Numerical Solution of the Linear Inverse Problem for the Euler–Darboux Equation

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Abstract—An inverse problem of the reconstruction of the right-hand side of the Euler–Darboux equation is studied. This problem is equivalent to the Volterra integral equation of the third kind with the operator of multiplication by a smooth nonincreasing function. Numerical solution of this problem is constructed using an integral representation of the solution of the inverse problem, the regularization method, and the method of quadratures. The convergence and stability of the numerical method is proved.

Keywords: inverse problem, Volterra equation of the third kind, regularization, numerical solution, stability.

The inverse problem for the Euler–Darboux equation arising in the theory of degenerating equations was investigated in [1]. In [2, 3], the regularizability of a similar problem in a more general setting was proved. Equivalency of these inverse problems to the Volterra integral equations of the third kind was proved. A numerical solution to the integral Volterra equation of the third kind was first constructed in [4] based on the regularized equation. In this paper, we consider the numerical solution of the inverse problem of the reconstruction of the right-hand side of the Euler–Darboux equation. Here, the integral representation of a solution to the inverse problem plays an important role.

1. STATEMENT OF THE PROBLEM

Let U be the space of functions $u(x, y)$ from the class $C(\overline{\Omega}) \cap C^{1,1}(\Omega)$, where $\Omega = \{0 < x < y < 1\}$, satisfying the boundary conditions

$$u(0, y) = \psi(y), \quad u(x, 1) = \phi(x), \tag{1}$$

and F be the space of the continuous functions $f(x, y)$ that are represented as

$$f(x, y) = -\beta_2(1-y)^{\alpha} y^{\beta_0 - 1} v(y) - y^{\beta_1}(y-x)^{\beta_0 - \beta_1} K(y, x) v(x),$$

where

$$\beta_2 = 1 + \beta_1 - \beta_0, \quad 1 < \beta_1 < \beta_0 < 2 \le \alpha, \quad L_0 = \partial^2/\partial x \partial y - \beta_1 (y-x)^{-1} \partial/\partial y.$$

We want to find a pair (u, f), $u \in U, f \in F$ such that

$$(L_0 u)(x, y) = f(x, y)/(y-x)^{\beta_0}, \tag{2}$$

provided that the following condition is satisfied for the given functions.

Condition 1. $\psi(y) \in C^2\ (0 < y \le 1)$, $\psi'(0) = 0$, $\phi(x) \in C^1\ (0 \le x < 1)$, and $K(y, x) \in C^{1,1}(\overline{\Omega})$.

The solution to problem (1), (2) can be represented (see [1]) in the integral form

$$u(x, y) = \phi(x) - \int_y^1 \eta^{\beta_1}(\eta - x)^{-\beta_1}\left(\psi'(\eta) - (1-\eta)^{\alpha} v(\eta)d\eta - \int_0^x K(\eta, t)v(t)dt\right)d\eta$$

$$- \int_y^1 (1-\eta)^{\alpha}\eta^{\beta_1 - 1}(\eta - x)^{1-\beta_1} v(\eta)d\eta.$$

Taking into account that $u \in U$ if and only if the function $v(x)$ is a solution to the Volterra integral equation of the third kind (see [1, 3])

$$(1-x)^{\alpha} v(x) + \int_0^x K(x, t)v(t)dt = \psi'(x), \tag{3}$$

the representation for $u(x, y)$ can be written as

$$u(x, y) = \phi(x) - \int_y^1 \frac{\eta^{\beta_1 - 1}(1-\eta)^{\alpha}}{(\eta - x)^{\beta_1 - 1}} v(\eta)d\eta - \int_y^1 \frac{\eta^{\beta_1}}{(\eta - x)^{\beta_1}} \int_x^{\eta} K(\eta, t)v(t)dtd\eta. \tag{4}$$

Recall some useful results from [4]. Let I be the identity operator,

$$(Av)(x) = (1-x)^{\alpha} v(x),$$

$$(Lv)(x) = \int_0^x L(x, t)v(t)dt, \quad (Gv)(x) = \int_0^x G(t)v(t)dt, \quad G(t) = C_0(1-t)^{\alpha} + K(t, t),$$

$$g(x) = \psi'(x) + C_0\psi(x), \quad L(x, t) = K(t, t) - K(x, t) - C_0\int_x^t K(s, t)ds, \quad 0 < C_0 = \text{const},$$

and let the following condition be fulfilled.

Condition 2. For positive $\theta_1 < 1$ and C_0, the inequalities

$$0 < d_1 \le G(x), \quad 0 \le \theta_1 G(x) - \alpha(1-x)^{\alpha - 1}$$

hold.

If Condition 2 is satisfied, then the system of integral equations

$$u_{\varepsilon}(x, y) = \phi(x) - \int_y^1 \left(\frac{\eta^{\beta_1 - 1}(1-\eta)^{\alpha}}{(\eta - x)^{\beta_1 - 1}} v_{\varepsilon}(\eta) - \frac{\eta^{\beta_1}}{(\eta - x)^{\beta_1}} \int_x^{\eta} K(\eta, t)v_{\varepsilon}(t)dt\right)d\eta, \tag{5}$$

$$(\varepsilon I + A)v_{\varepsilon}(x) + (Gv_{\varepsilon})(x) = (Lv_{\varepsilon})(x) + g(x) \tag{6}$$

with the small parameter $\varepsilon \in (0, 1)$ has a unique solution that converges uniformly as $\varepsilon \longrightarrow 0$ to the solution of system (3), (4).

