ON THE CONNECTION OF FRACTIONAL MOMENTS OF THE RIEMANN ZETA-FUNCTION WITH ITS ZEROS

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ABSTRACT. The paper presents a lower estimate for the number of zeros of the Riemann zeta-function on a segment of the critical line.

1. Introduction

Let s be a complex variable, $s = \sigma + it$.

Definition 1.1. For $\sigma > 1$, the Riemann zeta-function $\zeta(s)$ is given by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

The function $\zeta(s)$ satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{1.1}$$

In the theory of zeta-function, one of the directions of studies is the study of its zero distribution. Denote by $N_0(T)$ the number of zeros of $\zeta(1/2+it)$, $0 < t \le T$, $T > T_0 > 0$.

Definition 1.2. The Hardy function $Z(t)$ is given by

$$Z(t) = e^{i\theta(t)}\zeta\left(\frac{1}{2}+it\right), \tag{1.2}$$

where

$$e^{i\theta(t)} = \pi^{-it/2}\frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right|}.$$

From the functional equation (1.1), it follows that the function $Z(t)$ assumes real values for real t. The real roots of the function $Z(t)$ are the zeros of the zeta-function on the critical line. Studying zeros of $\zeta(s)$ on the critical line reduces to studying the real zeros of the function $Z(t)$.

In [1], Hardy and Littlewood proved the inequality

$$N_0(T) \gg T,$$

where the constant in the Vinogradov sign is absolute.

In 1942, in [5], A. Selberg obtained the following estimate, which is correct with respect to the order:

$$N_0(T) \gg T \log T.$$

In 2002, in [3], A. A. Karatsuba proposed a new technique for estimating $N_0(T)$; on this base, he obtained the inequality

$$N_0(T) \gg T \log^{1/8} T.$$

The approach of Karatsuba consists in the fact that to the Hardy–Littlewood method, he added correct estimates for the fractional moments having the form

$$I_l(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/l} dt,$$

where l is a positive integer.

Note that the result of Karatsuba is less exact than that of Selberg, but more exact than the result of Hardy and Littlewood.

In the present work, we improve the result of Karatsuba. Such an improvement is obtained owing to the circumstance that we use the lower estimates for $I_l(T)$ for all rational positive l and not only for natural l, as in [3]. The following theorem is the main assertion of the paper.

Theorem 1.1. *Let $\varepsilon > 0$ be an arbitrarily small number. There exists a positive constant $c_0 = c_0(\varepsilon)$ such that the following inequality holds for $T > T_0 > 0$:*

$$N_0(T) \geq c_0 T \log^{1/4-\varepsilon} T.$$

2. Lemmas

Lemma 2.1. *Let $\zeta\left(\frac{1}{2} + it\right)$ be the Riemann zeta-function. Then the following inequalities hold:*

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/l} dt \gg T \log^{1/l^2} T, \quad l \text{ is a rational number;}$$

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2/l} dt \ll T \log^{1/l^2} T, \quad l \text{ is a natural number.}$$

The proof is given in [2].

Lemma 2.2. *Let $T \leq t \leq T + H$, $H \leq \frac{T}{\log T}$, $T \geq T_0 > 0$, and let the function $Z(t)$ be defined by relation (1.2). Then the following formula holds:*

$$Z(t) = 2 \sum_{n \leq P} \frac{1}{\sqrt{n}} \cos(t \log P - t \log n + \varphi(t)) + O\big(H^3 T^{-7/4}\big) + O\big(T^{-1/4} \log T\big),$$

where

$$P = \sqrt{\frac{T}{2\pi}}, \quad \varphi(t) = -\frac{T}{2} - \frac{\pi}{8} + \frac{(t-T)^2}{4t}.$$

Proof. Simplifying the approximate functional equation of [4], for $T \leq t \leq T + H$, we obtain the assertion of the lemma.

Lemma 2.3. *Let $a(n)$ be arbitrary complex numbers, $0 < X < X_1 \le 2X$ and $3 \le N < N_1 \le 2N$. The following inequality holds:*

$$\int_X^{X_1} \left| \sum_{N<n\le N_1} a(n) n^{it} \right|^2 dt \le (X + 32N\log N) \sum_{N<n\le N_1} |a(n)|^2.$$

For the proof, see [6].

Lemma 2.4. *Let $H > 10$, $H < H_1 \le 2H$, $0 < \Delta < 1$, and $H^{5/6} < P < H^{7/8}$. Then the following estimates hold:*

$$\int_H^{H_1} \left| \sum_{n\le P\Delta} \frac{\lambda_1}{\sqrt{n}\log\left(\dfrac{P}{n}\right)} n^{it} \right|^2 dt \ll H \log^{-1}\frac{1}{\Delta},$$

$$\int_H^{H_1} \left| \sum_{P\Delta<n\le P} \frac{\lambda_2}{\sqrt{n}} n^{it} \right|^2 dt \ll H \log\frac{1}{\Delta},$$

where λ_1 and λ_2 are complex numbers such that $|\lambda_1| = |\lambda_2| = 1$.

