ON THE DISTRIBUTION OF NORMS OF PRIME IDEALS OF A GIVEN CLASS IN ARITHMETIC PROGRESSIONS

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Let $\mathcal{C}$ be a class of ideals of the ring of algebraic numbers of an imaginary quadratic field. Let l and q be relatively prime integers, $1 \leqslant q \leqslant \log^{A_1} x$, and $A_1 > 1$. An asymptotic formula for the number $\pi_1(x, q, l, \mathcal{C})$ of prime ideals belonging to the class $\mathcal{C}$ whose norms do not exceed x and lie in an arithmetic progression is obtained in this paper. Bibliography: 6 titles.

Introduction

In solving additive problems with prime numbers by the circle method, the law of distribution of prime numbers in arithmetic progressions plays an important role. For example, the following formula, known in the literature as the Siegel–Walfisz formula, is frequently used:

$$\pi(x, q, l) = \frac{\operatorname{Li} x}{\varphi(q)} + O(xe^{-c\sqrt{\log x}}), \tag{1}$$

where $\pi(x, q, l) = \sum\limits_{\substack{p \leqslant x \\ p \equiv l \ (\mathrm{mod}\ q)}} 1$, $\operatorname{Li} x = \int_2^x \frac{dx}{\log x}$, l and q are integer, relatively prime numbers, $1 \leqslant q \leqslant \log^{A_1} x$, $A_1 > 1$, and $c > 0$ depends only on A_1.

For example, see [1, p. 154] for its proof.

Under the same conditions on l and q, a formula similar to (1) is valid for the function $\pi_1(x, q, l) = \sum\limits_{\substack{N(P) \leqslant x \\ N(P) \equiv l\,(\mathrm{mod}\ q)}} 1$, where summation ranges over prime ideals P of the ring of integral algebraic numbers of an imaginary quadratic field and $N(P)$ is the norm of an ideal P.

If the discriminant of the field does not grow with x, then the derivation of the formula for $\pi_1(x, q, l)$ coincides in essence with the derivation of the formula for $\pi(x, q, l)$.

In the problems on the representation of prime numbers by quadratic forms, an asymptotic formula is needed for the number of prime ideals the norms of which do not exceed x and lie in an arithmetic progression, but the ideals themselves belong to a certain fixed class of ideals. The present paper is devoted to the derivation of such a formula.

If the quadratic field contains many classes, then our problem differs significantly from the problem of obtaining the asymptotics for $\pi_1(x, q, l)$, because the problem of whether an ideal belongs to a given class of ideals does not reduce to the problem of whether the norm of an ideal belongs to a residue class with respect to a certain modulus (see [2, p. 271]).

We sketch the line of argument. Let $\mathcal{C}$ be a class of ideals, $\pi_1(x, q, l, \mathcal{C}) = \sum\limits_{\substack{P \in \mathcal{C} \\ M(P) \equiv l\ (\mathrm{mod}\ q) \\ N(P) \leqslant x}} 1$, $\mathcal{A}$ be ideal classes, $X(\mathcal{A})$ be characters of the ideal class group, X_0 be a principal character of the ideal class group, h be its order, $\chi(n)$ be Dirichlet characters modulo q, and χ_1 be a character of the quadratic field. Let $(l, q) = 1$, $1 \leqslant q \leqslant \log^{A_1} x$, $A_1 > 1$, and $c > 0$. Then

$$\pi_1(x, q, l, \mathcal{C}) = \frac{1}{\varphi(q)} \sum_{\substack{P \in \mathcal{C} \\ N(P) \leqslant x}} 1 + \frac{1}{h\varphi(q)} \sum_{\chi \neq \chi_0} \overline{\chi}(l) \sum_{p \leqslant x} \big(1 + \chi_1(p)\big)\chi(p)$$

$$= \frac{1}{h\varphi(q)} \sum_{X \neq X_0} \overline{X}(\mathcal{C}) \sum_{\chi \neq \chi_0} \overline{\chi}(l) \sum_{N(P) \leqslant x} X(P)\chi(N(P)) + O(\sqrt{x}\log x). \tag{2}$$

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The formula
$$\frac{1}{\varphi(q)} \sum_{\substack{P\in C\\ N(P)\leqslant x}} 1 = \frac{\mathrm{Li}\, x}{h\varphi(q)} + O(xe^{-c\sqrt{\log x}}) \quad (c>0)$$
is well known (for example, see [3, Chap. 7]).

Let χ_2 be a Dirichlet character modulo q. It is well known that
$$\sum_{p\leqslant x} \chi_2(p) = \begin{cases} \mathrm{Li}\, x + O(xe^{-c\sqrt{\log x}}) & \text{if } \chi_2 \text{ is a principal character,}\\ O(xe^{-c\sqrt{\log x}}) & \text{if } \chi_2 \text{ is a nonprincipal character}\end{cases}$$
(for a proof, see, for example, [1, Chap. IX]).

Hence we have
$$\frac{1}{h\varphi(q)} \sum_{\chi\neq\chi_0} \overline{\chi}(l) \sum_{p\leqslant x} (1+\chi_1(p))\chi(p) = \frac{\chi_1(l)}{h\varphi(q)}\chi(q;D,0)\mathrm{Li}\, x + O(xe^{-c\sqrt{\log x}}),$$
where $\chi(q;D,0)=1$ if $D\mid q$ and $\chi(q;D,0)=0$ if $D\nmid q$, where D is the modulus of the character χ_1 equal to the absolute value of the discriminant of the quadratic field.

