *and a number* $l \in \mathbb{C}$ *satisfies the following condition*: $|l| = 1$ *if* $\lambda \neq \mu$, *and* $l$ *is an arbitrary number if* $\lambda = \mu$. *Then the problem*

$$(f_\lambda + \overline{g_\mu} + lg_\mu)|_\Gamma = \varphi(t), \qquad f_\lambda, g_\mu \in C^{1,\sigma}(\overline{D}) \tag{6}$$

*is solvable up to a constant for any boundary function* $\varphi(t) \in C^{1,\sigma}(\Gamma)$.

**Proof.** The case of $\lambda = \mu$ in problem (6) for arbitrary $l \in \mathbb{C}$ is equivalent to the case of $\mu = \bar{\lambda}$ in problem (3) and hence follows from Lemma 1.

Let $\lambda \neq \mu$ and $l = 1$. We make the substitutions $f_\lambda = u + iv$, $g_\mu = p + iq$, and $\varphi = \varphi_1 + \varphi_2 i$ in problem (6) and obtain

$$(u + iv + p - iq + p + iq)|_\Gamma = (u + iv + 2p)|_\Gamma = \varphi_1 + \varphi_2 i, \qquad \varphi_1, \varphi_2 \in C^{1,\sigma}(\Gamma). \tag{7}$$

By (7), $v = \varphi_2 \in C^{1,\sigma}(\Gamma)$ on $\Gamma$. Hence one can reconstruct the function $f_\lambda = u + iv \in C^{1,\sigma}(\overline{D})$ (up to a constant) by Theorem 1. Then we find the boundary value of the real part $p$ of the function $g_\mu$ from Eq. (7); namely, $p|_\Gamma = (1/2)(\varphi_1 - u) \in C^{1,\sigma}(\Gamma)$. From the last equation, we reconstruct the function $g_\mu = p + iq \in C^{1,\sigma}(\overline{D})$ by Theorem 1.

Now let $l \in \mathbb{C}$ and $|l| = 1$. Take an $a \in \mathbb{C}$ such that $l = \bar{a}/a$. We multiply both sides of Eq. (6) by $a$ and obtain

$$(af_\lambda + a\overline{g_\mu} + \bar{a}g_\mu)|_\Gamma = a\varphi(t). \tag{8}$$

Set

$$f_\lambda^* = af_\lambda, \qquad g_\mu^* = \bar{a}g_\mu, \qquad \varphi^* = a\varphi. \tag{9}$$

(The notation $f_\lambda^*$ and $g_\mu^*$ for the functions is justified, because it is obvious that the classes of $\lambda$-, and $\mu$-holomorphic functions are invariant with respect to multiplication by scalars.) It is also obvious that $\varphi^*(t) \in C^{1,\sigma}(\Gamma)$ if $\varphi(t) \in C^{1,\sigma}(\Gamma)$. We substitute (9) into (8) and obtain the relation $(f_\lambda^* + \overline{g_\mu^*} + g_\mu^*)|_\Gamma = \varphi^*(t)$, which, together with the already established solvability of problem (6) for $l = 1$, implies the solvability of problem (6) for $|l| = 1$. The proof of the lemma is complete.

The same scheme can be used to prove the following assertion.

**Lemma 3.** *Let $\Gamma = \partial D$ be a Jordan curve, and let the assumptions of Lemma 2 be satisfied. Then the homogeneous problem*

$$(f_\lambda + \overline{g_\mu} + l g_\mu)|_\Gamma = 0, \qquad f_\lambda,\, g_\mu \in C(\overline{D}), \tag{10}$$

*has only the trivial solutions.*

**Proof.** Let $\lambda \neq \mu$, $l = 1$. We set $\varphi \equiv 0$ in Eq. (7) and find that $(u+iv+2p)|_\Gamma = 0$. It follows that $v = 0$ on $\Gamma$. Consequently, $u + iv = \text{ const }$ in $D$. Thus, $p = \text{ const }$ on $\Gamma$, and hence $p + iq = \text{ const}$ in $D$. The general case of $|l| = 1$ can be considered by analogy with Lemma 2.

Now let $\lambda = \mu$. The problem (10) acquires the form

$$(f_\lambda + \overline{g_\lambda} + l g_\lambda)|_\Gamma = 0.$$

Thus, we have the equality of two $\lambda$- and $\bar{\lambda}$-holomorphic functions on a Jordan contour $\Gamma$. This is only possible if these functions are constant [13, p. 123]. The proof of the lemma is complete.