Proof. For $P < H^{7/8}$ let us use Lemma 2.3 to the integrals from the condition of the lemma. Then

$$\int_H^{H_1} \left| \sum_{n\le P\Delta} \frac{\lambda_1}{\sqrt{n}\log\left(\frac{P}{n}\right)} n^{it} \right|^2 dt \le 2H \sum_{n\le P\Delta} \frac{1}{n\log^2\left(\frac{P}{n}\right)},$$

$$\int_H^{H_1} \left| \sum_{P\Delta<n\le P} \frac{\lambda_2}{\sqrt{n}} n^{it} \right|^2 dt \le 2H \sum_{P\Delta<n\le P} \frac{1}{n}.$$

Since the functions $f_1(x) = x\log^2(P/x)$ and $f_2(x) = x$ monotonically increase for $1 \le x \le P$, it follows that

$$\sum_{n\le P\Delta} \frac{1}{n\log^2\left(\dfrac{P}{n}\right)} \le \frac{1}{\log^2 P} + \int_1^{P\Delta} \frac{1}{x\log^2\left(\dfrac{P}{x}\right)}\, dx \ll \log^{-1}\frac{1}{\Delta},$$

$$\sum_{P\Delta<n\le P} \frac{1}{n} \ll \log\frac{1}{\Delta}.$$

The lemma is proved.

3. Proof of Theorem

Let the function $Z(t)$ be defined by formula (1.2), and let $T \ge T_0 > 0$, $0 < \varepsilon < \dfrac{1}{8}$, and $h = h(T) = c(\log T)^{-1/2+2\varepsilon}$, where c is a positive constant.

Let us consider two integrals $j_1(t)$ and $j_2(t)$.

$$j_1(t) = \int_0^h |Z(t+u)|\, du, \qquad j_2(t) = \left| \int_0^h Z(t+u)\, du \right|.$$

Denote by E the subset of the interval $(0, T)$ that consists of numbers t such that

$$j_2(t) < j_1(t).$$

Now, if $t \in E$, then on the interval $0 < u < h$, the function $Z(t+u)$ has a zero of odd order.

It is seen from the definition of the set E that

$$\int_0^T (j_1(t))^{1-4\varepsilon}\,dt \le \int_E (j_1(t))^{1-4\varepsilon}\,dt + \int_0^T (j_2(t))^{1-4\varepsilon}\,dt.$$

Let $\mu(E)$ be the measure of the set E; then the following inequality holds:

$$I_1 \le I_2 + \mu(E)^{4\varepsilon} I_3^{1-4\varepsilon}, \tag{3.1}$$

where

$$I_1 = \int_0^T (j_1(t))^{1-4\varepsilon}\,dt, \qquad I_2 = \int_0^T (j_2(t))^{1-4\varepsilon}\,dt, \qquad I_3 = \int_0^T j_1(t)\,dt.$$

1. Let us estimate I_1 from below.

Using the Hölder inequality, we obtain

$$h^{4\varepsilon}(j_1(t))^{1-4\varepsilon} \ge \int_0^h |Z(t+u)|^{1-4\varepsilon}\,du.$$

Using the latter inequality and the definition of the function $Z(t)$, we have the following chain of inequalities:

$$I_1 = \int_0^T (j_1(t))^{1-4\varepsilon} dt \ge h^{-4\varepsilon} \int_0^T \int_0^h |Z(t+u)|^{1-4\varepsilon} du dt \ge h^{1-4\varepsilon} \int_1^T |\zeta(1/2+it)|^{1-4\varepsilon} dt.$$

Let us apply Lemma 2.1 to the integral on the right-hand side of the inequality; for $l = \dfrac{2}{1-4\varepsilon}$, we have

$$I_1 \ge c_1 h^{1-4\varepsilon} T \ln^{\frac{(1-4\varepsilon)^2}{4}} T. \tag{3.2}$$

2. Let us change the order of integration in the integral I_3:

$$I_3 = \int_0^T \int_0^h |Z(t+u)|\,du\,dt = \int_0^h \int_u^{T+u} |Z(t)|\,dt\,du \le h \int_0^{T+1} |Z(t)|\,dt.$$