By formula (2), we obtain
$$\pi_1(x,q,l,\mathcal{C}) = \frac{1+\chi(q;D,0)\chi_1(l)}{h\varphi(q)}\mathrm{Li}\, x + \frac{1}{h\varphi(q)} \sum_{X\neq X_0} \overline{X}(\mathcal{C}) \sum_{\chi\neq\chi_0} \overline{\chi}(l) \sum_{N(P)\leqslant x} X(P)\chi(N(P)) + O(xe^{-c\sqrt{\log x}}). \quad (3)$$

Now, to obtain an asymptotics of the function $\pi_1(x,q,l,\mathcal{C})$ it is sufficient to estimate the sum
$$\sum_{X\neq X_0} \overline{X}(\mathcal{C}) \sum_{\chi\neq\chi_0} \overline{\chi}(l) \sum_{N(P)\leqslant x} X(P)\chi(N(P)).$$

This problem is solved in the present paper, the main result of which is given in the following theorem.

Theorem 1. *Let X be a nonprincipal character of the ideal class group of the ring of integral algebraic numbers of an imaginary quadratic field $\mathbb{Q}(\sqrt{d})$, and let χ be a nonprincipal primitive Dirichlet character $\mathrm{mod}\, r$, $1<r\leqslant \log^{A_1} x$, $A_1\geqslant 1$. Then the estimate*
$$\sum_{N(P)\leqslant x} X(P)\chi(N(P)) = O\Big(x\exp(-c\,(\log x)^{\frac{1}{20A_1}})\Big)$$
is valid, where $c=c(A_1)>0$.

The proof is based on the method of generating functions. Necessary information about the properties of the Dirichlet series
$$F(s,X,\chi) = \sum_{A} X(A)\chi(N(A))N(A)^{-s},$$
where summation is taken over all integral ideals A, is contained in the lemmas below.

We use the following notation in the paper:

X is a character of the ideal class group of the ring of integral algebraic numbers of a field $F=\mathbb{Q}(\sqrt{d})$, where d is a negative square-free number, h is the order of this group, δ_F is the discriminant of the field F, and $D=-\delta_F$;

χ_1 is the character of the quadratic field F defined in the following way:
$$\chi_1(n) = \begin{cases} \left(\dfrac{n}{|d|}\right) & \text{if } d\equiv 1 \pmod 4,\\ (-1)^{\frac{(n-1)}{2}}\left(\dfrac{n}{|d|}\right) & \text{if } d\equiv 3 \pmod 4,\\ (-1)^{\frac{n^2-1}{8}+\frac{n-1}{2}\frac{d'-1}{2}}\left(\dfrac{n}{|d'|}\right) & \text{if } d\equiv 2 \pmod 4,\ d=2d';\end{cases}$$

$$\chi(m;q,a)=\begin{cases}1 & \text{if} \quad m\equiv a \pmod q,\\ 0 & \text{otherwise};\end{cases}$$

$$G(r,u,\overline{n})=\sum_{m=1}^{r}\sum_{n=1}^{r}\exp(\frac{2\pi i}{r}(u(am^2+bmn+cn^2)+mn_1+nn_2))$$

is a double Gauss sum associated with integer, relatively prime numbers u and r, a quadratic form $am^2+bmn+cn^2$, and an integer vector $\overline{n}=\begin{pmatrix}n_1\\ n_2\end{pmatrix}$; $a(n)=a_X(n)=\sum\limits_{A,\ N(A)=n} X(A)$, $a_0(n)=a_{X_0}(n)$.

Lemmas

Let X is a nonprincipal character of the ideal class group. Let r and u be integer, relatively prime numbers. For $\operatorname{Re} s>1$, we define the Dirichlet series $F(s,r,u)$ and $F(s,r,u,\mathcal{A})$ by the relations

$$F(s,r,u)=\sum_{A}X(A)e^{2\pi i\frac{uN(A)}{r}}N(A)^{-s},\qquad F(s,r,u)=\sum_{\mathcal{A}}X(\mathcal{A})F(s,r,u,\mathcal{A}),$$

where

$$F(s,r,u,\mathcal{A})=\sum_{A\in\mathcal{A}}e^{2\pi i\frac{uN(A)}{r}}N(A)^{-s}.$$

Let $\mathcal{A}$ be a class of ideals, B be an ideal of the class $\mathcal{A}^{-1}$, and ξ represent integral algebraic numbers of the ideal B. It is known that if ξ ranges over all nonzero numbers of the ideal B, then the numbers $\frac{N(\xi)}{N(B)}$ range over the values of the norms of integral nonzero ideals of the class $\mathcal{A}$, and in the sequence $\frac{N(\xi)}{N(B)}$ the norm of each nonzero ideal of the class $\mathcal{A}$ occurs precisely two times (for example, see [4, Chap. 21]).

Define a binary quadratic form with integer coefficients $Q(\overline{n})=\frac{N(\xi)}{N(B)}=an_1^2+bn_1n_2+cn_2^2$, where $\overline{n}=\begin{pmatrix}n_1\\ n_2\end{pmatrix}$ is the vector of coordinates of a number ξ in an integral basis of the ideal B. Let $Q_1(\overline{n})=an_2^2-bn_1n_2+cn_1^2$.

