On Some New Classes of Pseudo-Differential Operators

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Abstract. We study Fredholm properties for a special class of elliptic pseudo-differential operators. Using a local principle we give boundedness theorems for such operators and describe their Fredholm properties in Sobolev-Slobodetskii spaces of a variable order. For a half space case we introduce a certain operator family which helps describing Fredholm properties.

INTRODUCTION

Pseudo-differential operators and equations were born in past century (see, for example, [1–3]) and now this theory lives and develops. In this paper we would like to enlarge class of spaces of spaces in which such pseudo-differential operators can act, and describe their Fredholm properties in these spaces. We use spaces of variable order and consider also pseudo-differential operators of variable order. First similar constructions were introduced in [4] (and further there were some generalizations and refinements), but our considerations contain new results related to boundedness and Fredholm properties and based on a local principle. Moreover we study a family of spaces of a variable order and intend to apply this methodology to appearing boundary value problems. Also we hope these consideration will be useful for more complicated situations when we deal with a cone instead of a half-space [5].

Local Sobolev–Slobodetskii Spaces

Let $s : \mathbb{R}^m \rightarrow \mathbb{R}$ be an arbitrary function satisfying the following conditions:

- the finite limit $\lim_{|x| \rightarrow +\infty} s(x)$ exists,
- the function $s(x)$ satisfies the Lipschitz condition in $\mathbb{R}^m$, i. e. there is a positive constant $C > 0$ such that
  $$|s(x_1) - s(x_2)| \leq C|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^m.$$

For fixed $x \in \mathbb{R}^m$ we introduce the following definition.

Definition 1 By definition a local Sobolev–Slobodetskii space $H^s(x)(\mathbb{R}^m)$ consists of distributions with finite value

$$
\|u\|_{s(x)} = \left( \int_{\mathbb{R}^m} (1 + |\xi|^{2s(x)}) |\hat{u}(\xi)|^2 d\xi \right)^{1/2},
$$

where $\hat{u}$ denotes the Fourier transform of the function $u$.

The value $\|u\|_{s(x)}$ is called a local $H^s$-norm of the function $u$.

For a brevity we will write $H^s(x)$ instead of $H^s(x)(\mathbb{R}^m)$, and if we speak on $H^s(x)$-functions with supports in a certain domain $D \subset \mathbb{R}^m$ then we write $H^s(x)(D)$. 

Definition 2. The universal Sobolev–Slobodetskii super-space $H^s_{\infty}$ is called the space which includes all local Sobolev–Slobodetskii spaces with finite norm

$$
\|u\|_{s_{\infty}} = \sup_{x \in \mathbb{R}^m} \|u\|_{s(x)}.
$$

The universal Sobolev–Slobodetskii subspace $H^s_0$ is called the space which is inside of all local Sobolev–Slobodetskii spaces with finite norm

$$
\|u\|_{s_0} = \inf_{x \in \mathbb{R}^m} \|u\|_{s(x)}.
$$

Obviously,

$$
\|u\|_{s_{\infty}} \leq \|u\|_{s_0} \leq \|u\|_{s_{\infty}}, \quad \forall x \in \mathbb{R}^m.
$$

Let us denote $S(\mathbb{R}^m)$ the Schwartz class of infinitely differentiable rapidly decreasing at infinity functions; this class is dense in each local Sobolev–Slobodetskii space [3].

Lemma 1. For $u \in S(\mathbb{R}^m)$ we have estimates

$$
|((1 + |\xi|)^{s_1} - (1 + |\xi|)^{s_2})| \leq c_1 |s_1 - s_2| \cdot (1 + |\xi|)^l,
$$

$$
\|u\|_{s_1(\xi)} - \|u\|_{s_2(\xi)} \leq c_2 |x_1 - x_2| \cdot \|u\|_{s_{\infty}}
$$

for a certain $l \in [s_1, s_2]$.