Apply Lemma 2.1 to the latter integral; for $l = 2$, we have

$$I_3 \le c_2 h T \log^{1/4} T. \tag{3.3}$$

3. Let us find an upper estimate for the integral I_2. First, let us use the Hölder inequality for $I_2^{2/(1-4\varepsilon)}$; we have

$$I_2^{2/(1-4\varepsilon)} \le T^{(1+4\varepsilon)}(1-4\varepsilon) \int_0^T j_2^2(t)\,dt = T^{(1+4\varepsilon)}(1-4\varepsilon)(A_1 + A_2),$$

where

$$A_1 = \int_0^{T/\log T} j_2^2(t)\,dt, \qquad A_2 = \int_{T/\log T}^{T} j_2^2(t)\,dt.$$

The following inequality holds:

$$j_2^2(t) = \left|\int_0^h Z(t+u)\,du\right|^2 \le h\int_0^h |Z(t+u)|^2\,du.$$

Using the latter inequality and Lemma 2.1 with $l=1$, for the integral A_1, we obtain

$$A_1 \le h^2 \int_0^{(T/\log T)+1} |\zeta(t)|^2\,dt \ll h^2 T \ll hT.$$

Take $H = \dfrac{T^{7/12}}{\log T}$, $g_1 = [T^{5/12}]$, $g_2 = [T^{5/12}\log T]$. The following inequality holds:

$$A_2 \le \sum_{g=g_1}^{g_2} \int_{gH}^{(g+1)H} j_2^2(t)\,dt.$$

Let us use Lemma 2.2 for $gH < t \le gH+H$ and $P = \sqrt{\dfrac{gH}{2\pi}}$, $\varphi(t) = -\dfrac{gH}{2} - \dfrac{\pi}{8} + \dfrac{(t-gH)^2}{4t}$; we have

$$Z(t+u) = 2\sum_{n\le P}\frac{1}{\sqrt{n}}\cos\left((t+u)\log\frac{P}{n} + \varphi(t)\right) + O\big(\log^{-9}T\big).$$

Denote $P_1 = Pe^{-1/h}$; then the integral $j_2(t)$ can be rewritten as follows:

$$j_2(t) \ll \left|\int_0^h \sum_{n\le P_1}\frac{1}{\sqrt{n}}\cos\left((t+u)\log\frac{P}{n}+\varphi(t)\right)du\right|$$

$$+\int_0^h \left|\sum_{P_1<n\le P}\frac{1}{\sqrt{n}}\cos\left((t+u)\log\frac{P}{n}+\varphi(t)\right)\right|du + \frac{h}{\log^3 T}.$$

Applying the Euler formula, we arrive at the relation

$$j_2(t) \ll \left|\sum_{n\le P_1}\frac{1}{\sqrt{n}\log\left(\frac{P}{n}\right)}n^{i(t+h)}\right| + \left|\sum_{n\le P_1}\frac{1}{\sqrt{n}\log\left(\frac{P}{n}\right)}n^{it}\right| + \int_0^h\left|\sum_{P_1<n\le P}\frac{1}{\sqrt{n}}n^{i(t+u)}\right|du + \frac{h}{\log^3 T}.$$

Using the latter inequality, we have

$$\int_{gH}^{(g+1)H} j_2^2(t)\,dt \ll \int_{gH}^{(g+1)H}\left|\sum_{n\le P_1}\frac{1}{\sqrt{n}\log\left(\frac{P}{n}\right)}n^{i(t+h)}\right|^2 dt + \int_{gH}^{(g+1)H}\left|\sum_{n\le P_1}\frac{1}{\sqrt{n}\log\left(\frac{P}{n}\right)}n^{it}\right|^2 dt$$

$$+h^2\int_{gH}^{(g+1)H}\left|\sum_{P_1<n\le P}\frac{1}{\sqrt{n}}n^{i(t+u_1)}\right|^2 dt + \frac{h^2 H}{\log^6 T},$$

where

$$0 \le u_1 \le h.$$

Using Lemma 2.4, we can estimate the integral A_2 as follows:

$$A_2 \ll hT.$$

Therefore,

$$I_2 \le c_3 h^{(1-4\varepsilon)/2} T. \tag{3.4}$$

4. Let us estimate the measure of the set E from below. Let us choose $c = \left(\frac{2c_3}{c_1}\right)^{2/(1-4\varepsilon)}$. Substituting estimates (3.2), (3.3), and (3.4) in inequality (3.1), we have

$$\mu(E) \ge c_4 T \ln^{-(1-4\varepsilon)/4} T, \quad c_4 > 0. \tag{3.5}$$

Let us partition the interval $(0, T)$ into intervals of the form $(hv, hv + h)$, $v = 0, 1, \dots, [T/h]$. In view of (3.5), among these intervals, at least $c_4 T(\log T)^{-(1-4\varepsilon)/4} h^{-1}$ of them contain points of the set E, and the interval $(hv, hv + h)$ contains a zero of the function $Z(t)$. Hence, on the interval $(0, T)$, the number of zeros of the function $Z(t)$ having odd order is no less than $T \log^{(1/4)-\varepsilon} T$. The theorem is proved.

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