## 4. MAIN THEOREM

Let us proceed to the statement of the main result of the paper, Theorem 4. Let $\overline{\mathbf{x}}_k$ be the complex conjugate of a vector $\mathbf{x}_k \in \mathbb{C}^n$. Let $\mathbf{y}_k \in \mathbb{C}^n$. We define $n \times n$ matrices $Q$, $J_1$, and $J$ by the formulas

$$\begin{gathered} Q = (\mathbf{x}_1, \ldots, \mathbf{x}_m, \overline{\mathbf{x}}_1 + l_1\mathbf{x}_1, \ldots, \overline{\mathbf{x}}_m + l_m\mathbf{x}_m, \mathbf{y}_1, \ldots, \mathbf{y}_{m_1}), \\ l_k \in \mathbb{C}, \qquad k = 1, \ldots, m, \qquad 2m + m_1 = n, \qquad \det Q \neq 0, \\ J_1 = \operatorname{diag}[\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m, \underbrace{\eta, \ldots, \eta}_{m_1}], \qquad J = QJ_1Q^{-1}. \end{gathered} \tag{11}$$

In accordance with the definition of $J$-analytic function, we assume that all eigenvalues of the matrix $J_1$ have nonzero imaginary parts. The following theorem holds.

**Theorem 4.** *Let $J = QJ_1Q^{-1}$, where the matrices $Q$ and $J_1$ are given by formulas (11). Further, assume that if $\lambda_k \neq \mu_k$, then $l_k = 0$ or $|l_k| = 1$. If $\lambda_k = \mu_k$, then the number $l_k \in \mathbb{C}$ may be arbitrary. Then the following assertions hold.*

*1. If $\Gamma = \partial D$ is a Lyapunov contour and $(\operatorname{Im}\lambda_k)(\operatorname{Im}\mu_k) > 0$, $k = 1, \ldots, m$, then the Schwarz problem (2) has a unique solution $\phi(z) \in H^\sigma(\overline{D})$ up to a constant vector for each boundary function $\psi(z) \in H^\sigma(\Gamma)$, $0 < \sigma < 1$.*

*2. If $\Gamma = \partial D$ is a Lyapunov contour, the function $\psi(z)$ belongs to the class $C^{1,\sigma}(\Gamma)$, and $|l_k| = 1$ whenever $\lambda_k \neq \mu_k$, then the Schwarz problem has a unique solution $\phi(z) \in C^{1,\sigma}(\overline{D})$.*

*3. If $\Gamma = \partial D$ is a Jordan curve and $|l_k| = 1$ whenever $\lambda_k \neq \mu_k$, then the homogeneous Schwarz problem (2) has only the trivial solutions in the function class $\phi(z) \in C(\overline{D})$.*

**Proof.** We substitute the expression $J = QJ_1Q^{-1}$ into Eq. (1), multiply both sides by the matrix $Q^{-1}$, and obtain

$$\frac{\partial}{\partial y}(Q^{-1}\phi) - J_1\frac{\partial}{\partial x}(Q^{-1}\phi) = 0. \tag{12}$$

We denote the vector function occurring in Eq. (12) by

$$\phi_0(z) = Q^{-1}\phi(z) = (f_1, \ldots, f_m, g_1, \ldots, g_m, h_1, \ldots, h_{m_1})^{\mathrm{T}}, \qquad 2m + m_1 = n. \tag{13}$$

It follows from Eq. (12) by virtue of the structure (11) of the matrix $J_1$ that $f_k$, $g_k$, and $h_k$ in Eq. (13) are arbitrary $\lambda_k$-, $\mu_k$-, and $\eta$-holomorphic functions, respectively.

Set

$$f_k(z) = u_k + iv_k, \quad g_k(z) = p_k + iq_k, \quad k = 1, \ldots, m; \quad h_k(z) = r_k + is_k, \quad k = 1, \ldots, m_1, \tag{14}$$

where the functions $u_k$, $v_k$, $p_k$, $q_k$, $r_k$, and $s_k$ are real. Then, in view of Eqs. (11), (13), and (14), the function

$$\phi(z) = Q\phi_0(z) = \begin{pmatrix} a_{11} & \dots & a_{1m} & \bar{a}_{11} + l_1 a_{11} & \dots & \bar{a}_{1m} + l_m a_{1m} & b_{11} & \dots & b_{1m_1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nm} & \bar{a}_{n1} + l_1 a_{n1} & \dots & \bar{a}_{nm} + l_m a_{nm} & b_{n1} & \dots & b_{nm_1} \end{pmatrix}$$
$$\times (u_1 + iv_1, \dots, u_m + iv_m, p_1 + iq_1, \dots, p_m + iq_m, r_1 + is_1, \dots, r_{m_1} + is_{m_1})^{\mathrm{T}} \tag{15}$$

is the general solution of Eq. (1) for the type of matrices under study.