Using the resolvent of the kernel $(-G(t)/[\varepsilon + p(t)])$, we write Eq. (6) as

$$v_{\varepsilon}(x) = (H_{\varepsilon}[Lv_{\varepsilon}])(x) + (H_{\varepsilon}g)(x), \tag{7}$$

where

$$(H_{\varepsilon}g)(x) = \frac{1}{\varepsilon + p(x)}\left(W_{\varepsilon}(x, 0)g(x) - \int_0^x W_{\varepsilon}(x, t)G(t)\frac{g(t) - g(x)}{\varepsilon + p(t)}dt\right),$$

$$W_{\varepsilon}(x, t) = \exp\left(-\int_t^x \frac{G(s)}{\varepsilon + p(s)}ds\right). \tag{8}$$

2. NUMERICAL SOLUTION

Define the uniform grid $\omega_{\tau,h} = \omega_\tau \times \omega_h$ on $\bar{\Omega}$ such that

$$\omega_\tau = \{y_n = \tau n, n = 0, 1, \ldots, N_0, \tau N_0 = 1\},$$

$$\omega_h = \{x_i = ih, i = 0, 1, \ldots, N_1, hN_1 = 1\}, \quad N_1 = k_0 N_0,$$

and $1 < k_0$ is a natural number. Denote by $C_{\tau,h}$ and C_h the spaces of the grid functions $u_{i,n} = u(x_i, y_n)$ and $v_i = v(x_i)$, respectively, with the norms

$$\|u_{i,n}\|_{\tau,h} = \max_{0 \le i \le n,\, 0 \le n \le N_0} |u_{i,n}|, \quad \|v_i\|_h = \max_{0 \le i \le N_1} |v_i|.$$

Setting $m_0 = k_0 m$, we approximate the integrals in (5) as

$$u^\varepsilon_{i,n} = \phi_i - \sum_{m=n+1}^{N_0} y_m^{\beta_1 - 1}\left(\tau^0_{i,m}(1 - y_m)^\alpha v_{\varepsilon,m} + \frac{y_m \tau_{i,m} h}{(y_m - x_i)^{\beta_1 - \alpha_0}} \sum_{j=i+1}^{m_0} K_{m,j} v_{\varepsilon,j}\right), \tag{9}$$

where

$$\tau_{i,m} = \frac{1}{1-\alpha_0}[(y_m - x_i)^{1-\alpha_0} - (y_{m-1} - x_i)^{1-\alpha_0}], \quad i = 1, 2, \ldots, n, \quad m = n+1, \ldots, N_0,$$

$$\tau^0_{i,m} = \frac{1}{2-\beta_1}[(y_m - x_i)^{2-\beta_1} - (y_{m-1} - x_i)^{2-\beta_1}], \quad i = 1, 2, \ldots, n, \quad m = n+1, \ldots, N_0.$$

Approximate the integrals in (7) using, for example, the right rectangular quadrature formula. Then, we obtain the system of linear algebraic equations

$$v_{\varepsilon,i} = -\frac{h}{\varepsilon + (1 - x_i)^\alpha} \sum_{j=1}^{i} W^{\varepsilon,h}_{i,j} G_j\left(h\sum_{k=1}^{j} L_{j,k} v_{\varepsilon,k} - h\sum_{k=1}^{i} L_{i,k} v_{\varepsilon,k} + g_j - g_i\right) + \Psi^{\varepsilon,h}_{i,1}\left(h\sum_{j=1}^{i} L_{i,j} v_{\varepsilon,j} + g_i\right), \tag{10}$$

$$i = 1, 2, \ldots, n, \quad n = 1, 2, \ldots, N_0,$$

where $L_{j,k} = L(x_j, x_k)$, $G_j = G(x_j)$, $v_{\varepsilon,j} = v_\varepsilon(x_j)$, $g_j = g(x_i)$, $x_j = jh$, $j = 1, 2, \ldots, i$,

$$W^{\varepsilon,h}_{i,j} = \frac{1}{\varepsilon + (1 - x_j)^\alpha} \exp\left(-h\sum_{s=j}^{i} \frac{G_s}{\varepsilon + (1 - x_s)^\alpha}\right),$$

$$L(x_i, x_k) = -K(x_i, x_k) + K(x_k, x_k) - C_0 h \sum_{s=k+1}^{i} K(x_s, x_k),$$

$$\Psi^{\varepsilon,h}_{i,1} = \frac{1}{\varepsilon + 1} \exp\left(-h\sum_{s=1}^{i} \frac{G_s - \alpha(1 - x_s)^{\alpha - 1}}{\varepsilon + (1 - x_s)^\alpha}\right).$$