Lemma 1. 1. *The function* $(\frac{\sqrt{|\delta_F|}r}{2\pi})^s\Gamma(s)F(s,r,u,\mathcal{A})$ *is analytic on the whole complex plane, except for the point* $s=1$, *at which it has a simple pole with residue* $\frac{\pi}{r^2}G(r,u,\overline{0})$.

2. *For* $\operatorname{Re} s<0$ *the identity*

$$\left(\frac{\sqrt{|\delta_F|}r}{2\pi}\right)^s\Gamma(s)F(s,r,u,\mathcal{A})=\frac{1}{2}\left(\frac{\sqrt{|\delta_F|}r}{2\pi}\right)^{1-s}\Gamma(1-s)\sum_{\overline{n}\neq\overline{0}}(Q_1(\overline{n}))^{s-1}\frac{G(r,u,\overline{n})}{r}$$

is valid.

3. *The function* $F(s,r,u)$ *is entire for any* r *and* u.

For a proof, see [5].

Corollary 1. *Let characters* X *and* χ *be nonprincipal and* χ *be primitive modulo* $r>1$. *Then*

$$F(s,X,\chi)=\sum_{A}X(A)\chi(N(A))N(A)^{-s}$$

is an entire function.

Proof. The following formula is valid:

$$F(s,X,\chi)=\frac{1}{\tau}\sum_{u=1}^{r}\overline{\chi}(u)F(s,r,u),$$

where τ is the Gauss sum associated with the character $\overline{\chi}$, $\operatorname{Re} s>1$. Now the corollary follows from Lemma 2.

Lemma 2. *Let ε be an arbitrary number in the interval $(0, 0.001)$, and let r and u be integer, relatively prime numbers, $r \geqslant 1$. Let χ be a primitive Dirichlet character modulo r.*

Then the estimates

$$\sum_{n\leqslant x} a(n)e^{2\pi i\frac{un}{r}} = O(x^{\frac{1}{3}+\varepsilon}r^{\frac{2}{3}}), \quad \sum_{n\leqslant x} a(n)\chi(n) = O(x^{\frac{1}{3}+\varepsilon}r^{\frac{7}{6}})$$

are valid.

Proof. The first estimate was obtained in [6], and the second estimate is a consequence of the first one and the known formula $\chi(n) = \frac{1}{\tau}\sum_{u=1}^{r}\overline{\chi}(u)e^{2\pi i\frac{un}{r}}$, where τ is the Gauss sum corresponding to the character $\overline{\chi}$.

Lemma 3. *The following asymptotic formula is valid:*

$$\sum_{\substack{(n,\delta_F)=1\\ n\leqslant x}} a_0(n) = c_0x + O(x^{\frac{1}{3}+\varepsilon}),$$

where $c_0 > 0$ and ε is an arbitrary positive number.

The asymptotic formula is derived essentially in the same way as is the estimate in Lemma 2. The difference is that the generating function

$$\sum_{\substack{n=1\\(n,p_1)=1}}^{\infty} a_0(n)n^{-s} = \prod_{p|\delta_F}(1-p^{-s})\zeta(s)L(s,\chi_1)$$

has a simple pole with residue $c_0 = \prod_{p|\delta_F}(\frac{p-1}{p})L(1,\chi_1)$ at the point $s = 1$, which implies the appearance of a principal term.

Lemma 4. *Let ε be an arbitrary number such that $0 < \varepsilon < 0.001$. Let X be a real nonprincipal character of the ideal class group of an imaginary quadratic field $\mathbb{Q}(\sqrt{d})$, let χ be a nonprincipal primitive character modulo r. If the character χ is real or the character $\chi^2\chi_1$ is principal, then there exists a constant $c > 0$, dependent only on ε, such that*

$$|F(1,X,\chi)| > \frac{c}{r^{3.5+\varepsilon}}.$$

Proof. Let $\operatorname{Re} s > 1$. The following relations are valid:

$$F(s,X,\chi) = \prod_{P|\delta_F}(1 - X(P)\chi(N(P))N(P)^{-s})^{-1}F_1(s,X,\chi)$$

and

$$F_1(s,X,\chi) = \prod_{\substack{(P,\delta_F)=1,\ N(P)=p\\ \chi_1(p)=1}}(1 - X(P)\chi(p)p^{-s})^{-1}\prod_{\substack{(P,\delta_F)=1,\ N(P)=p^2\\ \chi_1(p)=-1}}(1-\chi^2(p)p^{-2s})^{-1}.$$

This implies that it is sufficient to estimate $F_1(1,X,\chi)$ from below.

The conditions of the lemma imply that the coefficients of the Dirichlet series $F_1(s,X,\chi)$ are real. Indeed, if χ is a real character, then this is obvious, because $X(P) \in \mathbb{R}$. Let $\chi^2\chi_1$ be a principal character. Then $\chi^2(N(P))\chi_1(N(P))$ can take only the values 0 or 1. Therefore if $N(P) = p$ and $\chi_1(p) = 1$, then $\chi^2(p)$ can be equal either to 0 or to 1, i.e., $\chi(p)$ can be equal either to 0 or to 1, or to -1; if $N(P) = p^2$ and $\chi_1(p) = -1$, then $\chi^2(p)$ in this case can be equal either to 0 or to -1.