In view of the solution (15), the boundary condition (2) is equivalent to the following system of equations *on the contour* $\Gamma$ :

$$\begin{aligned} &\operatorname{Re}\left[\sum_{k=1}^{m}(a_{1k}(u_k + iv_k) + (\bar{a}_{1k} + l_k a_{1k})(p_k + iq_k)) + \sum_{k=1}^{m_1} b_{1k}(r_k + is_k)\right]\Bigg|_\Gamma = \psi_1(t), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\operatorname{Re}\left[\sum_{k=1}^{m}(a_{nk}(u_k + iv_k) + (\bar{a}_{nk} + l_k a_{nk})(p_k + iq_k)) + \sum_{k=1}^{m_1} b_{nk}(r_k + is_k)\right]\Bigg|_\Gamma = \psi_n(t). \end{aligned} \tag{16}$$

We introduce the following notation for the first $m$ columns of the matrix $Q$ in (11), or, which is the same, of the matrix $Q$ in (15) :

$$\mathbf{x}_k = \mathbf{x}'_k + \mathbf{x}''_k i, \qquad \mathbf{x}'_k, \mathbf{x}''_k \in \mathbb{R}^n, \qquad k = 1, \dots, m. \tag{17}$$

Let us treat system (16) as an inhomogeneous system of $n$ linear algebraic equations for the $2m$ variables $u_1, v_1, \dots, u_m, v_m$. Let us show that the $n \times 2m$ matrix

$$\widetilde{Q} = (\mathbf{x}'_1, -\mathbf{x}''_1, \dots, \mathbf{x}'_m, -\mathbf{x}''_m) \tag{18}$$

of this system contains at least one nonzero $2m \times 2m$ minor.

Assume the contrary. Then rank $\widetilde{Q} < 2m$ and hence

$$\alpha_1 \mathbf{x}'_1 + \beta_1 \mathbf{x}''_1 + \dots + \alpha_m \mathbf{x}'_m + \beta_m \mathbf{x}''_m = 0, \tag{19}$$

where the real numbers $\alpha_k$ and $\beta_k$ are not all zero simultaneously.

Since Eq. (19) can in view of notation (17) be rewritten in the form

$$\operatorname{Re}\left[(\alpha_1 - i\beta_1)\mathbf{x}_1 + \dots + (\alpha_m - i\beta_m)\mathbf{x}_m\right] = 0,$$

we have

$$(\alpha_1 - i\beta_1)\mathbf{x}_1 + \dots + (\alpha_m - i\beta_m)\mathbf{x}_m = i\mathbf{z}, \qquad \mathbf{z} \in \mathbb{R}^n. \tag{20}$$

We apply complex conjugation to both sides of (20) and obtain

$$(\alpha_1 + i\beta_1)\overline{\mathbf{x}}_1 + \dots + (\alpha_m + i\beta_m)\overline{\mathbf{x}}_m = -i\mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^n. \tag{21}$$

We add Eqs. (20) and (21) and find that a nontrivial linear combination of the vectors $\mathbf{x}_k$ and $\overline{\mathbf{x}}_k$ is zero; on the other hand, these vectors are linearly independent, which follows from the fact that the matrix $Q$ is nonsingular. This contradiction shows that rank $\widetilde{Q} = 2m$.

Let $\widetilde{Q}_1$ be a nonsingular $2m \times 2m$ submatrix of the matrix $\widetilde{Q}$ defined in (18). Note that if $\xi \in \mathbb{C}$, then $-\operatorname{Re}\xi + i\operatorname{Im}\xi = -\overline{\xi}$. Hence

$$\begin{aligned} &\operatorname{Re}\left[a\{-p - \operatorname{Re}[\bar{l}(p - iq)] + i(q + \operatorname{Im}[\bar{l}(p - iq)])\}\right] + \operatorname{Re}\left[(\bar{a} + la)(p + iq)\right] \\ &\quad = \operatorname{Re}\left[a(-p + iq)\right] - \operatorname{Re}\left[a\overline{\bar{l}(p - iq)}\right] + \operatorname{Re}\left[\bar{a}(p + iq)\right] + \operatorname{Re}\left[al(p + iq)\right] \\ &\quad = \operatorname{Re}\left[a(-p + iq)\right] - \operatorname{Re}\left[\overline{a(-p + iq)}\right] \pm \operatorname{Re}\left[al(p + iq)\right] = 0 \end{aligned} \tag{22}$$

for any numbers $a, l \in \mathbb{C}$.

Formula (22) permits one to find the solution of the nonsingular subsystem of (16) with matrix $\widetilde{Q}_1$ for the variables $u_1, v_1, \dots, u_m, v_m$. Indeed, let us make the following substitution in each of the equations in system (16) :

$$\begin{aligned} u_k &= u_k(p_k, q_k) = -p_k - \operatorname{Re}[\bar{l}_k(p_k - iq_k)], \\ v_k &= v_k(p_k, q_k) = q_k + \operatorname{Im}[\bar{l}_k(p_k - iq_k)], \qquad k = 1, \dots, m. \end{aligned} \tag{23}$$

Then, in view of (22),

$$\operatorname{Re}\{a_{1k}[u_k(p_k, q_k) + iv_k(p_k, q_k)] + (\bar{a}_{1k} + l_k a_{1k})(p_k + iq_k)\} = 0 \tag{24}$$

for each $k = 1, \dots, m$.