Since

$$\left(\sum_{s=k+1}^{i} K_{s,k}\right) = \left(\sum_{s=1}^{i} K_{s,k} - \sum_{s=1}^{k} K_{s,k}\right) = 0$$

for $k = i$, we have

$$\sum_{k=1}^{i} L_{i,k} v_{\varepsilon,k} = -\sum_{k=1}^{i} (K_{i,k} - K_{k,k}) v_{\varepsilon,k} - C_0 \sum_{k=1}^{i}\left(\sum_{s=k+1}^{i} K_{s,k}\right) v_{\varepsilon,k} = \sum_{k=1}^{i} L_{i,k} v_{\varepsilon,k}.$$

Thus (10) becomes

$$v_{\varepsilon,i} = -\frac{h}{\varepsilon + (1-x_i)^{\alpha}}\sum_{j=1}^{i} W_{i,j}^{\varepsilon,h} G_j\left(h\sum_{k=1}^{j-1} L_{j,k}v_{\varepsilon,k} - h\sum_{k=1}^{i-1} L_{i,k}v_{\varepsilon,k} + g_j - g_i\right) + \Psi_{i,1}^{\varepsilon,h}\left(h\sum_{j=1}^{i-1} L_{i,j}v_{\varepsilon,j} + g_i\right), \quad (11)$$

$$i = 1, 2, \ldots, n.$$

Below, to prove the convergence of the solution of system (9), (10) to the exact solution of inverse problem (1), (2), we will need the bound on the error of approximating the integrals in (5) by rule (9).

Introduce the notation

$$I_{i,n} = \int_{y_n}^{1} \frac{\eta^{\beta_1} d\eta}{(\eta - x_i)^{\beta_1}} \int_{x_i}^{\eta} K(\eta, t)v(t)dt, \quad I_{i,n}^{h,\tau} = \sum_{m=n+1}^{N_0} \frac{h y_m^{\beta_1}\tau_{i,m}}{(y_m - x_i)^{\beta_1 - \alpha_0}} \sum_{j=i+1}^{m_0} K_{m,j}v_j.$$

Lemma 1. *Let the functions* $K(x, y)$ *and* $v(x)$ *be continuously differentiable. Then, for any* $1 < \beta_1 < 2$, *there exists a number* $0 < \alpha_0 < 1$ *such that* $0 < \lambda_1 = 1 - \beta_1 + \alpha_0 < 1$ *and the inequality* $|I_{i,n} - I_{i,n}^{h,\tau}| \le O(h + \tau^{\lambda_1})$ holds.

Proof. Since $\alpha_0 < \beta_1$, we have $\lambda_1 < 1$. Moreover, for the functions $z_1(s) = \beta_1$ and $z_2(s) = 1 + s$, for any $1 < \beta_1 < 2$, there exists an interval $(s_1, s_2) \subset (0, 1)$ such that $z_1(s) < z_2(s)$ for any $s \in (s_1, s_2)$. Therefore, for $1 < \beta_1 < 2$, there exists a number $0 < \alpha_0 < 1$ such that $0 < \lambda_1$, and, for $x_i \le \eta \le y_m$, the inequality $[\gamma(x_i, y_m, \eta)]^{-1} \le N_0(\eta - x_i)^{\beta_1 - 1 - \alpha_0}$ holds, where $\gamma(x, y, \eta) = (\eta - x)^{1 + \alpha_0 - \beta_1} + \ldots + (y - x)^{1 + \alpha_0 - \beta_1}$, $\gamma(x, y, \eta)z_4(x, y, \eta) = \eta - y$, and $z_4(x, y, \eta) = (\eta - x)^{\beta_1 - \alpha_0} - (y - x)^{\beta_1 - \alpha_0}$. Taking into account this fact, we estimate the difference $I_{i,n} - I_{i,n}^{h,\tau}$. We have

$$|I_{i,n}^1| = \sum_{m=n+1}^{N_0} \int_{y_{m-1}}^{y_m} \frac{1}{(\eta - x_i)^{\alpha_0}} \left|\frac{y_m^{\beta_1}}{(\eta - x_i)^{\beta_1 - \alpha_0}} - \frac{y_m^{\beta_1}}{(y_m - x_i)^{\beta_1 - \alpha_0}}\right| \int_{x_i}^{\eta} |K(\eta, t)||v(t)|dtd\eta$$

$$+ \sum_{m=n+1}^{N_0} \int_{y_{m-1}}^{y_m} \frac{1}{(\eta - x_i)^{\alpha_0}} \frac{\eta^{\beta_1} - y_m^{\beta_1}}{(\eta - x_i)^{\beta_1 - \alpha_0}} \int_{x_i}^{\eta} |K(\eta, t)v(t)|dtd\eta$$

$$\le T_0 r_0 \sum_{m=n+1}^{N_0 z} \int_{y_{m-1}}^{y_m} \frac{d\eta}{(\eta - x_i)^{\alpha_0}} \left(\frac{y_m^{\beta_1}(y_m - \eta)(\eta - x_i)^{1 + \alpha_0 - \beta_1}}{(y_m - x_i)^{\beta_1 - \alpha_0}[(y_m - x_i)^{1 + \alpha_0 - \beta_1} + \ldots + (\eta - x_i)^{1 + \alpha_0 - \beta_1}]}\right.$$

