Acoustics Equations in Poroelastic Media

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Abstract—We study the acoustics equations in poroelastic media which were obtained by the author previously in result of homogenization of the exact dimensionless equations describing the joint motion of an elastic solid skeleton and a viscous fluid in the pores on the microscopic level. A small parameter in this model is the ratio ε of the average size l of the pores to the characteristic size L of the physical region under consideration. The homogenized equations (the limit regimes of the exact model as ε tends to zero) depend on the dimensionless parameters of the model, which depend on the small parameter, and are small or large quantities as ε tends to zero. On assuming that the solid skeleton is periodic, we analyze the concrete form of acoustics equations for the simplest periodic structures.

Keywords: *Stokes' and Lamé's equations, two-scale convergence, acoustics equations*

INTRODUCTION

Modern geophysics, studying the propagation of perturbations in the natural underground rock, includes various physical and mathematical models describing these processes. For a long time the main model was the system of Lamé's equations of linear elasticity. However, starting with the article by Biot [1], geophysics tends more and more to grapple with the fact that the presence of pores and cracks filled with fluids (liquids or gases) plays an essential role in modelling natural rock, i.e., that underground media is poroelastic. The phenomenological models of [1] are certain combinations of Lamé's equations and the equations describing fluid dynamics in pores (for instance, the Darcy system of filtration equations).

It is well known that the main drawback of complicated phenomenological models is the presence of phenomenological constants (or functions) which must be somehow determined, and it is also necessary to critically examine the structure of the differential equations itself. In order to understand the gist of this type of models, R. Burridge and J. B. Keller proposed in [2] a completely natural approach to their justification. Namely, they considered a complete mathematical model describing on the microscopic level the behavior of the mixture of an elastic solid skeleton and a fluid filling the pores. This model is plausible since it consists of Lamé's equations, Stokes' equations, and the well-known conservation laws on the solid-fluid boundary, as well as includes minimal number of the phenomenological constants (Lamé's constants, viscosity, and the speed of sound in the fluid) which are sufficiently reliably determined experimentally.

The complete model includes some small and large fast oscillating coefficients that depend on a small parameter ε which is the dimensionless pore size. The second assumption of the authors, also quite natural, is that all phenomenological models correctly describing this process must somehow follow from the main model as the small parameter tends to zero. Thus, the correct phenomenological models are some homogenization of the complete mathematical model describing the process on the microscopic level. For a periodic system of pores, using a method of two-scale decomposition, the authors formally derived the Biot poroelasticity equations.

Later some rigorous derivation of the homogenized equations appeared in [3–5]. As a rule, the derivation of the homogenized equations is not accompanied by their mathematical analysis. This is

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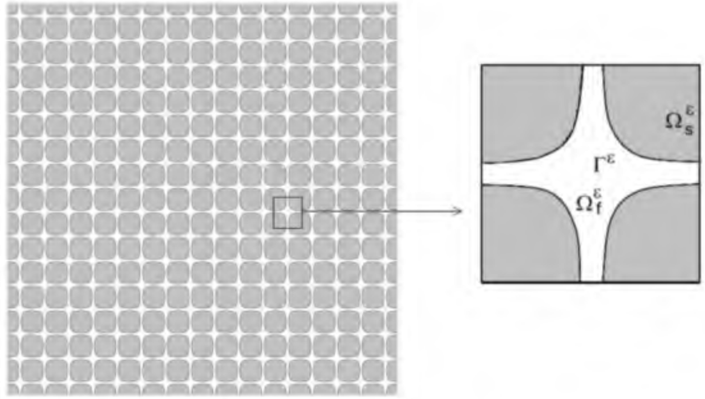

Fig. 1. Geometry of a pore space

natural since, in the general case, this analysis is technically complicate, and, in principle, a sufficiently difficult separate problem.

The goal of this article is to derive the homogenized equations for the simplest geometries of the pore space. It turns out that, even for these simplest cases, the arising initial boundary value problems are new and absolutely unstudied. For instance, for the system of alternating fluid-solid layers no appropriate phenomenological models exist, and numerical analysis of the problem is performed only on the microscopic level (see [6]).

1. STATEMENT OF THE PROBLEM

Following [2], consider an elastic deformable medium Ω with a system of pores filled with a slightly viscous liquid. The solid component Ω_s of this medium is called the *solid skeleton*, while the domain Ω_f occupied by the fluid is the *pore space*. Let $Y = (0,1)^3 \subset \mathbb{R}^3$. Let Y_s denote the solid part of Y; and Y_f, its liquid part that is the open complement to Y_s in Y. Suppose that the boundary $\gamma = \partial Y_f \cap \partial Y_s$ between the liquid and solid components is a Lipschitz continuous surface (see Fig. 1).

In the dimensionless variables (without primes)

$$\mathbf{x}' = L\mathbf{x}, \qquad t' = \tau t, \qquad \mathbf{w}' = \frac{L^2}{g\tau^2}\mathbf{w}, \qquad \rho_s' = \rho_0\rho_s, \qquad \rho_f' = \rho_0\rho_f,$$

the pore space Ω_f^ε is a periodic repetition of an elementary cell εY_f, the solid skeleton Ω_s^ε is a periodic repetition of an elementary cell εY_s, while the Lipschitz continuous boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is a periodic repetition in Ω of the boundary $\varepsilon\gamma$. The differential equations of the mathematical model for small deviations of the dimensionless displacements w in the domain $\Omega \subset \mathbb{R}^3$ for $t > 0$ are of the form [4]

$$\rho^\varepsilon \frac{\partial^2 \mathbf{w}}{\partial t^2} = \operatorname{div} \mathbb{P}, \tag{1.1}$$

$$\frac{\partial p_f}{\partial t} + \alpha_{p,f} \operatorname{div} \frac{\partial \mathbf{w}}{\partial t} = 0, \tag{1.2}$$

$$\frac{\partial p_s}{\partial t} + \alpha_{p,s} \operatorname{div} \frac{\partial \mathbf{w}}{\partial t} = 0, \tag{1.3}$$

where

$$\mathbb{P} = \chi^\varepsilon \left(\alpha_\mu \mathbb{D}\left(x, \frac{\partial \mathbf{w}}{\partial t}\right) - p_f \mathbb{I} \right) + (1 - \chi^\varepsilon)(\alpha_\lambda \mathbb{D}(x, \mathbf{w}) - p_s \mathbb{I}),$$

$$\mathbb{D}(x, \mathbf{w}) = \frac{1}{2}(\nabla \mathbf{w} + (\nabla \mathbf{w})^T), \qquad \rho^\varepsilon = \chi^\varepsilon \rho_f + (1 - \chi^\varepsilon)\rho_s.$$