Let $H_x$ be a family of Hilbert spaces parametrized by points $x \in \mathbb{R}^m$. We denote $\|u\|_x$ the norm of element $u \in H^s(x)$ and assume that $S$ is a dense subset in $H^s(x)$, $\forall x \in \mathbb{R}^m$. For $u \in S$ we define the functional

$$
f(x, u) = \|u\|_x.
$$

Definition 3. We say that a family of Hilbert spaces $\{H_x\}_{x \in \mathbb{R}^m}$ is a local continuous family in the point $x_0 \in \mathbb{R}^m$ if for fixed $u \in S$ the functional $f(x, u)$ is continuous in the point $x_0$.

Lemma 2. The family $H^s(x)$ is a local continuous family.

**Operators of a Variable Order**

Definition 4. Given function $A(x, \xi)$ defined in $\mathbb{R}^m \times \mathbb{R}^m$ a pseudo-differential operator $A$ is called an operator of the following type

$$
(Au)(x) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} A(x, \xi) e^{i(x-y)\xi} \rho(y) dyd\xi, \quad x \in \mathbb{R}^m.
$$

The function $A(x, \xi)$ is called a symbol of the operator $A$.

Let $\alpha : \mathbb{R}^m \to \mathbb{R}$ be a function with the same properties as $s(x)$.

Definition 5. The class $E_{\alpha(x)}$ consists of functions $A(x, \xi)$ defined in $\mathbb{R}^m \times \mathbb{R}^m$ and satisfying the conditions

$$
c_1 (1 + |\xi|)^{\alpha(x)} \leq |A(x, \xi)| \leq c_2 (1 + |\xi|)^{\alpha(x)}
$$

and for each point $x_0 \in \mathbb{R}^m$ there exists a neighborhood $U_{x_0}$ such that for all $x \in U_{x_0}$ the following inequality

$$
|A(x, \xi) - A(x_0, \xi)| \leq c_3 |x - x_0| (1 + |\xi|)^{\alpha(x)}
$$

holds, where $c_1, c_2, c_3$ are positive constants. The function $\alpha : \mathbb{R}^m \to \mathbb{R}$ is called a variable order of a pseudo-differential operator (symbol).

If $x_0$ is an infinity then we need to require satisfying the following inequality

$$
|A(x, \xi) - A(\infty, \xi)| \leq c_3 |x|^{-1} (1 + |\xi|)^{\alpha(x)}
$$

instead of the inequality (3).

Example. A very simple example is so called fractional Laplacian of a variable order. Its symbol is

$$
A(x, \xi) = (1 + \xi_1^2 + \xi_2^2 + \cdots + \xi_m^2)^{\alpha(x)}.
$$

All conditions mentioned above are satisfied.
Boundedness Theorems

If we fix \( x_0 \in \mathbb{R}^m \) then the operator \( A_{x_0} \) is an operator with the symbol \( A(x_0, \xi) \).

**Lemma 3** If \( A(x_0, \xi) \in E_{s(a)}(x_0) \) then the operator \( A_{x_0} \) is bounded in local Sobolev–Slobodetskii space, \( A_{x_0} : H^{s(x_0)} \rightarrow H^{s(x_0)} \).

This property is called a local boundedness.

**Definition 6** An operator \( A \) with the symbol \( A(x, \xi) \) is called a local bounded operator if for each point \( x_0 \in \mathbb{R}^m \) the operator \( A_{x_0} : H^{s(x)} \rightarrow H^{s(x)} \) with the symbol \( A(x_0, \xi) \) is bounded.

**Theorem 4** If \( A(x, \xi) \in E_{s(a)} \) then the function \( x_0 \mapsto \|A_{x_0}\|_{s(x_0)} \) is continuous at each point \( x_0 \in \mathbb{R}^m \) including the infinity.

**Remark 1** It means the property of a local boundedness is conserved in enough small neighborhood.

**Theorem 5** If an operator \( A \) with the symbol \( A(x, \xi) \) is local bounded then it is bounded \( A : H^M \rightarrow H^{s-a} \).