$$\left. + (\eta - x_i)^{1 + \alpha_0 - \beta_1}\beta_1\tau\right) \le r_0 T_0 (N_0 \tau^{1 + \alpha_0 - \beta_1} + \beta_1 \tau);$$

$$|I_{i,n}^2| \le \sum_{m=n+1}^{N_0} \int_{y_{m-1}}^{y_m} \frac{1}{(\eta - x_i)^{\alpha_0}} \frac{y_m^{\beta_1}}{(y_m - x_i)^{\beta_1 - \alpha_0}} r_0 (y_m - \eta)[T_1(\eta - x_i) + T_0]d\eta \le r_0(T_0\tau + T_0\tau^{1 + \alpha_0 - \beta_1}),$$

where $|v(x)| \le r_0 = \text{const}$,

$$T_0 = \max_D |K(x, t)|, \quad T_1 = \max_D |K'_x(x, t)|, \quad D = \{0 \le t \le x \le 1\}.$$

Thus,

$$\left|I_{i,n}-I_{i,n}^{h,\tau}\right| \le \left|I_{i,n}^{1}\right| + \left|I_{i,n}^{2}\right| + \left|I_{i,n}^{3}-I_{i,n}^{h,\tau}\right| = \left|I_{i,n}^{1}\right| + \left|I_{i,n}^{2}\right|$$

$$+\sum_{m=n+1}^{N_0}\int_{y_{m-1}}^{y_m}\frac{d\eta}{(\eta-x_i)^{\alpha_0}}\frac{y_m^{\beta_1}}{(y_m-x_i)^{\beta_1-\alpha_0}}\sum_{j=i+1}^{m_0}\int_{x_{j-1}}^{x_j}|K(y_m,t)v(t)-K(y_m,x_j)v(x_j)|dt \le M_1h + M_2\tau^{\lambda_1},$$

where

$$M_1 = \max_{\bar{\Omega}}\left|\frac{\partial}{\partial t}(K(y,t)v(t))\right|, \quad M_2 = r_0[T_1 + T_0(\beta_1 + N_0 + 1)].$$

The lemma is proved.

The stability of the solution to system (10) with respect to the right-hand side, which is measured in $C^1[0, 1]$ metric, is proved below.

Lemma 2. *Under Conditions* 1 *and* 2, *the solution to system* (11) *satisfies the bound*

$$\|v_{\varepsilon,i}\|_h \le (Q_0 + Q_1)M_3, \text{ where } Q_0 = \max_{[0,1]}|\psi'(x)|, \quad Q_1 = \max_{[0,1]}|\psi''(x)|,$$

and M_3 *is a constant independent of* h *and* ε.

Proof. Using Condition 1, we obtain

$$|L_{i,k}-L_{j,k}| \le |K_{i,k}-K_{j,k}| + C_0h\sum_{m=j+1}^{i}|K_{j,k}| \le (T_1 + C_0T_0)(hi-hj),$$

$$|L_{i,k}| \le |K_{i,k}-K_{k,k}| + C_0h\sum_{m=k+1}^{i}|K_{j,k}| \le (T_1 + C_0T_0)(hi-hk).$$

Therefore, we have

$$\left|h\sum_{k=1}^{j-1}L_{j,k}v_{\varepsilon,k} - h\sum_{k=1}^{i-1}L_{i,k}v_{\varepsilon,k}\right| \le h\sum_{k=1}^{j-1}|L_{j,k}-L_{i,k}||v_{\varepsilon,k}| + h\sum_{k=1}^{i-1}|L_{i,k}||v_{\varepsilon,k}|$$

$$\le 2(T_1 + C_0T_0)(hi-hj)h\sum_{k=1}^{i-1}|v_{\varepsilon,k}|.$$

Since $(1-x_k)^\alpha \le (1-x_j)^\alpha$ for all $j \le k$, we have, taking into account Condition 2,

$$\frac{hi-hj}{\varepsilon + (1-x_j)^\alpha} \le h\sum_{k=j}^{i}\frac{1}{\varepsilon + (1-x_k)^\alpha} \le d_1^{-1}\sum_{k=j}^{i}\frac{G_k}{\varepsilon + (1-x_k)^\alpha}.$$

Based on the estimates given above, we obtain from (11)

$$\left|-\frac{h}{\varepsilon + (1-x_i)^\alpha}\sum_{j=1}^{i}W_{i,j}^{\varepsilon,h}G_j\left(h\sum_{k=1}^{j-1}L_{j,k}v_{\varepsilon,k} - h\sum_{k=1}^{i-1}L_{i,k}v_{\varepsilon,k} + g_j - g_i\right)\right|$$

$$\le \frac{h}{\varepsilon + (1-x_i)^\alpha}\sum_{j=1}^{i}W_{i,j}^{\varepsilon,h}G_j\left(2T_2h\sum_{k=1}^{i-1}|v_{\varepsilon,k}| + Q_1 + C_0Q_0\right)(hi-hj)$$

$$\le \left(2T_2h\sum_{k=1}^{i-1}|v_{\varepsilon,k}| + Q_1 + C_0Q_0\right)\frac{T_3h}{\varepsilon + (1-x_i)^\alpha}\sum_{j=1}^{i}\frac{hi-hj}{\varepsilon + (1-x_j)^\alpha}\exp\left(-h\sum_{s=j}^{i}\frac{G_s}{\varepsilon + (1-x_s)^\alpha}\right)$$