**Remark 2.** Without loss of generality, we assume that the matrix $\widetilde{Q}_1$ corresponds to the first $2m$ equations in system (16).

Thus, the *unique* solution of the inhomogeneous subsystem with matrix $\widetilde{Q}_1$, $\det \widetilde{Q}_1 \neq 0$, of the linear algebraic system (16) for the variables $u_k, v_k$, $k = 1, \dots, m$, can be found in the form

$$\begin{aligned} u_k &= -p_k - \operatorname{Re}[\bar{l}_k(p_k - iq_k)] + w_k(\psi_1, \dots, \psi_n, r_1, s_1, \dots, r_{m_1}, s_{m_1}), \\ v_k &= q_k + \operatorname{Im}[\bar{l}_k(p_k - iq_k)] + \widetilde{w}_k(\psi_1, \dots, \psi_n, r_1, s_1, \dots, r_{m_1}, s_{m_1}), \end{aligned} \tag{25}$$

where $w_k(\cdot)$ and $\widetilde{w}_k(\cdot)$, $k = 1, \dots, m$, are linear functions of their arguments. Here

$$(w_1, \dots, w_m, \widetilde{w}_1, \dots, \widetilde{w}_m)^{\mathrm{T}} = (\widetilde{Q}_1)^{-1}(\psi_1 + \varkappa_1(r_k, s_k), \dots, \psi_{2m} + \varkappa_{2m}(r_k, s_k))^{\mathrm{T}}, \tag{26}$$

where $\varkappa_j(\cdot)$ are linear functions of the variables $r_k$ and $s_k$, $k = 1, \dots, m_1$.

Indeed, the unknowns $u_k$, $v_k$, $p_k$, and $q_k$ cancel out by virtue of Eqs. (23) and (24) after the substitution of the expressions (25) into the first $2m$ equations in system (16). As a result, we obtain an inhomogeneous system for the variables $w_k$ and $\widetilde{w}_k$, whose matrix coincides with $\widetilde{Q}_1$; hence Eq. (26) follows.

For each $k = 1, \dots, m$, let us multiply the second equation in (25) by $i$ and add it to the first equation. Then, in view of the relation $-\operatorname{Re}\xi + i\operatorname{Im}\xi = -\overline{\xi}$ and notation (14), we have

$$\begin{aligned} u_k + iv_k - \overline{p_k + iq_k} - \overline{[\bar{l}_k(p_k - iq_k)]} &= u_k + iv_k - \overline{p_k + iq_k} - l_k(p_k + iq_k) \\ &= (f_k - \overline{g_k} - l_k g_k)|_\Gamma = w_k + i\widetilde{w}_k = \varphi_k, \end{aligned} \tag{27}$$

where we recall that $f_k$ and $g_k$ are arbitrary $\lambda_k$- and $\mu_k$-holomorphic functions, respectively.

Without loss of generality, we can assume that the signs in front of the functions $g_k$ in Eq. (27) are "+." Further, all transformations carried out above are invertible. Hence if $m_1 = 0$, i.e., $n = 2m$, in Eqs. (11), then, in view of relations (25)–(27), the Schwarz problem (2) is equivalent to the following system of $m$ independent scalar boundary value problems for functional equations:

$$(f_k + \overline{g_k} + l_k g_k)|_\Gamma = w_k + i\widetilde{w}_k = \varphi_k, \quad k = 1, \dots, m, \quad \text{where} \quad \begin{cases} f_k, g_k \in H^\sigma(\overline{D}) & \text{for} \quad \varphi_k \in H^\sigma(\Gamma), \\ f_k, g_k \in C^{1,\sigma}(\overline{D}) & \text{for} \quad \varphi_k \in C^{1,\sigma}(\Gamma), \end{cases} \tag{28}$$

and the class of the boundary $\Gamma$ and the values of $l_k \in \mathbb{C}$ depend on which of the assertions of the theorem is considered.

For $m_1 = 0$, the functions $\varkappa_j(\cdot)$ are lacking in Eq. (26). Hence the following three assertions hold for $n = 2m$.

1°. Assume that the boundary function $\psi$ in condition (2) belongs to the class $H^\sigma(\Gamma)$, where $\Gamma$ is a Lyapunov curve. All the functions $\varphi_k = w_k + i\widetilde{w}_k$ in problems (28) belong to the class $H^\sigma(\Gamma)$ as well by virtue of Eq. (26). Hence the assertion of part 1 of Theorem 4 follows from Lemmas 1 and 2 or from Theorem 2.