**Remark 2** It seems it is enough to require a local boundedness on a certain countable set including the infinity.

### Compact Operators in Spaces \( H^{s(x)} \)

Let \( \{T_x : H^{s(x)} \rightarrow H^{s(x)}\}_{x \in \mathbb{R}^m} \) be a family of linear bounded operators.

**Lemma 6** If the linear bounded operator \( T_{x_0} : H^{s(x)} \rightarrow H^{s(x)} \) is compact then there is a neighborhood \( U_{x_0} \) of the point \( x_0 \) such that for all \( x_1 \in U_{x_0} \), the operator \( T_{x_1} : H^{s(x)} \rightarrow H^{s(x)} \) is compact.

Let \( \psi_{x_0}(x) \in C_0^\infty(\mathbb{R}^m) \) be a function equals to 1 in some neighborhood of the point \( x_0 \).

**Corollary 7** An operator with the symbol \( \psi_{x_0}(x)A(x, \xi) - A(x_0, \xi) \) is compact in some neighborhood of the point \( x_0 \) as operator \( H^{s(x)} \rightarrow H^{s(x)-a(x)} \).

### Fredholmness

**Definition 7** An operator \( A_{x_0} \) with the symbol \( A(x_0, \xi) \) (\( x_0 \in \mathbb{R}^m \) is fixed) we call the local representative of the operator \( A \) in the point \( x_0 \).

**Theorem 8** If local representatives \( A_{x_0} : H^{s(x_0)} \rightarrow H^{s(x_0)-a(x_0)} \) of the operator \( A \) are invertible at each point of \( \mathbb{R}^m \) including the infinity then the operator \( A : H^M \rightarrow H^{s(a)} \) has a Fredholm property.

**Remark 3** It is enough to require an invertibility for the local representatives in a certain dense set.

### A HALF-SPACE CASE

**Definition 8** Let \( A(x, \xi) \) be a function defined in \( \mathbb{R}^n \times \mathbb{R}^m \). A pseudo-differential operator \( A \) in a half-space with the symbol \( A(x, \xi) \) is called an operator of the following type

\[
(Au)(x) = \int_{\mathbb{R}^m} \left[ \int_{\mathbb{R}^n} A(x, \xi) e^{i(x-y)\xi} u(y) \, dy \right] \, d\xi, \quad x \in \mathbb{R}_+^n.
\]

We consider the following equation in a half-space

\[
(Au)(x) = v(x), \quad x \in \mathbb{R}_+^n,
\]

related to such an operator.
According to [3] we introduce the space \( H^{s}(\mathbb{R}^m) \) of functions from \( H^{s}(\mathbb{R}^m) \) with support in \( \mathbb{R}^m \), this is space of solutions, and the space \( H^{p,q}(\mathbb{R}^m) \) of functions (distributions) from \( S^{*}(\mathbb{R}^m) \) with support in \( \mathbb{R}^m \), these must admit analytical continuation in a whole \( H^{p,q}(\mathbb{R}^m) \) with finite norm

\[
\| \psi \|_{s(x)} = \inf \| \ell \psi \|_{s(x)},
\]

where \( \infimum \) is taken over all continuations \( \ell \).

For studying a Fredholmness of the equation (6) we use a local principle. We extract two types of local representatives.

1) For inner point \( x_0 \in \mathbb{R}^m \) this is well-known operator

\[
(A_{x_0}u)(x) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} A(x_0, \xi) e^{i(x,y)\xi} u(y) dy d\xi, \quad x \in \mathbb{R}^m.
\]

2) For boundary point \( x_0' = (x_0', 0) \in \mathbb{R}^{m-1} \) (we use notation \( x_0 = (x_0', x_0^0) \) for the \( x_0 \in \mathbb{R}^m \) ) it will be the operator

\[
(A_{x_0'}u)(x) = \int_{\mathbb{R}^m} \left[ \int_{\mathbb{R}^m} A(x_0', \xi) e^{i(x,y)\xi} u(y) d\xi \right] dy, \quad x \in \mathbb{R}^m.
\]

Thus, we have two operator families.