Here $\mathbb{I}$ is the spherical tensor; $\chi^\varepsilon(\mathbf{x})$ is a given characteristic function of the pore space $\Omega_f^\varepsilon \subset \Omega$,

$$\chi^\varepsilon(\mathbf{x}) = \chi\left(\frac{\mathbf{x}}{\varepsilon}\right),$$

while $\chi(\mathbf{y})$ is the 1-periodic extension to $\mathbb{R}^3$ of the characteristic function of Y_f in Y; p_f is pressure in the liquid; p_s is pressure in the solid skeleton,

$$\alpha_\mu = \frac{2\mu\tau}{L^2\rho_0}, \qquad \alpha_\lambda = \frac{2\lambda\tau^2}{L^2\rho_0}, \qquad \alpha_{p,f} = \rho_f c_f^2 \frac{\tau^2}{L^2}, \qquad \alpha_{p,s} = \rho_s c_s^2 \frac{\tau^2}{L^2}.$$

Here μ is the viscosity of the fluid, λ is Lamé's constant of the solid skeleton, L is the characteristic size of the domain in which we study the physical process, τ is the characteristic time of the process, ρ_f and ρ_s are the dimensionless densities of the fluid and solid relative to the density of water ρ_0, c_f is the speed of sound in the fluid, c_s is the speed of compression wave in the solid skeleton, and g is the free fall acceleration.

We understand (1.1) in the sense of distributions. It means that the displacement vector $\mathbf{w}$ of the solid component satisfies the Lamé equation

$$\rho_s \frac{\partial^2 \mathbf{w}}{\partial t^2} = \alpha_\lambda \mathbf{w} - \nabla p_s$$

in the solid skeleton Ω_s^ε (where $\chi^\varepsilon = 0$), while the vector velocity $\mathbf{v} = \dfrac{\partial \mathbf{w}}{\partial t}$ of the fluid component satisfies the Stokes equation

$$\rho_f \frac{\partial \mathbf{v}}{\partial t} = \alpha_\mu \Delta \mathbf{u} - \nabla p_f$$

in the pore space Ω_f^ε (where $\chi^\varepsilon = 1$). Meanwhile, on the common boundary $\Gamma^\varepsilon = \partial\Omega_f^\varepsilon \cap \partial\Omega_s^\varepsilon$ between the solid skeleton and pore space the displacement vector $\mathbf{w}$ satisfies the continuity condition

$$[\mathbf{w}](\mathbf{x}_0, t) = 0, \qquad \mathbf{x}_0 \in \Gamma, \quad t \geq 0,$$

and the momentum conservation law

$$[\mathbb{P} \cdot \mathbf{n}](\mathbf{x}_0, t) = 0, \qquad \mathbf{x}_0 \in \Gamma, \quad t \geq 0,$$

where $\mathbf{n}(\mathbf{x}_0)$ is the unit normal vector to the boundary at $\mathbf{x}_0 \in \Gamma$, and

$$[\varphi](\mathbf{x}_0, t) = \varphi_{(s)}(\mathbf{x}_0, t) - \varphi_{(f)}(\mathbf{x}_0, t),$$

$$\varphi_{(s)}(\mathbf{x}_0, t) = \lim_{\mathbf{x} \to \mathbf{x}_0,\ \mathbf{x} \in \Omega_s} \varphi(\mathbf{x}, t), \qquad \varphi_{(f)}(\mathbf{x}_0, t) = \lim_{\mathbf{x} \to \mathbf{x}_0,\ \mathbf{x} \in \Omega_f} \varphi(\mathbf{x}, t).$$

More exactly, (1.1) means the validity of the integral identities

$$\int_{\Omega_T} \left(\rho^\varepsilon \frac{\partial^2 \mathbf{w}}{\partial t^2} \cdot \varphi + \mathbb{P} : \mathbb{D}(x, \varphi) \right) dx\, dt = 0$$

for all smooth vector functions $\varphi = \varphi(\mathbf{x}, t)$ finite in the domain $\Omega_T = \Omega \times (0, T)$.

In this identity $\mathbb{A} : \mathbb{B}$ stands for the convolution of two matrices with respect to both indices:

$$\mathbb{A} : \mathbb{B} = \mathrm{tr}\,(\mathbb{B}^* \circ \mathbb{A}) = \sum_{i,j=1}^{3} A_{ij} B_{ji}.$$

Henceforth, we assume that $\Omega = \mathbb{R}^3$. Then the problem is only completed with the initial conditions

$$\mathbf{w}(\mathbf{x}, 0) = \mathbf{w}_0, \qquad \frac{\partial \mathbf{w}}{\partial t}(\mathbf{x}, 0) = \mathbf{v}_0, \qquad \mathbf{x} \in \mathbb{R}^3, \tag{1.4}$$

for the displacement vector of the continuous medium.