For $\frac{1}{2} \leqslant t < 1$, consider the function

$$H(t) = \sum_A\Big(\sum_{B|A}X(B)\chi(N(B))\Big)\chi_1^2(N(A))t^{N(A)}$$

(A ranges over integral ideals and B ranges over the set of divisors of A).

If $A = P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_n^{\alpha_n}$ is the canonical decomposition of an ideal A into the product of prime ideals, then

$$\sum_{B|A} X(B)\chi(N(B)) = \prod_{j=1}^{n} (1 + X(P_j)\chi(N(P_j)) + \cdots + X^{\alpha_j}(P_j)\chi^{\alpha_j}(N(P_j)));$$

therefore,

$$\sum_{B|A} X(B)\chi(N(B)) \geqslant 0, \quad \sum_{B|A^2} X(B)\chi(N(B)) \geqslant 1.$$

Hence

$$H(t) \geqslant \sum_{A} \chi_1^2(N(A)) t^{N(A)^2} = \sum_{n=1}^{\infty} a_0(n)\chi_1^2(n) t^{n^2}.$$

Using Lemma 3 and the formula of partial summation, we have

$$H(t) = c_0 \int_0^{\infty} t^{x^2} dx + O\Big(\int_1^{\infty} x^{-2/3+\varepsilon_1} \exp\Big(-x\sqrt{\ln \frac{1}{t}} \Big)^2 dx \Big) = \frac{c_0\sqrt{\pi}}{2\sqrt{\ln \frac{1}{t}}} + O\Big(\Big(\ln \frac{1}{t} \Big)^{-1/6-\varepsilon_1} \Big) > \frac{c_1}{\sqrt{\ln \frac{1}{t}}}$$

(we assume that t is sufficiently close to 1); $0 < \varepsilon_1 < 0.001$. We transform $H(t)$ in the form

$$H(t) = \sum_{m=1}^{\infty} b(m) \sum_{n=1}^{\infty} a_0(n)\chi_1^2(n) t^{mn},$$

where $b(m) = a(m)\chi(m)\chi_1^2(m)$.

Using Lemma 3 and the formula of partial summation, we obtain

$$\sum_{n=1}^{\infty} a_0(n)\chi_1^2(n) t^{mn} = \frac{c_0}{m \ln \frac{1}{t}} + c_0 t^m - c_0 \int_0^1 t^{mx} dx + \int_1^{\infty} R_0(x) m \ln \frac{1}{t} t^{mx} dx,$$

where $R_0(x)$ is the remainder in the asymptotic formula of Lemma 3.

Thus,

$$H(t) = c_0 \frac{F_1(1, X, \chi)}{\ln \frac{1}{t}} + H_1(t) - H_2(t) + H_3(t),$$

where

$$H_1(t) = c_0 \sum_{m=1}^{\infty} b(m) t^m, \quad H_2(t) = c_0 \int_0^1 \sum_{m=1}^{\infty} b(m) t^{mx} dx,$$

and

$$H_3(t) = O\Big(\int_1^{\infty} |R_0(x)| |\sum_{m=1}^{\infty} b(m) m \ln \frac{1}{t} t^{mx}| dx \Big).$$

We estimate $H_1(t)$ and $H_2(t)$. Let $0 < x \leqslant 1$. Make use of the formula of partial summation and Lemma 2:

$$\sum_{m=1}^{\infty} b(m) t^{mx} = -\int_1^{\infty} \sum_{m \leqslant u} b(m) dt^{ux} = O(r^{\frac{7}{6}} \int_0^{\infty} u^{-\frac{2}{3}+\varepsilon_1} t^{ux} du) = O\Big(\Big(x \ln \frac{1}{t} \Big)^{-\frac{1}{3}-\varepsilon_1} r^{\frac{7}{6}} \Big).$$

Hence we obtain

$$H_1(t) = O\Big(\Big(x \ln \frac{1}{t} \Big)^{-\frac{1}{3}-\varepsilon_1} r^{\frac{7}{6}} \Big), \quad H_2(t) = O\Big(\Big(x \ln \frac{1}{t} \Big)^{-\frac{1}{3}-\varepsilon_1} r^{\frac{7}{6}} \Big).$$

Estimate $H_3(t)$. We have

$$\sum_{m=1}^{\infty} b(m) m t^{mx} = -\int_1^{\infty} \sum_{m \leqslant u} b(m)(1 + ux \ln t) t^{ux}$$

$$= O\Big(r^{\frac{7}{6}}\int_0^\infty u^{\frac{1}{3}+\varepsilon_1}\Big(1+ux\ln\frac{1}{t}\Big)t^{ux}du\Big) = O\Big(\Big(x\ln\frac{1}{t}\Big)^{-\frac{4}{3}-\varepsilon_1}r^{\frac{7}{6}}\Big).$$

Estimating $R_0(x)$ by Lemma 3 as $R_0(x) = O(x^{\frac{1}{3}+\frac{\varepsilon_1}{2}})$, we arrive at the relation

$$H_3(t) = O\Big(\Big(\ln\frac{1}{t}\Big)^{-\frac{1}{3}-\varepsilon_1}r^{\frac{7}{6}}\Big).$$

Thus,

$$\frac{c_1}{\sqrt{\ln\frac{1}{t}}} < c_0\frac{F_1(1,X,\chi)}{\ln\frac{1}{t}} + c_2r^{7/6}\Big(\ln\frac{1}{t}\Big)^{-1/3-\varepsilon_1}.$$

We take

$$t = 1 - c_3r^{-\frac{7}{1-6\varepsilon_1}},$$

where $c_3>0$ is small to an extent that $\frac{c_1}{\sqrt{\ln\frac{1}{t}}} > 2c_2r^{7/6}(\ln\frac{1}{t})^{-1/3-\varepsilon_1}$.