2°. Assume that the boundary function $\psi$ in condition (2) belongs to the class $C^{1,\sigma}(\Gamma)$, where $\Gamma$ is a Lyapunov curve. Then the functions $\varphi_k = w_k + i\widetilde{w}_k$ in problems (28) belong to the class $C^{1,\sigma}(\Gamma)$ as well by virtue of Eq. (26). Hence the assertion of part 2 of Theorem 4 follows from Lemmas 1 and 2.

3°. Assume that the curve $\Gamma$ in condition (2) is a Jordan curve and $\psi \equiv 0$. Then all the functions $\varphi_k$ in problems (28) are identically zero by virtue of Eq. (26). As a result, the assertion of part 3 of Theorem 4 follows from Lemma 3.

Now consider the more complicated case of $m_1 = n - 2m > 0$. Then the boundary functions $w_k$ and $\widetilde{w}_k$, $k = 1, \ldots, 2m$, in Eq. (26) additionally depend on the a priori unknown values of the functions $r_k$ and $s_k$ on the contour $\Gamma$. To sidestep this difficulty, we make the following transformations.

We substitute the solutions (25) with *known* functions $w_k$ and $\widetilde{w}_k$ determined by Eq. (26) into the remaining $m_1$ equations in system (16). Then, by virtue of Eq. (24), we obtain the following linear algebraic system after transposing the functions $\psi_k$ to the right-hand side:

$$\begin{gathered} L_1(r_1, s_1, \ldots, r_{m_1}, s_{m_1}) = L_1'(\psi_1, \ldots, \psi_n), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ L_{m_1}(r_1, s_1, \ldots, r_{m_1}, s_{m_1}) = L_{m_1}'(\psi_1, \ldots, \psi_n), \end{gathered} \tag{29}$$

where $L_k(\cdot)$ and $L_k'(\cdot)$ are linear functions of their arguments.

Since the functions $\psi_k$ are defined on the contour $\Gamma$, we can rewrite system (29) in view of notation (14) in the form

$$\begin{gathered} L_1'(\cdot)|_\Gamma = L_1(\cdot)|_\Gamma = \operatorname{Re}[c_{11}h_1 + \cdots + c_{1m_1}h_{m_1}]|_\Gamma = \operatorname{Re}\xi_1(z)|_\Gamma, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ L_{m_1}'(\cdot)|_\Gamma = L_{m_1}(\cdot)|_\Gamma = \operatorname{Re}[c_{m_11}h_1 + \cdots + c_{m_1m_1}h_{m_1}]|_\Gamma = \operatorname{Re}\xi_{m_1}(z)|_\Gamma, \end{gathered} \tag{30}$$

where $c_{kj} \in \mathbb{C}$ and all the functions $h_k$ and $xi_k$, $k = 1, \ldots, m_1$, are $\eta$-holomorphic in $D$ according to (11) and (13). Equations (30) mean that the following algebraic system holds for the variable functions $h_k = h_k(z)$, $k = 1, \ldots, m_1$, defined in the domain $D$:

$$\begin{gathered} c_{11}h_1 + \cdots + c_{1m_1}h_{m_1} = \xi_1(z) + c_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ c_{m_11}h_1 + \cdots + c_{m_1m_1}h_{m_1} = \xi_{m_1}(z) + c_{m_1}, \end{gathered} \tag{31}$$

where $c_1, \ldots, c_{m_1} \in \mathbb{C}$.

The right-hand side of system (31) is known. Indeed, the functions $L_k'(\cdot) \in H^\sigma(\Gamma)$ in system (29) depend only on the functions $\psi_1, \ldots, \psi_n$ and on the matrix $Q$ defined in (11); i.e., they are given by the assumptions of the theorem. Hence the $\eta$-holomorphic functions $\xi_k(z)$, $k = 1, \ldots, m_1$, can be reconstructed according to Theorem 1 from the boundary values $L_k'(\cdot)$ of their real parts known from system (30).

To find $h_k$ from system (31), let us show that its determinant is nonzero,

$$\Delta_2 = \begin{vmatrix} c_{11} & \ldots & c_{1m_1} \\ \ldots & \ldots & \ldots \\ c_{m_11} & \ldots & c_{m_1m_1} \end{vmatrix} \neq 0. \tag{32}$$

Assume the contrary: $\Delta_2 = 0$. Note that the entries of the determinant $\Delta_2$ are uniquely determined by the matrix $Q$, which is a Jordan basis of the matrix $J$. Further, the determinant $\Delta_2$ is independent of the matrix $J_1$ in (11); i.e., it depends neither on the eigenvalues of $J$ nor on the contour $\Gamma$.

Hence let $\Gamma$ be a Lyapunov contour. Consider the homogeneous Schwarz problem ($\psi \equiv 0$) for a matrix $J^*$ with the same Jordan basis $Q$ as $J$ and with all eigenvalues lying above the real axis. Then all functions $L'_k(\cdot)$ in systems (29) and (30) are zero, and hence we assume without loss of generality that the right-hand side of system (31) is zero.