Suppose that the dimensionless parameters α_μ, α_λ, $\alpha_{p,f}$, and $\alpha_{p,s}$ depend on the small parameter ε, and there exist finite or infinite limits

$$\lim_{\varepsilon \searrow 0} \alpha_\mu(\varepsilon) = \mu_0, \qquad \lim_{\varepsilon \searrow 0} \alpha_\lambda(\varepsilon) = \lambda_0, \qquad \lim_{\varepsilon \searrow 0} \alpha_{p,f}(\varepsilon) = c_f^*,$$

$$\lim_{\varepsilon \searrow 0} \alpha_{p,s}(\varepsilon) = c_s^*, \qquad \lim_{\varepsilon \searrow 0} \frac{\alpha_\mu}{\varepsilon^2} = \mu_1, \qquad \lim_{\varepsilon \searrow 0} \frac{\alpha_\lambda}{\varepsilon^2} = \lambda_1.$$

In addition, assume that

$$\mu_0 = 0, \qquad 0 < c_f^*,\ c_s^* < \infty. \tag{1.5}$$

All possible limit regimes for (1.1)–(1.4) are described in [4, 5]. These regimes depend on a physical criterion λ_0 characterizing the elastic property of the solid skeleton. We can roughly divide all ground materials occurring in practice into the three groups:

(1) $\lambda_0 = \infty$ for an absolutely rigid solid skeleton;

(2) $0 < \lambda_0 < \infty$ for a weakly deformable solid skeleton;

(3) $\lambda_0 = 0$ for an extremely deformable solid skeleton.

Refer to the corresponding porous media as *absolutely rigid solid*, *elastic*, and *extremely elastic* porous ground.

In this article, we calculate the coefficients of the homogenized systems of equations for two simplest geometries of the pore space.

Thus, suppose that the elementary cell Y_f is described by one of the two geometries:

Geometry (A). The set Y_f is the layer

$$Y_f = \{y \in \mathbb{R}^3 \ : \ 0 < y_1 < m, \quad 0 < y_2,\, y_3 < 1\},$$

where $m < 1$ is the porosity of the pore space.

Geometry (B). The set Y_f consists of the three disjoint cylinders of the same radius with the axes parallel to the corresponding axes of the rectangular coordinate system:

$$Y_f = Y_f^{(1)} \cup Y_f^{(2)} \cup Y_f^{(3)},$$
$$Y_f^{(1)} = \{y \in \mathbb{R}^3 : 0 < y_1 < 1,\ y_2^2 + y_3^2 < r^2\},$$
$$Y_f^{(2)} = \{y \in \mathbb{R}^3 : 0 < y_2 < 1,\ y_1^2 + (y_3 - a)^2 < r^2\},$$
$$Y_f^{(3)} = \{y \in \mathbb{R}^3 : 0 < y_3 < 1,\ (y_1 - a)^2 + (y_2 - a)^2 < r^2\},$$

where $3\pi r^2 = m$ with $2r < a < 1 - r$.

The resulting systems of equations are quite complicated, and in the general case rigorous mathematical results for them are lacking. For instance, in all physical situations under consideration and the main geometric structures of the pore space, the equation

$$\frac{\partial p}{\partial t} = \int_0^t b(t-\tau)\Delta p(\mathbf{x},\tau)\, d\tau \tag{1.6}$$

is a typical one, where a positive infinitely differentiable function $b(t)$ of $t > 0$ is determined only by the geometry of the pore space. This is an integrodifferential equation, and no rigorous mathematical results are presently available. More complicated physical processes are described by systems including (1.6), for instance, the system

$$v = m\frac{\partial u}{\partial t} - \int_0^t b_{1,A}(t-\tau)\frac{\partial p}{\partial x}(x,\tau)\, d\tau,$$
$$\rho_f \frac{\partial v}{\partial t} + \rho_s(1-m)\frac{\partial^2 u}{\partial t^2} = \frac{(1-m)\lambda_0(\lambda_0 + 2c_s)}{(\lambda_0 + c_s)}\frac{\partial^2 u}{\partial x^2} - \frac{(m\lambda_0 + c_s)}{m(\lambda_0 + c_s)}\frac{\partial p}{\partial x}, \tag{1.7}$$
$$\left(\frac{1}{c_f} + \frac{(1-m)}{m(\lambda_0 + c_s)}\right)\frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} + \frac{(1-m)c_s}{(\lambda_0 + c_s)}\frac{\partial^2 u}{\partial x \partial t} = 0.$$

2. ACOUSTICS IN AN ABSOLUTELY RIGID POROUS GROUND: $\lambda_0 = \infty$

An absolutely rigid body is characterized by $\lambda_0 = \infty$; in this case, the displacement of the solid skeleton tends to zero, while the limiting velocity $\mathbf{v}$ and pressure p_f in the liquid satisfy the acoustics equations [1]

$$m\mathbf{v}(\mathbf{x},t) = -\int_0^t \mathbb{B}_1(t-\tau)\cdot\nabla p_f(\mathbf{x},\tau)\,d\tau, \tag{2.1}$$

$$\frac{\partial p_f}{\partial t} + c_f^* \operatorname{div}\mathbf{v} = 0 \tag{2.2}$$

if $0 < \mu_1 < \infty$, and the system of acoustics equations consisting of the continuity equation (2.2) and

$$m\rho_f\frac{\partial \mathbf{v}}{\partial t} = -(m\mathbb{I} - \mathbb{B}_2)\cdot\nabla p_f \tag{2.3}$$

if $\mu_1 = 0$.