$$\leq \left(2T_2 h\sum_{k=1}^{i-1}|v_{\varepsilon,k}| + Q_1 + C_0Q_0\right)T_3(d_1\theta_2)^{-2}\sum_{\nu=0}^{\mu}e^{-\theta_1\nu},$$

where

$$\mu = h\sum_{s=1}^{i}\frac{G_s}{\varepsilon+(1-x_s)^{\alpha}},\quad T_2 = T_1 + C_0T_0,\quad T_3 = 2\max_{[0,1]}|G(x)|/e^2,\quad \theta_2 = 1-\theta_1.$$

The expression

$$S_\mu = \sum_{\nu=0}^{\mu}e^{-\theta_1\nu}$$

is majorized by the convergent series $\sum_{\nu=0}^{\infty}e^{-\theta_1\nu}$. Therefore, $S_\mu < +\infty$. Similarly, using Conditions 1 and 2, we obtain the following inequality for the second term in (11):

$$\left|\Psi_{i,1}^{\varepsilon,h}\left(h\sum_{j=1}^{i-1}L_{i,j}v_{\varepsilon,j} + g_i\right)\right| \leq \frac{1}{\varepsilon+1}\left(T_2h\sum_{j=1}^{i-1}|v_{\varepsilon,j}| + (1+C_0)Q_0\right).$$

Using the notation $T_4 = T_2(2(d_1\theta_2)^{-2}S_0T_3 + 1)$, $d_2 = (1 + C_0)[1 + (d_1\theta_2)^{-2}S_0T_3]$, and $S_0 = \sup|S_\mu|$, we obtain from system (11) that

$$|v_{\varepsilon,i}| \leq T_4h\sum_{j=1}^{i-1}|v_{\varepsilon,j}| + (Q_1+Q_0)d_2.$$

This immediately (see [5, p. 21]) entails the bound

$$\|v_{\varepsilon,i}\|_h \leq [(Q_1+Q_0)d_2]\exp(T_4),$$

which proves the lemma.

The following lemma can be proved in a similar way.

Lemma 3. *If Conditions* 1, 2 *are fulfilled and* $v(x) \in C^1[0, 1]$, *then the bound*

$$\|H_\varepsilon^h[v_i]\|_h \leq Q_3, \tag{12}$$

holds, where

$$H_\varepsilon^h[v_i] = -\frac{h}{\varepsilon+(1-x_i)^{\alpha}}\sum_{j=1}^{i}W_{i,j}^{\varepsilon,h}G_j(v_j - v_i) + \Psi_{i,1}^{\varepsilon,h}v_i,$$

$$|v(x)| \leq r_0,\quad |v'(x)| \leq r_1,\quad r_0, r_1 = \text{const},\quad Q_3 = d_1^{-1}T_3S_0r_1 + 2r_0.$$

Lemmas 1 and 2 imply the stability of the solution of difference scheme (9), (10) with respect to the boundary functions if $\phi(x)$ and $\psi(y)$ are measured in the norms of the spaces $C[0, 1]$ and $C^2[0, 1]$, respectively.

Lemma 4. *If Conditions* 1 *and* 2 *hold, then, for* $\sigma_0 = 1 - 2\sigma$, $\varepsilon = O(h^{\sigma})$, *and* $\sigma > 0$, *it holds that*

$$\left|\int_0^x\frac{G(s)+\alpha(1-s)^{\alpha-1}}{\varepsilon+(1-s)^{\alpha}}ds - h\sum_{k=1}^{i}\frac{G_k+\alpha(1-x_k)^{\alpha-1}}{\varepsilon+(1-x_k)^{\alpha-1}}\right| \leq O(h^{\sigma_0}). \tag{13}$$

Proof. In view of Conditions 1 and 2, we have

$$\left|\int_0^x\frac{G(s)+\alpha(1-s)^{\alpha-1}}{\varepsilon+(1-s)^{\alpha}}ds - h\sum_{k=1}^{i}\frac{G_k+\alpha(1-x_k)^{\alpha-1}}{\varepsilon+(1-x_k)^{\alpha-1}}\right|$$

$$\le \sum_{k=1}^{i} \left| \int_{x_{k-1}}^{x_k} \left(\frac{G(s) + \alpha(1-s)^{\alpha-1}}{\varepsilon + (1-s)^{\alpha}} - \frac{G_k + \alpha(1-x_k)^{\alpha-1}}{\varepsilon + (1-x_k)^{\alpha-1}} \right) ds \right|$$

$$\le [\|G'(x)\|_C + \alpha(\alpha-1)] \frac{h}{\varepsilon + (1-x_i)^{\alpha}} + [\|G(x)\|_C + \alpha] \frac{\alpha h}{(\varepsilon + (1-x_i)^{\alpha})^2},$$

and for $\varepsilon = O(h^{\sigma})$, this yields (13). The lemma is proved.