The resulting homogeneous system (31), whose determinant (32) is zero by assumption, has *infinitely many linearly independent* solutions for the $\eta$-holomorphic functions $h_k(z) \in H^\sigma(\overline{D}\,)$. Such solutions can readily be found, say, in the form of polynomials $h_k = r_k + is_k = a(x+\eta y)^n$, $a \in \mathbb{C}$, $n = 1, 2, \ldots$ Let us substitute the boundary values of these solutions into the right-hand side of Eq. (26) and hence find the functions $w_k, \widetilde{w}_k$ for problems (28).

Further, we construct the solutions of problems (28) with the use of Theorem 2 of Lemmas 1 and 2 depending on the numbers $l_k$. Then we find the function $\phi(z)$ by formula (15). It is nonconstant, because so are the functions $h_k$. One can construct infinitely many such solutions $\phi(z)$ of the homogeneous Schwarz problem, which contradicts Theorem 3. The resulting contradiction means that $\Delta_2 \neq 0$.

Thus, system (31) uniquely (up to a constant) determines the $\eta$-holomorphic functions $h_k = r_k + is_k$, $k = 1, \ldots, m_1$. They belong to the classes $H^\sigma(\overline{D})$ or $C^{1,\sigma}(\overline{D})$ depending by virtue of system (29) on the class of the boundary vector function $\psi$.

Let us substitute the *boundary values* of the functions $r_k$ and $s_k$ into the right-hand side of Eq. (26). Further, we substitute the resulting functions $w_k$ and $\widetilde{w}_k$ into Eq. (25) and find the solution of system (16) *on the contour* $\Gamma$. Indeed, as was mentioned above, the functions (25) are a solution of the first $2m$ equations in this system, because the unknowns cancel each other out. As to the remaining $m_1$ equations in system (16), the functions (25) are their solution, because the left- and right-hand sides coincide on the contour $\Gamma$.

Further, to determine the functions $u_k$ and $v_k$, we use system (28), in which the boundary values of the functions $\varphi_k$ are known owing to Eqs. (25). Then we carry out the same argument as in 1°–3° above. The proof of the theorem is complete.

By way of remark, note the following. Straightforward computations show that if $\mathbf{x} \in \mathbb{C}^n$ (or $\mathbf{x} \in \mathbb{R}^n$), $l \in \mathbb{C}$, and $|l| = 1$, then the vector $\overline{\mathbf{x}} + l\mathbf{x}$ is a multiple of a real vector.

## 5. SOME COROLLARIES OF THE SCHEME OF PROOF OF THE MAIN THEOREM

Let us apply Theorem 4 to matrices $J \in \mathbb{C}^{2\times 2}$ with distinct eigenvalues $\lambda$ and $\mu$. Let $Q = (\mathbf{x}, \mathbf{y})$ be a Jordan basis of the matrix $J$.

First, assume that the matrix $Q$ is real. We make the substitutions $\phi = Q\phi^*$ and $J = Q^* J_1 (Q^*)^{-1}$ in Eq. (1), where $J_1 = \operatorname{diag}[\lambda, \mu]$. Then the Schwarz problem (2) splits into two independent boundary value problems for $\lambda$- and $\mu$-holomorphic functions, respectively, which can be solved with the use of Theorem 1.

Now assume that only the vector $\mathbf{y}$ belongs to $\mathbb{R}^2$. We write it in the form $\mathbf{y} = a\overline{\mathbf{x}} + b\mathbf{x}$, $a, b \in \mathbb{C}$. Here $a \neq 0$. (Otherwise, the vector $\mathbf{y}$ would not be real.) By Cramer's formulas, from the last relation we obtain

$$a = \frac{\det(\mathbf{y},\mathbf{x})}{\det(\overline{\mathbf{x}},\mathbf{x})}, \qquad b = \frac{\det(\overline{\mathbf{x}},\mathbf{y})}{\det(\overline{\mathbf{x}},\mathbf{x})}, \qquad l = \frac{b}{a} = \frac{\det(\overline{\mathbf{x}},\mathbf{y})}{\det(\mathbf{y},\mathbf{x})} = -\frac{\det(\overline{\mathbf{x}},\mathbf{y})}{\det(\mathbf{x},\mathbf{y})}, \qquad \mathbf{y} \in \mathbb{R}^2,$$

whence it follows that $|l| = 1$.

As a result, we have the relation $\overline{\mathbf{x}} + ba^{-1}\mathbf{x} = a^{-1}\mathbf{y}$, or $\overline{\mathbf{x}} + l\,\mathbf{x} = a^{-1}\mathbf{y}$, where $l \in \mathbb{C}$, $|l| = 1$. The matrix $Q = (\mathbf{x}, a^{-1}\mathbf{y})$ is a Jordan basis of the same matrix $J$. Thus, the matrix $J$ admits the representation (11). Hence, in view of Theorem 1 and part 2 of Theorem 4, we have proved the following theorem.