In (2.1),

$$\mathbb{B}_1(t) = \sum_{i=1}^{3}\left(\int_{Y_f}\mathbf{V}^{(i)}(\mathbf{y},t)\,dy\right)\otimes\mathbf{e}_i \equiv \sum_{i=1}^{3}\langle\mathbf{V}^{(i)}(\mathbf{y},t)\rangle_{Y_f}\otimes\mathbf{e}_i, \tag{2.4}$$

where $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ are the unit basis vectors of the Cartesian coordinate system; for given vectors $\mathbf{a}$ and $\mathbf{b}$, the skeleton $\mathbf{a}\otimes\mathbf{b}$ is fined as

$$(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$$

for an arbitrary vector $\mathbf{c}$, while the functions $\mathbf{V}^{(i)}$, $i = 1,2,3$, are 1-periodic in $\mathbf{y}$ solutions to the initial boundary value problem

$$\rho_f\frac{\partial \mathbf{V}^{(i)}}{\partial t} - \mu_1\Delta\mathbf{V}^{(i)} + \nabla R^{(i)} = 0, \qquad \mathbf{y}\in Y_f,\ \ t>0, \tag{2.5}$$

$$\operatorname{div}_y\mathbf{V}^{(i)} = 0, \qquad \mathbf{y}\in Y_f,\ \ t>0, \tag{2.6}$$

$$\mathbf{V}^{(i)}(\mathbf{y},t) = 0, \qquad \mathbf{y}\in\gamma,\ \ t>0, \tag{2.7}$$

$$m\rho_f\mathbf{V}^{(i)}(\mathbf{y},0) = \mathbf{e}_i, \qquad \mathbf{y}\in Y_f. \tag{2.8}$$

The matrix $\mathbb{B}_2$ in (2.3) is determined similarly:

$$\mathbb{B}_2 = \sum_{i=1}^{3}\langle\nabla R_i(\mathbf{y})\rangle_{Y_f}\otimes\mathbf{e}_i, \tag{2.9}$$

where the functions R_i are the 1-periodic in $\mathbf{y}$ solutions to the boundary value problem

$$\Delta_y R_i = 0, \qquad \mathbf{y}\in Y_f, \tag{2.10}$$

$$\nabla_y R_i\cdot\mathbf{n} = \frac{1}{m}\mathbf{n}\cdot\mathbf{e}_i, \qquad \mathbf{y}\in\gamma. \tag{2.11}$$

It is easy to see that the system (2.2), (2.3) reduces to one hyperbolic second order equation with constant coefficients. Thus, we pay the main attention to (2.1), (2.2).

In geometry (A), the initial boundary value problem (2.5)–(2.8) for $i = 2,3$ reduces to the initial boundary value problem

$$\rho_f\frac{\partial V}{\partial t} - \mu_1\frac{\partial^2 V}{\partial y_1^2} = 0, \qquad 0<y_1<m, \quad t>0, \tag{2.12}$$

$$V(0,t) = V(m,t) = 0, \qquad t>0, \tag{2.13}$$

$$m\rho_f V(y_1,0) = 1, \qquad 0<y_1<m, \tag{2.14}$$

for the one-dimensional heat equation if we put

$$\mathbf{V}^{(i)} = V(y_1, t)\mathbf{e}_i, \qquad R^{(i)} = 0.$$

For $i = 1$, the situation is different. Namely, in this case, the unique solution to (2.5)–(2.8) is given by the functions

$$\mathbf{V}^{(1)} = \mathbf{0}, \qquad R^{(1)} = 0.$$

Seemingly, the existence of the unique zero solution contradicts the inhomogeneous initial condition (2.8). In reality, the peculiarity of the problem is such that $\{\mathbf{V}^{(1)},\, R^1\}$ satisfies (2.5) and the initial condition (2.8) in the sense of the integral identities

$$\int\limits_0^T \int\limits_{Y_f} \left(-\rho_f \mathbf{V}^{(1)} \frac{\partial \boldsymbol{\varphi}}{\partial t} + \mu_1 \nabla \mathbf{V}^{(1)} : \nabla \boldsymbol{\varphi} \right) dy\, dt = \frac{1}{m} \int\limits_{Y_f} \mathbf{e}_1 \cdot \boldsymbol{\varphi}(\mathbf{y}, 0)\, dy \tag{2.15}$$

for all smooth solenoidal functions $\boldsymbol{\varphi}$; i.e., at the initial moment, the solution to (2.15) coincides with the projection of the initial function onto the subspace of solenoidal functions. Since $\mathbf{e}_1 = \nabla \phi$ for this geometry, where the scalar function ϕ that is 1-periodic in $\mathbf{y}$ coincides with y_1 on Y_f; therefore,

$$\int\limits_{Y_f} \mathbf{e}_1 \cdot \boldsymbol{\varphi}(\mathbf{y}, 0)\, dy = 0,$$

and the unique solution to (2.15) is $\mathbf{V}^{(1)} = \mathbf{0}$.

Summarizing the argument, we have

$$\mathbb{B}_{1,A}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_{1,A}(t) & 0 \\ 0 & 0 & b_{1,A}(t) \end{pmatrix}, \tag{2.16}$$

where

$$b_{1,A}(t) = \int\limits_{Y_f} V(\mathbf{y}, t)\, dy = \langle V(\mathbf{y}, t) \rangle_{Y_f},$$

and (2.1), (2.2) reduces to

$$\frac{\partial p_f}{\partial t} = c_f^* \int\limits_0^t \operatorname{div}\left(\mathbb{B}_{1,A}(t - \tau) \cdot \nabla p_f(\mathbf{x}, \tau) \right) d\tau,$$

or

$$\frac{\partial p_f}{\partial t} = c_f^* \int\limits_0^t b_{1,A}(t - \tau) \left(\frac{\partial^2 p_f}{\partial x_2^2} + \frac{\partial^2 p_f}{\partial x_3^2} \right) (\mathbf{x}, \tau)\, d\tau. \tag{2.17}$$