Theorem 1. *Let Conditions* 1 *and* 2 *be fulfilled and* $\varepsilon = O(h^{\sigma})$. *Then, for any* $1 < \beta_1 < 2$, *there exists a number* $0 < \alpha_0 < 1$ *such that* $0 < \lambda_1 = 1 - \beta_1 + \alpha_0 < 1$ *and the solution to system* (9), (10) *uniformly converges to the exact solution to system* (3), (4) *for all* $0 < \sigma \le 1/2$ *as* $h \longrightarrow 0$ *and* $\tau \longrightarrow 0$. *Moreover, the following bound holds*:

$$\|u_{i,n}^{\varepsilon} - u_{i,n}\|_{\tau,h} \le O(h^{\sigma} + \tau^{\lambda_1}).$$

Proof. Equation (3) is equivalent (see [3]) to the equation $Av + Gv = Lv + g$ that follows from (6) for $\varepsilon = 0$. The last equation is reduced to

$$v(x) = (H_{\varepsilon}[Lv])(x) - \varepsilon(H_{\varepsilon}v)(x) + (H_{\varepsilon}g)(x), \tag{14}$$

where H_{ε} is defined by formula (8). Setting $x = x_i$ in (14) and $x = x_i$, $y = y_n$ in (4) and using the right rectangular quadrature formula for integrals, we obtain the system of equations

$$u_{i,n} = \phi_i - \sum_{m=n+1}^{N_0} \left(\tau_{i,m}^0 (1-y_m)^{\alpha} v_m + \frac{h y_m^{\beta_1} \tau_{i,m}}{(y_m - x_i)^{\beta_1 - \alpha_0}} \sum_{j=i+1}^{m_0} K_{m,j} v_j \right) + R_{i,n}^1, \tag{15}$$

$$\begin{aligned} v_i = &-\frac{h}{\varepsilon + (1-x_i)^{\alpha}} \sum_{j=1}^{i} W_{i,j}^{\varepsilon,h} G_j \left(h \sum_{k=1}^{j-1} L_{j,k} v_k - h \sum_{k=1}^{i-1} L_{j,k} v_k - \varepsilon(v_j - v_i) + g_j - g_i \right) \\ &+ \Psi_{i,1}^{\varepsilon,h} \left(h \sum_{j=1}^{i-1} L_{i,j} v_j + g_i - \varepsilon v_i \right) + R_i, \quad i = 1, 2, \ldots, n, \quad n = 1, 2, \ldots, N_0, \end{aligned} \tag{16}$$

where $R_{i,n}^1$ and R_i are the sums of the residual terms of the integrals in (4) and (14). Define the error vectors

$$\xi_{i,n}^{\tau,h} = u_{i,n}^{\varepsilon} - u_{i,n}, \quad \eta_{\varepsilon,i}^h = v_{\varepsilon,i} - v_i, \quad i = 1, 2, \ldots, n, \quad n = 1, 2, \ldots, N_0.$$

Then, it follows from (9), (11) and (15), (16) that

$$\xi_{i,n}^{\varepsilon} = -\sum_{m=n+1}^{N_0} \left(\tau_{i,m}^0 (1-y_m)^{\alpha} \eta_{\varepsilon,m}^h + \frac{h y_m^{\beta_1} \tau_{i,m}}{(y_m - x_i)^{\beta_1 - \alpha_0}} \sum_{j=i+1}^{m_0} K_{m,j} \eta_{\varepsilon,j}^h \right) - R_{i,n}^1, \tag{17}$$

$$\begin{aligned} \eta_{\varepsilon,j}^h = &-\frac{h}{\varepsilon + (1-x_i)^{\alpha}} \sum_{j=1}^{i} W_{i,j}^{\varepsilon,h} G_j \left(h \sum_{k=1}^{j-1} L_{j,k} \eta_{\varepsilon,k}^h - h \sum_{k=1}^{i-1} L_{i,k} \eta_{\varepsilon,k}^h - \varepsilon(v_j - v_i) \right) \\ &+ \Psi_{i,1}^{\varepsilon,h} \left(h \sum_{j=1}^{i-1} L_{i,j} \eta_{\varepsilon,k}^h - \varepsilon v_i \right) - R_i, \quad i = 1, 2, \ldots, n, \quad n = 1, 2, \ldots, N_0. \end{aligned} \tag{18}$$

Repeating the calculations from the proof of Lemma 2 for (17), (18), we obtain the inequalities

$$\begin{aligned} |\xi_{i,n}^{\varepsilon}| &\le \sum_{m=n+1}^{N_0} \left(\tau_{i,m}^0 (1-y_m)^{\alpha} |\eta_{\varepsilon,m}^h| + \frac{h T_0 y_m^{\beta_1} \tau_{i,m}}{(y_m - x_i)^{\beta_1 - \alpha_0}} \sum_{j=i+1}^{m} |\eta_{\varepsilon,j}^h| \right) + |R_{i,n}^1|, \\ |\eta_{\varepsilon,i}^h| &\le T_4 h \sum_{k=1}^{i-1} |\eta_{\varepsilon,k}^h| + \varepsilon |H_{\varepsilon}^h v_i| + |R_i|, \quad i = 1, 2, \ldots, n, \quad n = 1, 2, \ldots, N_0. \end{aligned} \tag{19}$$