**Theorem 5.** *Assume that $\Gamma = \partial D$ is a Lyapunov contour and the matrix $J \in \mathbb{C}^{2\times 2}$ has distinct eigenvalues and at least one real eigenvector. Then the Schwarz problem has a unique solution $\phi(z) \in C^{1,\sigma}(\overline{D}\,)$ for any boundary function $\psi(z) \in C^{1,\sigma}(\Gamma)$.*

The matrix $J_1$ in Eqs. (11) was chose to be diagonal. However, the scheme of proof of Theorem 4 also applies to some nondiagonal matrices $J_1$, i.e., matrices $J = QJ_1Q^{-1}$ with nondiagonal Jordan form.

The description of the general case is rather awkward, and hence we restrict ourselves to dimension $n = 4$. Consider the matrix $J = QJ_1Q^{-1}$, where

$$J_1 = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 1 & \mu \end{pmatrix}, \qquad Q = (\mathbf{x}, \mathbf{y}, \overline{\mathbf{x}} + l_1\mathbf{x}, \overline{\mathbf{y}} + l_2\mathbf{y}), \qquad l_1, l_2 \in \mathbb{C}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{C}^4,$$
$$(\operatorname{Im}\lambda)(\operatorname{Im}\mu) > 0, \qquad \begin{cases} |l_1| = 1, \quad |l_2| \in \{0,1\} & \text{for } \lambda \neq \mu, \\ l_1, l_2 \text{ are arbitrary numbers} & \text{for } \lambda = \mu. \end{cases} \tag{33}$$

Let us show that the following theorem holds.

**Theorem 6.** *Let $J = QJ_1Q^{-1}$, where the matrices $J_1$ and $Q$ are defined in* (33). *Further, assume that $\Gamma = \partial D$ is a Lyapunov contour and the boundary function $\psi(z)$ belongs to the class $C^{1,\sigma}(\Gamma)$. Then the Schwarz problem* (2) *has a unique solution $\phi(z) \in H^\sigma(\overline{D})$.*

**Proof.** The proof is based on that of Theorem 4 and is carried out by the same scheme. Set

$$\phi_0(z) = Q^{-1}\phi(z) = (f_1,\, f_2,\, g_1,\, g_2)^{\mathrm{T}} \tag{34}$$

by analogy with (13).

In this case, the functions $f_1$ and $g_1$ in (34) are arbitrary $\lambda$- and $\mu$-holomorphic functions, respectively, by virtue of the structure (33) of the matrix $J_1$. Further,

$$f_2(z) = y\frac{\partial f_1(z)}{\partial x} + h_1(z), \qquad g_2(z) = y\frac{\partial g_1(z)}{\partial x} + h_2(z), \tag{35}$$

where again $h_1$ and $h_2$ are arbitrary $\lambda$- and $\mu$-holomorphic functions, respectively.

Here we deal with the case of $n = 2m$ in the proof of Theorem 4. Hence, just as in the proof of Theorem 4, we obtain a system similar to system (28) for $n = 4$. It consists of the following two functional equations:

$$\begin{aligned} (f_1 + \overline{g_1} + l_1g_1)|_\Gamma &= \varphi_1(\psi_1, \dots, \psi_n), \qquad |l_1| = 1, \\ (f_2 + \overline{g_2} + l_2g_2)|_\Gamma &= \varphi_2(\psi_1, \dots, \psi_n), \qquad |l_2| = \{0,1\}. \end{aligned} \tag{36}$$

Indeed, the derivation of Eqs. (28) only uses the fact that $f_k$ and $g_k$ are complex functions; none of their other properties were used.

By Lemma 2, the first equation in system (36) for $|l_1| = 1$ has a unique solution $f_1, g_1 \in C^{1,\sigma}(\overline{D})$. By definition, this means that

$$f_1' = \frac{\partial f_1(z)}{\partial x} \in H^\sigma(\overline{D}), \qquad g_1' = \frac{\partial g_1(z)}{\partial x} \in H^\sigma(\overline{D}). \tag{37}$$

In view of Eqs. (37) and (35), the second equation in system (36) can be written in the form

$$(yf_1' + h_1 + y\overline{g_1'} + \overline{h_2} + l_2yg_1' + l_2h_2)|_\Gamma = \varphi_2;$$

i.e.,

$$(h_1 + \overline{h_2} + l_2h_2)|_\Gamma = \varphi_2 - (yf_1' + y\overline{g_1'} + l_2yg_1')|_\Gamma \in H^\sigma(\Gamma). \tag{38}$$

It remains to note that problem (38) has a unique solution $h_1, h_2$ in the function class $H^\sigma(\overline{D})$ by Lemma 2 for $|l_2| = 1$ and by Theorem 2 for $l_2 = 0$. In conclusion, we construct the desired solution of the Schwarz problem (2) by formula (15).