Then the inequality

$$F(1,X,\chi) > c_4r^{-3.5-50\varepsilon_1}$$

is valid, where $c_4>0$. We put $\varepsilon=50\varepsilon_1$, completing the proof of the lemma.

Lemma 5. *For any X and χ, in the domain $\sigma \geqslant 1-\log^{-1}(r|t|+3r)$ the following inequality holds:*

$$|F(s,X,\chi)| \leqslant c_5\log^2(r|t|+3r). \tag{4}$$

Proof. Let $s=\sigma+it$, $\sigma\geqslant 1-\log^{-1}(r|t|+3r)$. Since the character X is nonprincipal, the function $F(s,X,\chi)$ is represented by the Dirichlet series $F(s,X,\chi)=\sum_{n=1}^{\infty}a(n)\chi(n)n^{-s}$ (see Lemma 2) in the half-plane $\operatorname{Re}s>\frac{1}{3}$.

Let $N=[(r|t|+3r)^2]+1$. By Lemma 2 and the formula of partial summation, we have

$$\sum_{n=N+1}^{\infty}a(n)\chi(n)n^{-s} = s\int_N^\infty\sum_{N<n\leqslant x}a(n)\chi(n)x^{-s-1}dx \ll \frac{(r|t|+3r)^{7/6}}{N^{2/3}}N^{\varepsilon} \ll 1.$$

The sum $\sum_{n=1}^{N}a(n)\chi(n)n^{-s}$ is estimated with the help of the inequality $|a(n)|\leqslant\tau(n)$:

$$\Big|\sum_{n=1}^{N}a(n)\chi(n)n^{-s}\Big| \leqslant \Big(\sum_{n=1}^{N}n^{-\sigma}\Big)^2 \ll \Big(\sum_{n=1}^{N}n^{-1}\Big)^2 \ll \log^2(r|t|+3r).$$

Corollary 2. *Let the assumptions of Lemma 4 be satisfied. Then there exists a constant $c_1=c_1(\varepsilon)$ such that $F(s,X,\chi)\neq 0$ in the circle*

$$|s-1| < c_1r^{-3.5-\varepsilon}\log^{-3}(4r).$$

Proof. Let $|s-1|\leqslant\frac{c}{2c_5+c}r^{-3.5-\varepsilon}\ln^{-3}4r$, where c is a constant from Lemma 4 and c_5 is a constant from Lemma 5. Then

$$F(s,X,\chi) = F(1,X,\chi) + \sum_{n=1}^{\infty}b_n(s-1)^n,$$

where

$$b_n = \frac{1}{2\pi i}\int_C\frac{F(s,X,\chi)}{(s-1)^{n+1}}ds,$$

C is the circle of radius $\ln^{-1}4r$ with center at the point $s=1$.

Since the inequality

$$\sigma \geqslant 1 - \frac{1}{\ln 4r} \geqslant 1 - \frac{1}{\ln(r|t| + 3r)}$$

is valid for any s from C, we have

$$|b_n| \leqslant c_5 \ln^{2+n} 4r$$

in view of (4); consequently,

$$\left| \sum_{n=1}^{\infty} b_n (s-1)^n \right| \leqslant \sum_{n=1}^{\infty} |b_n||s-1|^n \leqslant c_5 \sum_{n=1}^{\infty} \left(\frac{c}{c+2c_5} \right)^n \frac{(\ln 4r)^{n+2}}{(r^{3.5+\varepsilon})^n (\ln 4r)^{3n}}$$

$$\leqslant c_5 r^{-3.5-\varepsilon} \sum_{n=1}^{\infty} \left(\frac{c}{c+2c_5} \right)^n = \frac{c}{2} r^{-3.5-\varepsilon}.$$

Thus, for $|s-1| \leqslant \frac{c}{2c_5+c} r^{-3.5-\varepsilon} \ln^{-3} 4r$ the inequality

$$|F(s, X, \chi)| \geqslant |F(1, X, \chi)| - |\sum_{n=1}^{\infty} b_n (s-1)^n| \geqslant \frac{c}{2} r^{-3.5-\varepsilon} > 0$$

is valid, as was to be proved. □

Lemma 6. *Let $F(s)$ be a function analytic in the circle $|s - s_0| \leqslant r$, $F(s_0) \neq 0$, and let $|\frac{F(s)}{F(s_0)}| \leqslant M$ in this circle.*

If $F(s) \neq 0$ in the domain $|s - s_0| \leqslant \frac{r}{2}$, $\mathrm{Re}\,(s - s_0) \geqslant 0$, then

$$(a) \qquad \mathrm{Re}\, \frac{F'(s_0)}{F(s_0)} \geqslant -\frac{4}{r} \ln M,$$

$$(b) \qquad \mathrm{Re}\, \frac{F'(s_0)}{F(s_0)} \geqslant -\frac{4}{r} \ln M + \mathrm{Re}\, \frac{1}{s_0 - \rho},$$

where ρ is any zero of $F(s)$ in the domain $|s - s_0| \leqslant \frac{r}{2}$, $\mathrm{Re}\,(\rho - s_0) \leqslant 0$.