In geometry (B), the domain Y_f is the union of disjoint domains $Y_f^{(j)}$, $j = 1, 2, 3$. Thus, the initial boundary value problem (2.5)–(2.8) for $\{\mathbf{V}^{(i)}, R^{(i)}\}$ in Y_f splits into the three independent initial boundary value problems

$$\rho_f \frac{\partial \mathbf{V}^{(i,j)}}{\partial t} - \mu_1 \Delta \mathbf{V}^{(i,j)} + \nabla R^{(i,j)} = 0, \qquad \mathbf{y} \in Y_f^{(j)}, \quad t > 0, \tag{2.18}$$

$$\operatorname{div}_y \mathbf{V}^{(i,j)} = 0, \qquad \mathbf{y} \in Y_f^{(j)}, \quad t > 0, \tag{2.19}$$

$$\mathbf{V}^{(i,j)}(\mathbf{y}, t) = 0, \qquad \mathbf{y} \in \gamma^{(j)}, \quad t > 0, \tag{2.20}$$

$$\rho_f \mathbf{V}^{(i,j)}(\mathbf{y}, 0) = \mathbf{e}_i, \qquad \mathbf{y} \in Y_f^{(j)}, \tag{2.21}$$

for the functions $\{\mathbf{V}^{(i,j)}, R^{(i,j)}\}$ in $Y_f^{(j)}$, $j = 1, 2, 3$, where

$$\mathbf{V}^{(i)} = \sum_{j=1}^{3} \mathbf{V}^{(i,j)}, \qquad R^{(i)} = \sum_{j=1}^{3} R^{(i,j)}.$$

In a similar fashion, the initial boundary value problem (2.5)–(2.8) for the functions $\{\mathbf{V}^{(j,j)}, R^{(j,j)}\}$ in the domain $Y_f^{(j)}$ reduces to the heat equation

$$\rho_f \frac{\partial U}{\partial t} - \mu_1 \left(\frac{\partial^2 U}{\partial z_1^2} + \frac{\partial^2 U}{\partial z_2^2} \right) = 0 \tag{2.22}$$

in the domain $Z = \{(z_1, z_2) : z_1^2 + z_2^2 < r^2\}$ for $t > 0$ with the homogeneous boundary condition

$$U(z_1, z_2, t) = 0 \tag{2.23}$$

on $\partial Z = \{(z_1, z_2) : z_1^2 + z_2^2 = r^2\}$ for $t > 0$ and the inhomogeneous initial condition

$$m\rho_f U(z_1, z_2, 0) = 1, \qquad z_1^2 + z_2^2 < r^2, \tag{2.24}$$

if we put

$$\mathbf{V}^{(j,j)}(\mathbf{y}, t) = U(z_1, z_2, t)\mathbf{e}_j, \qquad R^{(j,j)} = 0,$$

where

$$\mathbf{y} = \mathbb{A}^j \cdot \mathbf{z} + \mathbf{a}^j$$

is a linear change of variables taking the cylinder $\{\mathbf{z} \in \mathbb{R}^3 : (z_1, z_2) \in Z,\ 0 < z_3 < 1\}$ into the cylinder $Y_f^{(j)}$. Meanwhile, as in the case of geometry (A), the unique solution to the initial boundary value problem (2.18)–(2.21) for $i \neq j$ is $\mathbf{V}^{(i,j)} = \mathbf{0}$, $R^{(i,j)} = 0$.

Therefore,

$$\mathbb{B}_{1,B}(t) = \begin{pmatrix} b_{1,B}(t) & 0 & 0 \\ 0 & b_{1,B}(t) & 0 \\ 0 & 0 & b_{1,B}(t) \end{pmatrix}, \tag{2.25}$$

where

$$b_{1,B}(t) = \int_Z U(z_1, z_2, t)\, dz_1\, dz_2 = \langle U(z_1, z_2, t) \rangle_Z,$$

and (2.1), (2.2) reduces to

$$\frac{\partial p_f}{\partial t} = c_f^* \int_0^t b_{1,B}(t - \tau) \Delta p_f(\mathbf{x}, \tau)\, d\tau.$$

3. ACOUSTICS IN A WEAKLY DEFORMABLE POROUS GROUND: $0 < \lambda_0 < \infty$

As above, consider only the case

$$0 < \mu_1 < \infty.$$

If $0 < \lambda_0 < \infty$ then the displacement of the solid skeleton $\mathbf{u}$ is already nonzero, while the pressure p_f and the velocity $\mathbf{v}$ of the fluid component satisfy the acoustics equations

$$\mathbf{v} = m \frac{\partial \mathbf{u}}{\partial t} - \int_0^t \mathbb{B}_1(t - \tau) \cdot \nabla p_f(\mathbf{x}, \tau)\, d\tau, \tag{3.1}$$

$$\frac{1}{c_f^*} \frac{\partial p_f}{\partial t} + \frac{1}{c_s^*} \frac{\partial p_s}{\partial t} + \operatorname{div} \mathbf{v} = (m - 1) \operatorname{div} \frac{\partial \mathbf{u}}{\partial t} \tag{3.2}$$

related to the averaged Lamé system for the displacement $\mathbf{u}$ of the solid skeleton:

$$\rho_f \frac{\partial \mathbf{v}}{\partial t} + \rho_s(1-m)\,\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathrm{div}\,\big(\mathbb{A}^s : \mathbb{D}(x,\mathbf{u}) + \mathbb{C}_0^s p_f\big), \tag{3.3}$$

$$\frac{1}{c_s^*}\,p_s + \mathbb{E}_0^s : \mathbb{D}(x,\mathbf{u}) + b_0^s \mathrm{div}\,\mathbf{u} + c_0^s p_f = 0. \tag{3.4}$$