Applying the difference analogue of the Gronwall–Bellmann lemma (see [5, pp. 20, 21]) to the last inequality, we have

$$\left|\eta^h_{\varepsilon,j}\right| \le (\varepsilon\left|H^h_\varepsilon v_i\right| + |R_i|)\exp(T_4);$$

consequently, using bound (12), we obtain

$$\left\|\eta^h_{\varepsilon,i}\right\|_h \le (Q_3\varepsilon + \|R_i\|_h)\exp(T_4). \tag{20}$$

Now, we obtain a bound on the residual term R_i. From Condition 1, we have

$$|R_{i,1}| = \left|-\frac{1}{\varepsilon+(1-x_i)^\alpha}\sum_{j=1}^{i}\int_{x_{j-1}}^{x_j} W_i^\varepsilon G(t)\left(\int_0^t L(t,s)v(s)ds - \int_0^{x_j} L(x_j,s)v(s)ds\right)dt\right|$$

$$\le \frac{2T_2r_0h}{\varepsilon+(1-x_i)^\alpha}\sum_{j=1}^{i}\int_{x_{j-1}}^{x_j}\exp\left(-\int_t^{x_i}\frac{G(s)}{\varepsilon+(1-s)^\alpha}ds\right)\frac{G(t)}{\varepsilon+(1-t)^\alpha}dt \le \frac{2hT_2r_0d_3}{\varepsilon+(1-x_i)^\alpha},$$

$$|R_{i,2}| = \left|\sum_{j=1}^{i}\int_{x_{j-1}}^{x_j}\frac{(W_i^\varepsilon - W_{i,j}^\varepsilon)G(t)}{\varepsilon+(1-x_i)^\alpha}\left(\int_0^{x_j} L(x_j,s)v(s)ds - \int_0^{x_i} L(x_i,s)v(s)ds\right)dt\right|$$

$$\le \frac{6T_2T_3r_0h}{\varepsilon+(1-x_i)^\alpha}\sum_{j=1}^{i} W_{i,j}^\varepsilon(x_i-x_j) \le 6d_1^{-1}T_2T_3r_0d_4\frac{h}{\varepsilon+(1-x_i)^\alpha},$$

$$|R_{i,3}| = \left|\sum_{j=1}^{i}\int_{x_{j-1}}^{x_j}\frac{(W_{i,j}^\varepsilon - W_{i,j}^{\varepsilon,h})G(t)}{\varepsilon+(1-x_i)^\alpha}\left(\int_0^{x_j} L(x_j,s)v(s)ds - \int_0^{x_i} L(x_i,s)v(s)ds\right)dt\right|$$

$$\le \frac{2T_2r_0h}{\varepsilon+(1-x_i)^\alpha}\sum_{j=1}^{i} W_{i,j}^\varepsilon(x_i-x_j)\left|1-\exp\left(\int_{x_j}^{x_i}\frac{G(s)}{\varepsilon+(1-s)^\alpha}ds - \sum_{k=j}^{i}\frac{hG_k}{\varepsilon+(1-x_k)^\alpha}\right)\right|,$$

where

$$d_3 = \sup_{\eta\ge 0}\sum_{\nu=0}^{\eta} e^{-\nu}, \quad \eta = \int_{x_1}^{x_i}\frac{G(s)}{\varepsilon+(1-s)^\alpha}ds, \quad d_4 = \sup_{\eta\ge 0}\sum_{\nu=0}^{\eta} e^{-\nu}\nu.$$

According to Lemma 4, the expression

$$R_{i,0} = \int_{x_j}^{x_i}\frac{G(s)}{\varepsilon+(1-s)^\alpha}ds - h\sum_{k=j}^{i}\frac{G_k}{\varepsilon+(1-x_k)^\alpha}$$

is bounded if $\varepsilon = O(h^\sigma)$ and $0 < \sigma \le 1/2$. Then, Theorem 1 implies $|1 - \exp(R_{i,0})| \le d_5 = \text{const}$. Thus,

$$|R_{i,3}| \le d_1^{-1}T_2T_3r_0d_4d_5\frac{h}{\varepsilon+(1-x_i)^\alpha},$$

$$|R_{i,4}| = \left|\sum_{j=1}^{i}\int_{x_{j-1}}^{x_j}\frac{W_{i,j}^{\varepsilon,h}[G(t)-G_j]}{\varepsilon+(1-x_i)^\alpha}\left(\int_0^{x_j} L(x_j,s)v(s)ds - \int_0^{x_i} L(x_i,s)v(s)ds\right)dt\right|$$

$$\le \frac{2T_2r_0}{\varepsilon+(1-x_i)^\alpha}\sum_{j=1}^{i} W_{i,j}^{\varepsilon,h}(x_i-x_j)\int_{x_j}^{x_i}[G(t)-G_j]dt \le d_1^{-1}T_2T_5r_0d_4\frac{h^2}{\varepsilon+(1-x_i)^\alpha},$$

$$|R_{i,5}| \le \frac{1}{\varepsilon + (1-x_i)^\alpha} \sum_{j=1}^{i} \int_{x_{j-1}}^{x_j} W_{i,j}^{\varepsilon,h} G_j \left(\left| \int_0^{x_j} L(x_j, s) v(s) ds - h \sum_{k=1}^{j} L_{j,k} v_k \right| \right.$$

$$\left. + \left| \int_0^{x_i} L(x_i, s) v(s) ds - h \sum_{k=1}^{i} L_{i,k} v_k \right| \right) dt \le d_1^{-1} T_6 T_3 d_4 \frac{h}{\varepsilon + (1-x_i)^\alpha},$$

where

$$T_5 = \max_{[0,1]} |G'(x)|, \quad T_6 = \max_{\bar{\Omega}} \left| \frac{\partial}{\partial t}(L(x,t) v(t)) \right|.$$