For a proof, see, for example, [1, Chap. VI].

Lemma 7. *Let ε be an arbitrary number satisfying $0 < \varepsilon < 0.001$, $s = \sigma + it$.*

There exists a constant $d_0 > 0$, dependent only on ε, such that $F(s, X, \chi) \neq 0$ in the domain $\sigma \geqslant 1 - \frac{d_0}{V \log(V + e^{10})}$, where

$$V = \begin{cases} \max(\ln^2(r|t| + 3r), 2c_1^{-1} r^{3.5+\varepsilon} \ln^3(4r)), \\ \ln^2(r|t| + 3r); \end{cases}$$

we take the upper formula if the character χ is real or the character $\chi^2\chi_1$ is principal, and the lower formula otherwise; c_1 is a constant from Corollary 2.

Proof. Consider the function $F(s, X^2, \chi^2)$. If the character X^2 is nonprincipal, then, by Corollary 1, this function is entire. If the character X^2 is principal (the character X is real), then the relation

$$F(s, X^2, \chi^2) = L(s, \chi^2) L(s, \chi^2 \chi_1)$$

holds. This implies that if X is a real character, then $F(s, X^2, \chi^2)$ has a simple pole at the point $s = 1$ either if the character χ is real, or if the character $\chi^2\chi_1$ is principal. First let either the character χ be real, or the character $\chi^2\chi_1$ be principal.

By Corollary 2, we may assume that

$$|t| > \frac{c_1}{2r^{3.5+\varepsilon} \ln^3 4r}.$$

Indeed, if

$$|t| \leqslant \frac{c_1}{2r^{3.5+\varepsilon} \ln^3 4r}, \quad |\sigma - 1| \leqslant \frac{c_1}{2r^{3.5+\varepsilon} \ln^3 4r},$$

then

$$|s-1| \leqslant \frac{c_1}{r^{3.5+\varepsilon}\ln^3 4r}$$

and $F(s,X,\chi)\neq 0$.

Therefore, if $F(s,X,\chi)=0$ and

$$|t| \leqslant \frac{c_1}{2r^{3.5+\varepsilon}\ln^3 4r},$$

then

$$\sigma \leqslant 1-\frac{c_1}{2r^{3.5+\varepsilon}\ln^3 4r}$$

(we used the fact that the function $F(s,X,\chi)$ does not have zeros to the right of the unit line, because it has an Euler product).

Let $\rho=\sigma+it$ be a zero of the function $F(s,X,\chi)$,

$$|t| > \frac{c_1}{2r^{3.5+\varepsilon}\ln^3 4r},\quad \sigma = 1-\frac{x}{V\ln(V+e^{10})},\quad 0<x\leqslant 1,$$

$$V=\max(\ln^2(r|t|+3r), 2c_1^{-1}r^{3.5+\varepsilon}\ln^3(4r)).$$

We need to prove that $x\geqslant d_0>0$.

Let

$$s_0=\sigma_0+it,\quad \sigma_0=\frac{4x}{V\ln(V+e^{10})}.$$

Consider the circle of radius $v=\frac{1}{V}$ with center at the point s_0. Since,

$$v\leqslant \frac{c_1}{2r^{3.5+\varepsilon}\ln^3 4r}\leqslant |t|<2|t|,$$

it follows that 1 does not lie in the circle of radius v with center at the point $s_1=\sigma_0+2it$. Therefore the function $F(s,X^2,\chi^2)$ is regular in the circle $|s-s_1|\leqslant v$.

We estimate $\frac{1}{|F(s_0,X,\chi)|}$ from above:

$$\frac{1}{|F(s_0,X,\chi)|}\leqslant \prod_P (1+N(P)^{-\sigma_0})\leqslant \prod_p (1+p^{-\sigma_0})^2<\zeta^2(\sigma_0)\ll(\sigma_0-1)^{-2}.$$

This estimate and inequality (4) imply that

$$(1)\quad \left|\frac{F(s,X,\chi)}{F(s_0,X,\chi)}\right|\leqslant M=O(\ln^2(r|t|+3r)(\sigma_0-1)^{-2})\quad\text{for}\quad |s-s_0|\leqslant v,$$

$$(2)\quad \left|\frac{F(s,X^2,\chi^2)}{F(s_1,X^2,\chi^2)}\right|\leqslant M=O(\ln^2(r|t|+3r)(\sigma_0-1)^{-2})\quad\text{for}\quad |s-s_1|\leqslant v.$$

Note that $|\rho-s_0|\leqslant v/2$. Indeed,

$$|\rho-s_0|=\frac{5x}{V\ln(V+e^{10})}\leqslant\frac{5}{V\ln(V+e^{10})}\leqslant\frac{1}{2V}=\frac{v}{2}.$$