In turn, the last equation is a corollary of the macroscopic equations

$$\begin{gathered} p = p_f + p_s, \\ \rho_f \frac{\partial \mathbf{v}}{\partial t} + \rho_s(1-m)\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathrm{div}_x(\lambda_0((1-m)\mathbb{D}(x,\mathbf{u}) + \langle \mathbb{D}(y,\mathbf{U})\rangle_{Y_s}) - p\mathbb{I}), \end{gathered} \tag{3.5}$$

$$\frac{1}{c_s^*}\,p_s + (1-m)\mathrm{div}\,\mathbf{u} + \langle \mathrm{div}_y \mathbf{U}\rangle_{Y_s} = 0 \tag{3.6}$$

and the microscopic equations

$$\mathrm{div}\,v_y\left(\lambda_0(1-\chi)(\mathbb{D}(y,\mathbf{U}) + \mathbb{D}(x,\mathbf{u})) - \left(P_s + \frac{1}{m}p_f\chi\right)\mathbb{I}\right) = 0, \tag{3.7}$$

$$\frac{1}{c_s^*}P_s + (1-\chi)(\mathrm{div}\,\mathbf{u} + \mathrm{div}_y \mathbf{U}) = 0 \tag{3.8}$$

(see [1]). For instance, we obtain (3.3) from (3.5) and (3.6) by expressing in the last equations

$$\langle \mathbb{D}(y,\mathbf{U})\rangle_{Y_s} = \int_{Y_s} \mathbb{D}(y,\mathbf{U})\,dy, \qquad \langle \mathrm{div}_y \mathbf{U}\rangle_{Y_s}$$

as

$$\lambda_0 \langle \mathbb{D}(y,\mathbf{U})\rangle_{Y_s} = \mathbb{A}_1^s : \mathbb{D}(x,\mathbf{u}) + C_1^s p_f,$$
$$\langle \mathrm{div}_y \mathbf{U}\rangle_{Y_s} = \mathbb{E}_0^s : \mathbb{D}(x,\mathbf{u}) + \big(b_0^s - 1 + m\big)\mathrm{div}\,\mathbf{u} + c_0 p_f$$

in result of solving (3.7) and (3.8). These equations are sufficiently easy to solve only for geometry (A). Namely, by the uniqueness theorem and the structure of Y_s, the solution $\mathbf{U}$ to (3.7), (3.8) depends only on y_1 (and the variables $(\mathbf{x}, t)$ as parameters). Put

$$d_{ij}(\mathbf{x},t) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}(\mathbf{x},t) + \frac{\partial u_j}{\partial x_i}(\mathbf{x},t)\right), \quad i,j = 1,2,3, \qquad \mathbf{u} = (u_1,u_2,u_3).$$

Then we express (3.7), from which the pressure P_s is excluded using (3.8), as three scalar equations

$$\frac{\partial}{\partial y_1}\left((1-\chi(y_1))\left((\lambda_0 + c_s^*)\frac{\partial U_1}{\partial y_1} + \lambda_0 d_{11} + c_s^* \mathrm{div}\,\mathbf{u} + \frac{1}{m}p_f\right)\right) = 0,$$
$$\frac{\partial}{\partial y_1}\left(\lambda_0(1-\chi(y_1))\left(\frac{1}{2}\frac{\partial U_2}{\partial y_1} + d_{21}\right)\right) = 0,$$
$$\frac{\partial}{\partial y_1}\left(\lambda_0(1-\chi(y_1))\left(\frac{1}{2}\frac{\partial U_3}{\partial y_1} + d_{31}\right)\right) = 0,$$

whose solution we can write explicitly. The averaged values of the tensor $\mathbb{D}(y,\mathbf{U})$ satisfy

$$\left\langle \frac{\partial U_1}{\partial y_1}\right\rangle_{Y_s} = -\frac{(1-m)}{(\lambda_0 + c_s^*)}\left(\lambda_0 d_{11} + c_s^* \mathrm{div}\,\mathbf{u} + \frac{1}{m}p_f\right), \qquad \left\langle \frac{\partial U_1}{\partial y_i}\right\rangle_{Y_s} = 0, \qquad i = 2,3,$$
$$\frac{1}{2}\left\langle \frac{\partial U_2}{\partial y_1}\right\rangle_{Y_s} = -(1-m)d_{12}, \qquad \left\langle \frac{\partial U_2}{\partial y_i}\right\rangle_{Y_s} = 0, \qquad i = 2,3,$$
$$\frac{1}{2}\left\langle \frac{\partial U_3}{\partial y_1}\right\rangle_{Y_s} = -(1-m)d_{13}, \qquad \left\langle \frac{\partial U_3}{\partial y_i}\right\rangle_{Y_s} = 0, \qquad i = 2,3.$$

In addition, from the last relations and (3.6) we find the concrete form of (3.4):

$$\frac{1}{c_s^*}p_s - \frac{\lambda_0(1-m)}{(\lambda_0+c_s^*)}\left(\frac{\partial u_1}{\partial x_1} - \operatorname{div}\mathbf{u}\right) - \frac{(1-m)}{m(\lambda_0+c_s^*)}p_f = 0. \tag{3.9}$$

Furthermore, for the matrices $\langle \mathbb{D}(y,\mathbf{U})\rangle_{Y_s}$ and $\mathbb{D}(x,\mathbf{u})$, we have

$$\mathbb{D}_0 \equiv (1-m)\mathbf{D}(x,\mathbf{u}) + \langle \mathbb{D}(y,\mathbf{U})\rangle_{Y_s}$$

$$= \begin{pmatrix} (1-m)d_{11} + \left\langle \dfrac{\partial U_1}{\partial y_1}\right\rangle_{Y_s} & 0 & 0 \\ 0 & (1-m)d_{22} & (1-m)d_{23} \\ 0 & (1-m)d_{32} & (1-m)d_{33} \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{c_s^*(1-m)}{(\lambda_0+c_s^*)}\left(\operatorname{div}\mathbf{u} - \dfrac{\partial u_1}{\partial x_1}\right) - \dfrac{(1-m)}{m(\lambda_0+c_s^*)}p_f & 0 & 0 \\ 0 & (1-m)d_{22} & (1-m)d_{23} \\ 0 & (1-m)d_{32} & (1-m)d_{33} \end{pmatrix}.$$