The case of $\lambda = \mu$ can be treated in a similar way with the use of Lemma 1. The proof of the theorem is complete.

**Remark 3.** Obviously, Theorem 6 can be generalized to the dimensions $n = 4k$, $k = 1, 2, \ldots$ In this case, the main diagonal of the Jordan form of the matrix $J$ must contain blocks $J_1$ of the form (33), and the matrix of the Jordan basis should be constructed of matrices $Q$ of the form (33). Then the Schwarz problem splits into $k$ independent problems similar to those considered in the proof of Theorem 6.

## 6. CONSTRUCTION OF NONTRIVIAL SOLUTIONS OF THE HOMOGENEOUS SCHWARZ PROBLEM IN AN ARBITRARY DOMAIN

As a corollary of the proof of Theorem 6, we present an algorithm for constructing nontrivial solutions of the homogeneous Schwarz problem. Let $n = 4$. We slightly change the matrix $J = QJ_1Q^{-1}$ in (33); namely, we set

$$J_1 = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 0 & \bar{\lambda} & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}, \qquad Q = (\mathbf{x}, \mathbf{y}, \overline{\mathbf{x}}, \overline{\mathbf{y}} + l_2\mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in \mathbb{C}^4, \qquad |l_2| = 1. \tag{39}$$

Here both the case of $\lambda = \mu$ and the case of $\lambda \neq \mu$ are possible. Let the Schwarz problem (2) be homogeneous; i.e., $\psi \equiv 0$. Here we also have the representation (34), and system (36) acquires the form

$$(f_1 + \overline{g_1})|_\Gamma = 0, \qquad (f_2 + \overline{g_2} + l_2 g_2)|_\Gamma = 0. \tag{40}$$

By virtue of the representation (34), $f_1$ and $g_1$ in Eqs. (40) are arbitrary $\lambda$- and $\bar{\lambda}$-holomorphic functions, respectively; $g_2$ is an arbitrary $\mu$-holomorphic function, and $g_2 = h_2$ and $f_2 = yf_1' + h_1$ in Eqs. (35), where $h_1$ is an arbitrary $\lambda$-holomorphic function.

Let $D \subset \mathbb{R}^2$ be an arbitrary domain bounded by a Lyapunov contour $\Gamma$. Note that the first equation in system (40) has infinitely many solutions $f_1$, $g_1$ in which the sum $f_1 + \overline{g_1}$ is identically zero, $f_1 + \overline{g_1} \equiv 0$. One can take, say,

$$f_1 = (x + \lambda y)^n, \qquad g_1 = -\overline{(x + \lambda y)^n}, \qquad n = 1, 2, \ldots$$

In view of the substitution $g_2 = h_2$, $f_2 = yf_1' + h_1$, the second equation in system (40) acquires the form

$$(h_1 + \overline{h_2} + l_2 h_2)|_\Gamma = -yf_1'|_\Gamma \in H^\sigma(\Gamma). \tag{41}$$

Problem (41) has a (unique) solution for the functions $h_1, h_2 \in H^\sigma(\overline{D})$ for $|l_2| = 1$ and $\lambda \neq \mu$ by Lemma 2 and for arbitrary $l_2 \in \mathbb{C}$ and $\lambda = \mu$ by Lemma 1. One can also set $l_2 = 0$ and apply Theorem 2.

Obviously, one can construct infinitely many such solutions of the functional equation (41) in one and the same domain $D$ depending on the choice of the functions $f_1, g_1 \in C^{1,\sigma}(\overline{D})$. Each of these solutions gives a nontrivial solution $\phi(z) \in H^\sigma(\overline{D})$ of the homogeneous Schwarz problem by formula (15). The vector functions $\phi(z)$ are linearly independent as well for an appropriate choice of the functions $f_1$ and $g_1$.

Further, let us prove that none of these solutions has the property $\operatorname{Re}\phi(z) \equiv 0$.

Indeed, assume the contrary. Then $\operatorname{Re}\phi|_\Gamma = 0$ on each contour $\Gamma$. Since the algorithm for the construction of the functions $\phi$ is invertible, we see that Eq. (41) also holds in this case on an arbitrary contour $\Gamma$ with the same functions $h_1$, $h_2$, and $f_1'$, which is impossible.

As a result, we have in particular proved that Theorem 3 is not true for arbitrary matrices $J$. Namely, the following theorem holds.

**Theorem 7.** *There exist matrices $J \in \mathbb{C}^{4\times 4}$ with eigenvalues lying on different sides of the real axis and such that the corresponding homogeneous Schwarz problem in any domain $D$ bounded by a Lyapunov contour has infinitely many linearly independent solutions in the function classes $\phi(z) \in H^\sigma(\overline{D})$.*

A method similar to that given in Remark 3 generalizes Theorem 7 and the algorithm for constructing nonuniqueness examples to dimensions $n = 4k$, $k = 1, 2, \ldots$

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