We apply Lemma 6, putting first $F(s)=F(s,X,\chi)$ and then $F(s)=F(s,X^2,\chi^2)$:

$$\mathrm{Re}\left\{-\frac{F'(s_0,X,\chi)}{F(s_0,X,\chi)}\right\}\leqslant\frac{4}{v}\ln M - Re\,\frac{1}{s_0-\rho},$$

$$\mathrm{Re}\left\{-\frac{F'(s_1,X^2,\chi^2)}{F(s_1,X^2,\chi^2)}\right\}\leqslant\frac{4}{v}\ln M.$$

We estimate $\ln M$ from above. We have

$$M \leqslant c_6 \frac{\ln^2(r|t|+3r)}{x^2} V^2 \ln^2(V+e^{10}),$$

where c_6 is a constant. Next,

$$\ln^2(r|t|+3r) \leqslant V \leqslant V+e^{10}, \quad \ln^2(V+e^{10}) \leqslant (V+e^{10})^2;$$

therefore,

$$M \leqslant c_6 \frac{(V+e^{10})^5}{x^2}, \quad \ln M \leqslant 5\ln(V+e^{10}) - 2\ln x + c_7, \tag{5}$$

where c_7 is a constant.

Let $\zeta_F(s) = \zeta(s)L(s,\chi_1)$ be the Dedekind zeta function of the field F.

The following inequality holds:

$$0 \leqslant 3\left\{-\frac{\zeta_F'(\sigma_0)}{\zeta_F(\sigma_0))}\right\} + 4\left\{-\operatorname{Re}\frac{F'(\sigma_0+it,X,\chi)}{F(\sigma_0+it,X,\chi)}\right\} + \left\{-\operatorname{Re}\frac{F'(\sigma_0+2it,X^2,\chi^2)}{F(\sigma_0+2it,X^2,\chi^2)}\right\}.$$

Using the relation $-\frac{\zeta_F'(\sigma_0)}{\zeta_F(\sigma_0)} = (\sigma_0-1)^{-1} + O(1)$ and the inequalities of Lemma 6, we derive that

$$0 \leqslant 20\ln M - \frac{1}{20x}\ln(V+e^{10}) + c_8,$$

where c_8 is a constant.

Hence inequality (5) implies that

$$0 \leqslant c_9 - x^{-1}\left(\frac{1}{20} + 40x\ln x\right), \tag{6}$$

where c_9 is a constant.

Since $\lim\limits_{x\to+0} x\ln x = 0$, we have just proved that $x > d_0 > 0$.

To complete the proof, it is necessary to verify that the function $F(s,X,\chi)$ does not have zeros on the unit straight line. Assume the contrary, i.e., $F(1+it,X,\chi)=0$ for $|t| > \frac{c_1}{2r^{3.5+\varepsilon}\ln^3 4r}$.

Let $s_0 = 1 + \frac{4x}{V\ln(V+e^{10})} + it$, where x is an arbitrary positive number, $v = V^{-1}$; then $|\rho - s_0| \leqslant v/2$, where $\rho = 1+it$.

Next, repeating the above argument once again, we arrive at inequality (6), which holds for any positive x. Passing to the limit as $x \to +0$, we obtain a contradiction, which proves the fact that $F(1+it,X,\chi) \neq 0$ for t real.

The case where the character X is complex or X is real, χ is complex, and $\chi^2\chi_1$ is nonprincipal is considered similarly. Then the function $F(s,X^2,\chi^2)$ is entire, and in choosing the parameters, we need not verify that the point $s=1$ does not lie in the circle $|s-s_1| \leqslant v$.

Putting $V = \ln^2(r|t|+3r)$ and $v = V^{-1}$ and repeating the above argument, we complete the proof of the lemma. □

Lemma 8. *Let $T \geqslant 2$, $r \leqslant \log^{A_1} T$, $A_1 > 0$ be a constant, $V_1 = \ln^2(rT+4r) + 2c_1^{-1}r^{3.5+\varepsilon}\ln^3(4r)$, where c_1 is a constant from Corollary 2 and d_0 is a constant from Lemma 7. Then in the rectangular*

$$\Pi = \left\{ s \;\middle|\; 1 - \frac{d_0}{3V_1} \leqslant \operatorname{Re}\ s \leqslant 2, \quad |\operatorname{Im}\ s| \leqslant T \right\},$$

the estimate

$$\frac{F'(s,X,\chi)}{F(s,X,\chi)} = O(\log^{B_1} T)$$

is valid, where $B_1 = B_1(A_1) > 0$.

Proof. Consider the rectangular

$$\Pi_1 = \left\{ s \;\middle|\; 1 - \frac{2d_0}{3V_1} < \operatorname{Re}\ s < 3, \quad |\operatorname{Im}\ s| < T+1 \right\}.$$

By Lemma 7, $F(s,X,\chi)\neq 0$ for $s\in\Pi_1$, and thus the function $\log F(s,X,\chi)$ is analytic in the domain Π_1 (by $\log z$ we mean the principal branch of the logarithmic function).

Let $s\in\Pi$. It is easy to see that the circle of radius $\frac{d_0}{3V_1}$ with center at the point s is contained in the domain Π_1. By the Cauchy formula, we have

$$\frac{F'(s,X,\chi)}{F(s,X,\chi)}=\frac{1}{2\pi i}\int_C\frac{\log F(w,X,\chi)}{(w-s)^2}dw,$$

where C is the circumference of radius $\frac{d_0}{3V_1}$ with center at the point s.