**Theorem 9**

If all operators of the family \( A_{x_0} : H^{p,q}(\mathbb{R}^m) \rightarrow H^{p,q}(\mathbb{R}^m) \) are bounded then the operator (5) is bounded as operator \( H^{p,q}(\mathbb{R}^m) \rightarrow H^{p,q}(\mathbb{R}^m) \).

If all operators of two operator family are invertible then the operator (5) is Fredholm operator as operator \( H^{p,q}(\mathbb{R}^m) \rightarrow H^{p,q}(\mathbb{R}^m) \).

**Variable Index of Factorization**

This is key point for studying pseudo-differential equations in a half-space. We will give a corresponding definition and show its applicability to the equation (7) with fixed \( x' \in \mathbb{R}^{m-1} \).

**Definition 9**

Factorization of the symbol \( A(x', \xi) \) on a variable \( \xi_m \) is called its representation in the form

\[
A(x', \xi, \xi_m) = A_+ (x', \xi, \xi_m) A_- (x', \xi, \xi_m),
\]

where the factors \( A_+ (A_-) \) admit analytical continuation on \( \xi_m \) in upper (lower) complex half-plane \( \xi_m = \pm \pi, \tau > 0 \), under almost all \( \xi' \in \mathbb{R}^{m-1} \) and satisfy estimates

\[
|A_+ (x', \xi, \xi_m)| \leq c_1 (1 + |\xi'| + |\xi_m| + |\tau|)^{\alpha(x')}, \quad |A_- (x', \xi, \xi_m)| \leq c_1 (1 + |\xi'| + |\xi_m| + |\tau|)^{\alpha(x') - \alpha(x')}, \quad \forall \tau \in \mathbb{R}.
\]

The function \( \alpha(x') \), \( x' \in \mathbb{R}^{m-1} \), is called a variable index of factorization.

Let us note that such factorization can be constructed effectively with a help of the Cauchy type integral [3]. Below we will assume that the function \( \alpha(x') \) has the same properties as \( s(x') \).

**A Local Solvability and Boundary Conditions**

Now there is a question how one can use the theorem 9. In other words we need conditions which guarantee an invertibility for local representatives of boundary operator.

**Theorem 10**

Let

\[
|\alpha(x') - s(x')| < 1/2, \quad \forall x' \in \mathbb{R}^{m-1}.
\]

Then the operator (7) is invertible as an operator \( H^{p,q}(\mathbb{R}^m) \rightarrow H^{p,q}(\mathbb{R}^m) \) under each fixed \( x' \).
Collecting the theorems 9, 10 we obtain a Fredholm property for pseudo-differential operator (equation) in a half-space.

**Theorem 11**  
Let \( A(x, \xi) \in E_{\gamma}(x) \). If 
\[
|\omega(x') - s(x')| < 1/2, \quad \forall x' \in \mathbb{R}^{m-1}, \quad \forall x' \in \mathbb{R}^{m-1},
\]
then the operator (5) has a Fredholm property as an operator \( H^m(\mathbb{R}^m_+) \to H^{\gamma-m/2}(\mathbb{R}^m_+) \).

There are situations for which the condition (8) does not hold. We consider here one of possible variants.

Let \( \omega(x') - s(x') = m + \delta, |\delta| < 1/2, m \in \mathbb{N}, \forall x' \in \mathbb{R}^{m-1} \). Now for fixed \( x' \) even the operator (7) is non-invertible, but we know a general solution of corresponding equation [3]. We will briefly describe its form to explain appearing boundary operators.

Thus, we consider an equation with the operator (7) and right-hand side \( f \in H^0(0, \omega(x'))(\mathbb{R}^m) \).