Therefore,

$$\mathbb{A}^s : \mathbb{D}(x,\mathbf{u}) + \mathbb{C}_0^s p_f = \begin{pmatrix} -\dfrac{1}{m}p_f & 0 & 0 \\ 0 & (1-m)d_{22} - p & (1-m)d_{23} \\ 0 & (1-m)d_{32} & (1-m)d_{33} - p \end{pmatrix}. \tag{3.10}$$

For geometry (A), the equation of motion of the liquid phase (3.1), (3.2) is very simple:

$$v_1 = m\frac{\partial u_1}{\partial t}, \qquad v_2 = m\frac{\partial u_2}{\partial t} - \int_0^t b_{1,A}(t-\tau)\frac{\partial p_f}{\partial x_2}(\mathbf{x},\tau)\,d\tau,$$

$$v_3 = m\frac{\partial u_3}{\partial t} - \int_0^t b_{1,A}(t-\tau)\,\frac{\partial p_f}{\partial x_3}(\mathbf{x},\tau)\,d\tau, \tag{3.11}$$

$$\frac{1}{c_f}\frac{\partial p_f}{\partial t} + \frac{1}{c_s}\frac{\partial p_s}{\partial t} + \operatorname{div}\mathbf{v} + (1-m)\operatorname{div}\frac{\partial \mathbf{u}}{\partial t} = 0.$$

By (3.9) and (3.10), we can express the vector equation (3.3) as the scalar equations

$$\hat{\rho}\frac{\partial^2 u_1}{\partial t^2} = -\frac{1}{m}\,\frac{\partial p_f}{\partial x_1},$$

$$\rho_f\frac{\partial v_2}{\partial t} + \rho_s(1-m)\frac{\partial^2 u_2}{\partial t^2} = (1-m)\lambda_0\left(\frac{\partial^2 u_2}{\partial x_2^2} + \frac{1}{2}\frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)\right) - \frac{\partial p}{\partial x_2},$$

$$\rho_f\frac{\partial v_3}{\partial t} + \rho_s(1-m)\frac{\partial^2 u_3}{\partial t^2} = (1-m)\lambda_0\left(\frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) + \frac{\partial^2 u_3}{\partial x_3^2}\right) - \frac{\partial p}{\partial x_3}, \tag{3.12}$$

$$p = p_f + p_s, \qquad \frac{1}{c_s}\,p_s + \frac{\lambda_0(1-m)}{(\lambda_0+c_s)}\,\operatorname{div}'\mathbf{u} - \frac{(1-m)}{m(\lambda_0+c_s)}\,p_f = 0,$$

where

$$\hat{\rho} = m\rho_f + (1-m)\rho_s, \qquad \operatorname{div}'\mathbf{u} = \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}.$$

The system (3.11), (3.12) is closed and strongly anisotropic in different directions. Thus, as we consider the Cauchy problem in which the initial perturbations depend only on x_1, this system reduces to the wave equation

$$\alpha \frac{\partial^2 p_f}{\partial t^2} = \frac{\partial^2 p_f}{\partial x_1^2} \tag{3.13}$$

for the pressure p_f in the fluid component and the equation

$$\frac{\partial^2 u_1}{\partial t \partial x_1} = -\frac{\alpha}{m\,\hat{\rho}} \frac{\partial p_f}{\partial t}, \tag{3.14}$$

where

$$\alpha = m\hat{\rho}\left(\frac{1}{c_s^*} + \frac{(1-m)}{m(\lambda_0 + c_s^*)}\right).$$

However, if the initial perturbations depend only on x_2 and x_3 then (3.12) and (3.13) reduce to

$$\begin{gathered}
\mathbf{v} = m\frac{\partial \mathbf{u}}{\partial t} - \int\limits_0^t b_{1,A}(t-\tau)(\nabla p_f \mathbf{x}, \tau)\, d\tau, \\
\rho_f \frac{\partial \mathbf{v}}{\partial t} + \rho_s(1-m)\frac{\partial^2 \mathbf{u}}{\partial t^2} = (1-m)\frac{\lambda_0}{2}(\Delta \mathbf{u} + \nabla(\mathbf{u})) - \nabla p, \\
\frac{1}{c_f}\frac{\partial p_f}{\partial t} + \frac{1}{c_s}\frac{\partial p_s}{\partial t} + \operatorname{div} \mathbf{v} + (1-m)\operatorname{div}\frac{\partial \mathbf{u}}{\partial t} = 0, \\
p = p_f + p_s, \qquad \frac{1}{c_s}p_s + \frac{\lambda_0(1-m)}{(\lambda_0 + c_s)}\operatorname{div} \mathbf{u} - \frac{(1-m)}{m(\lambda_0 + c_s)}p_f = 0,
\end{gathered} \tag{3.15}$$

where

$$\mathbf{v} = (0, v_2, v_3), \qquad \mathbf{u} = (0, u_2, u_3),$$

while all differential operators depend only on x_2 and x_3.

In the simplest case, when the initial perturbations depend only on x with $x = x_2$ or $x = x_3$, the system (3.20)–(3.22) for the functions p, u, and v with $p = p_f$, $(u, v) = (u_2, v_2)$ or $(u, v) = (u_3, v_3)$, reduces to

$$\begin{gathered}
v = m\,\frac{\partial u}{\partial t} - \int\limits_0^t b_{1,A}(t-\tau)\,\frac{\partial p}{\partial x}(x, \tau)\, d\tau, \\
\rho_f \frac{\partial v}{\partial t} + \rho_s(1-m)\,\frac{\partial^2 u}{\partial t^2} = \frac{(1-m)\lambda_0(\lambda_0 + 2c_s)}{(\lambda_0 + c_s)}\,\frac{\partial^2 u}{\partial x^2} - \frac{(m\lambda_0 + c_s)}{m(\lambda_0 + c_s)}\,\frac{\partial p}{\partial x}, \\
\left(\frac{1}{c_f} + \frac{(1-m)}{m(\lambda_0 + c_s)}\right)\frac{\partial p}{\partial t} + \frac{\partial v}{\partial x} + \frac{(1-m)c_s}{(\lambda_0 + c_s)}\,\frac{\partial^2 u}{\partial x \partial t} = 0.
\end{gathered} \tag{3.16}$$