Using Lemma 4 and the condition $0 < \sigma \le 1/2$, we obtain

$$|R_{i,6}| = \left| (\Psi_{i,1}^{\varepsilon} - \Psi_{i,1}^{\varepsilon,h}) \left(\int_0^{x_i} L(x_i, s) v(t) dt + g_i \right) \right| \le T_7 |\Psi_{i,1}^{\varepsilon} - \Psi_{i,1}^{\varepsilon,h}|$$

$$\le d_1^{-1} T_7 \Psi_{i,1,\theta}^{\varepsilon,h} \left(h \sum_{j=1}^{i} \frac{G_j}{\varepsilon + (1-x_j)^\alpha} \right) \left(T_8 h + T_9 \frac{h}{\varepsilon + (1-x_i)^\alpha} \right)$$

$$\le (d_1 \theta_1 e)^{-1} T_7 \left(T_8 h + T_9 \frac{h}{\varepsilon + (1-x_i)^\alpha} \right),$$

where $0 < \theta_1 < 1$, $T_8 = \alpha(1-\alpha) + T_5$, $T_9 = T_3 + \alpha$,

$$\Psi_{i,1,\theta}^{\varepsilon,h} = \exp\left(-h\theta_1 \sum_{j=1}^{i} \frac{G_j}{\varepsilon + (1-x_j)^\alpha} \right), \quad \left| \int_0^{x_i} L(x_i, s) v(t) dt + g_i \right| \le T_7,$$

$$|R_{i,7}| = \left| \Psi_{i,1}^{\varepsilon,h} \left(\int_0^{x_i} L(x_i, t) v(t) dt - h \sum_{j=1}^{i} L_{i,j} v_j \right) \right| \le T_6 h/2.$$

Since the residual term $R_i = \sum_{j=1}^{7} R_{i,j}$, we have from the bounds obtained above that

$$\|R_i\|_h = Q_5 h + Q_6 h/\varepsilon,$$

where Q_5 and Q_6 are positive numbers independent of ε and h. Then, (20) implies

$$\|\eta_{\varepsilon,i}^h\|_h \le \left(Q_3 \varepsilon + Q_5 h + \frac{h}{\varepsilon} Q_6 \right) \exp(T_4). \tag{21}$$

Thus, we have obtained a bound that is typical in the regularization theory (see [6, p. 12]). This bound requires that the small parameter be coordinated with the discretization step. It follows from (21) that if we take $\varepsilon = O(h^\sigma)$, then the difference scheme based on (9), (10) converges for all $0 < \sigma \le 1/2$. Moreover, we can find the relation between ε and h that minimizes the error of the approximate solution. Define ε as the minimum of the right-hand side of (21). We obtain $\sigma = \sqrt{Q_6 h / Q_3}$.

Using Lemma 1 and bound (21), we obtain from (19)

$$\|u_{i,n}^{\varepsilon} - u_{i,n}\|_{\tau,h} = \|\xi_{i,n}^{\varepsilon}\|_{\tau,h} \le (1 + T_0) \|\eta_{\varepsilon,n}^h\|_h + O(h + \tau^{\lambda_1}) \le O(h^\sigma + \tau^{\lambda_1}).$$

The theorem is proved.

Example. Let $f(x, y) = -\beta_2(1 - y)^3 y^{\beta_0 - 1} v(y) - 9y^{\beta_1}(y - x)^{\beta_0 - \beta_1} v(x)$, $\phi(x) = x^3 + 23/30$, and $\psi(y) = y^3/3 + 3y^5/5 - y^6/6$. Then (3) becomes

$$(1-x)^3 v(x) + 9\int_0^x v(t)dt = x^2 + 3x^4 - x^5. \tag{22}$$

The exact solution to integral equation (22) is $v(x) = x^2$. Then,

$$u(x, y) = \frac{23}{30} + x^3 - 3\int_y^1 \frac{\eta^{\beta_1 - 1}}{(\eta - x)^{\beta_1 - 1}}\left(\frac{1}{3}\eta^2(1-\eta)^3 + \eta^3 + \eta^2 x + \eta x^2\right)d\eta.$$

Calculations by formulas (9), (11) give the error $\|u_{i,n}^{\varepsilon} - u_{i,n}\|_{\tau,h} \le 0.13636$ for $h = \tau = 0.01$ and the error $\|u_{i,n}^{\varepsilon} - u_{i,n}\|_{\tau,h} \le 0.0857$ for $h = \tau = 0.005$.

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