Using Lemma 5, we have

$$|\log F(w,X,\chi)|\ll\log|F(w,X,\chi)|+1\ll\log\log T$$

(we have used that $\ln(rT+3r)\ll\log T$, because $r\leqslant\log^{A_1}T$ by condition).

Thus,

$$\left|\frac{F'(s,X,\chi)}{F(s,X,\chi)}\right|\ll\log\log T\left|\int_C\frac{dw}{(w-s)^2}\right|\ll V_1\log\log T\ll\log^{B_1}T,$$

as required. □

Proof of Theorem 1

It is sufficient to estimate the sum

$$\sum_{N(P)\leqslant x}X(P)\chi(N(P)),$$

where X is a nonprincipal character of the ideal class group and χ is a primitive Dirichlet character modulo r, $1<r\leqslant\log^{A_1}x$.

Let

$$\Lambda_1(A)=\begin{cases}\log N(P) & \text{if } A=P^k \text{ and } P \text{ is a prime ideal, } k\geqslant 1,\\ 0 & \text{otherwise.}\end{cases}$$

From the formula of partial summation, we have

$$\left|\sum_{N(P)\leqslant x}X(P)\chi(N(P)\right|\ll\left|\sum_{N(A)\leqslant y}\Lambda_1(A)X(A)\chi(N(A))\right|\log^{-1}x+\sqrt{x}\log x,$$

where y is a number in the interval $(\sqrt{x},x]$ such that the modulus of the sum $\sum_{N(A)\leqslant y}\Lambda_1(A)X(A)\chi(N(A))$ is maximal.

We estimate the last sum. We apply the Perron formula (for example, see [1, p. 75])

$$\sum_{N(A)\leqslant y}\Lambda_1(A)X(A)\chi(N(A))=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\Big(-\frac{F'(s,X,\chi)}{F(s,X,\chi)}\Big)\frac{y^s}{s}ds+O\Big(\frac{x\ln^2x}{T}\Big),$$

where $b=1+\frac{1}{\ln x}$ and $T=\exp{(\log x)^{\frac{1}{10A_1}}}$.

Let $V_1=\ln^2(rT+4r)+2c_1^{-1}r^{3.5+\varepsilon}\ln^3(4r)$, where c_1 is a constant from Corollary 2 and d_0 is a constant from Lemma 7.

Since, by Lemma 8, in the rectangular $\Pi=\{s\mid 1-\frac{d_0}{3V_1}\leqslant\text{Re }s\leqslant 2,|\text{Im }s|\leqslant T\}$ the function $-\frac{F'(s,X,\chi)}{F(s,X,\chi)}$ is analytic and satisfies the inequality $|-\frac{F'(s,X,\chi)}{F(s,X,\chi)}|\ll\log^{B_1}T$, we can use the Cauchy theorem and move the interval of integration to the left:

$$\sum_{N(A)\leqslant y}\Lambda_1(A)X(A)\chi(N(A))=\frac{1}{2\pi i}\int_{b_1-iT}^{b_1+iT}\Big(-\frac{F'(s,X,\chi)}{F(s,X,\chi)}\Big)\frac{y^s}{s}ds+O\Big(y\exp\Big(-0.5\Big(\log x\Big)^{\frac{1}{10A_1}}\Big)\Big),$$

$$\text{where}\quad b_1=1-\frac{d_0}{3V_1}.$$

We estimate y^{b_1} from above. It is easy to see that

$$V_1 \leqslant c_{10}(1+c_1^{-1})\log^{4A_1+2} T < c_{10}(1+c_1^{-1})\log^{\frac{3}{5}} x,$$

where $c_{10} > 1$ is an absolute constant.

Therefore, there exists a constant $c_{11} > 0$ such that

$$y^{b_1} \leqslant x^{b_1} < x\exp(-c_{11}(\log x)^{\frac{2}{5}}).$$

Making use of this inequality and the inequalities

$$\left| -\frac{F'(s,X,\chi)}{F(s,X,\chi)}\right| \ll \log^{B_1} T, \quad \int_{b_1-iT}^{b_1+iT} \frac{|ds|}{|s|} \ll \log T,$$

we obtain

$$\frac{1}{2\pi i}\int_{b_1-iT}^{b_1+iT}\left(-\frac{F'(s,X,\chi)}{F(s,X,\chi)}\right)\frac{y^s}{s}ds = O\Big(x\exp\Big(-c\Big(\log x\Big)^{\frac{1}{20A_1}}\Big)\Big),$$

i.e.,

$$\sum_{N(A)\leqslant y} \Lambda_1(A)X(A)\chi(N(A)) = O\Big(x\exp\Big(-c\Big(\log x\Big)^{\frac{1}{20A_1}}\Big)\Big),$$

where $c > 0$ is a constant. Theorem 1 is proved. $\square$

Corollary 3. *Let l and q be two relatively prime numbers, $1 \leqslant q \leqslant \log^{A_1} x$, and $A_1 > 1$. Then the asymptotic formula*

$$\pi_1(x,q,l,\mathcal{C}) = \frac{1+\chi(q;D,0)\chi_1(l)}{h\varphi(q)}\mathrm{Li}\; x + O\Big(x\exp\Big(-c\Big(\log x\Big)^{\frac{1}{20A_1}}\Big)\Big)$$

is valid, where $c = c(A_1) > 0$.

This follows immediately from Theorem 1 and formula (3).

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