4. ACOUSTICS IN AN EXTREMELY DEFORMABLE POROUS GROUND: $\lambda_0 = 0$

Here we consider only one physical case described by a new acoustics equation and, in our opinion, characteristic for acoustics in poroelastic media:

$$0 < \mu_1, \qquad \lambda_1 < \infty.$$

According to [5], for these criteria, the limit regime is described by the system

$$\frac{1}{m}\,p_f=\frac{1}{1-m}\,p_s, \tag{4.1}$$

$$\frac{1}{c_f^*}\,p_f+\frac{1}{c_s^*}\,p_s+\operatorname{div}\mathbf{w}=0, \tag{4.2}$$

$$\frac{\partial \mathbf{w}}{\partial t}=\int\limits_0^t \mathbb{B}_3(t-\tau)\cdot\nabla p_s(\mathbf{x},\tau)\,d\tau \tag{4.3}$$

for the displacement $\mathbf{w}$ of the continuous medium and the pressures p_f and p_s in the fluid and solid components. The matrix $\mathbb{B}_3(t)$ is determined by the solutions to the periodic initial boundary value problems

$$\tilde{\rho}\,\frac{\partial^2\mathbf{W}^{(i)}}{\partial t^2}=\operatorname{div}_y\left\{\mu_1\chi D\left(y,\frac{\partial \mathbf{W}^{(i)}}{\partial t}\right)+\lambda_1(1-\chi)D(y,\mathbf{W}^{(i)})-R^{(i)}\mathbb{I}\right\}, \qquad \mathbf{y}\in Y, \qquad t>0, \tag{4.4}$$

$$\operatorname{div}_y\mathbf{W}^{(i)}=0, \qquad \mathbf{y}\in Y, \qquad t>0, \tag{4.5}$$

$$\mathbf{W}^{(i)}(\mathbf{y},0)=0, \qquad \tilde{\rho}\,\frac{\partial \mathbf{W}^{(i)}}{\partial t}(\mathbf{y},0)=-\frac{1}{(1-m)}\,\mathbf{e}_i, \qquad \mathbf{y}\in Y, \tag{4.6}$$

using

$$\mathbb{B}_3(t)=\sum_{i=1}^{3}\left\langle\frac{\partial \mathbf{W}^{(i)}}{\partial t}(\mathbf{y},t)\right\rangle_Y\otimes\mathbf{e}_i,$$

where

$$\tilde{\rho}=\rho_f\chi+\rho_s(1-\chi).$$

As in Section 3, we consider only geometry (A) for which the last equations that determine $\mathbb{B}_3(t)$ take the simplest form. Namely, for geometry (A), the solutions to (4.4)–(4.6) for $i=2,3$ depend only on the spatial variable y_1. Indeed, putting $\mathbf{W}^{(i)}=W(y_1,t)\mathbf{e}_i$, we obtain the periodic initial boundary value problem for $W(y_1,t)$:

$$\tilde{\rho}\,\frac{\partial^2 W}{\partial t^2}=\frac{\partial}{\partial y_1}\left(\mu_1\chi\frac{\partial^2 W}{\partial y_1\partial t}+\lambda_1(1-\chi)\frac{\partial W}{\partial y_1}\right), \qquad 0<y_1<1, \qquad t>0,$$

$$W(y_1,0)=0, \qquad \tilde{\rho}\frac{\partial W}{\partial t}(y_1,0)=-\frac{1}{(1-m)}, \qquad 0<y_1<1,$$

which has the unique solution in the corresponding function space. Our statement now follows from the uniqueness for the original problem (4.4)–(4.6). For $i=1$, the situation is different. We know (see [5]) that (4.4)–(4.6) possesses a unique solution in this case as well, but the natural assumption that this solution depends only on the spatial variable y_1 leads to a contradiction. Therefore, here we see again the previous situation with the inhomogeneous Stokes problem (2.15) in the class of solenoidal functions. Recall that (4.4) is understood in the sense of distributions as the integral identity

$$\int\limits_0^T\int\limits_Y\left(-\tilde{\rho}\,\frac{\partial \mathbf{W}^{(1)}}{\partial t}\,\frac{\partial \boldsymbol{\varphi}}{\partial t}+\left(\mu_1\chi D\Big(y,\frac{\partial \mathbf{W}^{(1)}}{\partial t}\Big)+\lambda_1(1-\chi)D(y,\mathbf{W}^{(1)})\right)D(y,\boldsymbol{\varphi})\right)dy\,dt$$
$$=-\int\limits_Y\frac{1}{(1-m)}\,\mathbf{e}_1\cdot\boldsymbol{\varphi}(\mathbf{y},0)\,dy$$

for a solenoidal function $\mathbf{W}^{(1)}$ valid for every smooth solenoidal function $\boldsymbol{\varphi}$ equal to zero for $t = T$.

Since for this geometry $\mathbf{e}_1 = \nabla\phi(\mathbf{y})$, it follows that the right-hand side of this identity vanishes for all solenoidal functions $\boldsymbol{\varphi}$, and the unique solution to this identity is $\mathbf{W}^{(1)} \equiv 0$. Consequently,

$$\mathbb{B}_{3,A}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_{3,A}(t) & 0 \\ 0 & 0 & b_{3,A}(t) \end{pmatrix},$$

where

$$b_{3,A}(t) = \int\limits_Y \frac{\partial W}{\partial t}(y_1, t)\, dy_1.$$

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