We will remind that for fixed \( x'_0 \in \mathbb{R}^{m-1} \) a general solution of the equation
\[
(A_{\gamma'} u)(x) = f(x), \quad x \in \mathbb{R}^m
\]
with the operator (7) in Fourier image has the following form [3]
\[
\hat{u}(\xi, x'_0, \xi') = \hat{A}_m^{-1}(x'_0, \xi) \hat{P}_m(\xi) \hat{\Pi} \hat{P}_m^{-1}(x'_0, \xi) \hat{A}_m^{-1}(x'_0, \xi') \sum_{k=1}^{m} \hat{c}_k(x'_0, \xi') \xi^{-k-1}_m , \quad (9)
\]
where the operator \( \hat{\Pi}_x \) is the Cauchy type integral
\[
(\hat{\Pi}_x \hat{u})(\xi) = \frac{1}{2\pi i} \lim_{r \to 0^+} \int_{C} \hat{u}(\xi', \eta, \xi') d\eta,
\]
\( P_m(\xi) \in E_m \) is an arbitrary polynomial, \( c_k(x'_0, \xi') \) are arbitrary functions from \( H^{\alpha}(\mathbb{R}^{m-1}) \), \( s_k(x'_0) = s(x'_0) - \omega(x'_0) + k - 1/2, k = 1, \ldots, m \).

To determine uniquely \( c_k \) we choose \( m \) bounded pseudo-differential operators \( B_j : H^0(0, \omega(x'))(\mathbb{R}^m) \to H^{0-\alpha_j}(\mathbb{R}^m) \) with symbols \( B_j(x'_0, \xi), j = 1, \ldots, m \), and \( \gamma \) is a restriction operator on the hyperplane \( x_m = 0 \) so that operators \( \gamma B_j \) are bounded \( H^0(0, \omega(x'_0))(\mathbb{R}^m) \to H^{0-\alpha_j}(\mathbb{R}^m) \) under the condition \( s(x'_0) - \alpha_j(x'_0) - 1/2 > 0, \forall x'_0 \in \mathbb{R}^{m-1} \). Let us introduce a new local sobolev-Slobodetskii space as a direct sum
\[
H^m(0, \omega(x'_0))(\mathbb{R}^m) \cong H^m(\mathbb{R}^m_+) \oplus \sum_{j=1}^{m} H^{0-\alpha_j}(\mathbb{R}^{m-1})
\]
For the point \( x'_0 \in \mathbb{R}^{m-1} \) we introduce a local operator \( H^0(0, \omega(x'_0))(\mathbb{R}^m) \to H^m(0, \omega(x'_0))(\mathbb{R}^m) \) by the formula
\[
B_{x'_0} u = (A_{x'_0} u, \gamma B_1 u, \ldots, \gamma B_m u).
\]

Therefore, the operator \( B_{x'_0} \) is given by pasting \( m \) additional operators together with a family of additional spaces to the operator \( A_{x'_0} \) at each point \( x'_0 \in \mathbb{R}^{m-1} \).

We can conclude as above that under our assumptions all such operators are locally bounded, and it implies a boundedness of the (10).

One can study an invertibility of the local operator (10) by the Fourier transform, it permits to reduce an identification problem for the functions \( c_k \) to unique solvability of some \( m \times m \)-system of linear algebraic equations. Non-vanishing for a determinant of the latter system is necessary and sufficient condition for an invertibility of the local operator \( B_{x'_0} \).

**Remark 4**  
The left case \( \omega(x') - s(x') = -m + \delta, |\delta| < 1/2, m \in \mathbb{N}, \forall x' \in \mathbb{R}^{m-1} \), can’t considered analogically. For this case we extend the domain of \( A_{x'_0} \), and we past to the domain additional local Sobolev-Slobodetskii spaces with additional unknowns in these spaces.
CONCLUSION

It seems a lot of results obtained by a local principle can be transferred on operators and equations of a variable order. In our opinion more interesting case is studying such operators and equations on manifolds with non-smooth boundaries like [5–9].

ACKNOWLEDGMENTS

This work was supported by the State contract of the Russian Ministry of Education and Science (contract No 1.7311.2017/8.9).

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