QUASI-INVERSION METHOD FOR AN EVOLUTIONARY EQUATION OF FRACTIONAL ORDER

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ABSTRACT. Let B be a linear closed densely defined operator on a Banach space having no resolvent in general. The paper studies the Cauchy-type problem for the evolutionary fractional-order equation $D^\alpha Bu(t)$, $t > 0$, where $0 < \alpha \leqslant 1$, $D^\alpha u(t)$ is the Riemann–Liouville fractional derivative of order α.

1. Introduction. Statement of the Problem

In a Banach space E, let us consider the following Cauchy-type problem:

$$D^\alpha u(t) = Bu(t), \quad t > 0, \tag{1.1}$$

$$\lim_{t\to 0} I^{1-\alpha} u(t) = u_0, \tag{1.2}$$

where $0 < \alpha \leqslant 1$, $I^{1-\alpha}u(t) = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} u(s)\, ds$ is the left-side fractional Riemann–Liouville integral of order $1-\alpha$ ($I^{1-\alpha}$ is the identity operator for $\alpha = 1$), $D^\alpha u(t) = \dfrac{d}{dt} I^{1-\alpha} u(t)$ is the left-side Riemann–Liouville derivative of order α, $\Gamma(\cdot)$ is the gamma-function, and B is a linear closed densely defined unbounded operator.

The case where the problem (1.1), (1.2) is well posed was considered for $0 < \alpha < 1$ in [5, 9]. In these works, it is assumed that the operator B has the resolvent $(\lambda I - B)^{-1}$ that decays in a certain way for Re $\lambda > \omega$ and as $|\lambda| \to \infty$. For $\alpha = 1$, for the uniform well-posedness of the problem (1.1), (1.2), it is required that the operator B be the generator of a C_0-semigroup [6, 7, 10, 11]. The Cauchy problem with Caputo fractional derivative was studied in [1].

We do not assume the existence of a resolvent for the operator B, but impose a certain condition that means that the operator B is subordinated to the generator of the C_0-semigroup in some sense.

Naturally, with such an operator, the problem (1.1), (1.2) is not well-posed in general. For example, for $\alpha = 1$, the problem (1.1), (1.2) considered on a finite interval contains the direct, as well as inverse, Cauchy problems.

For a number of ill-posed problems of mathematical physics, the work [12] elaborates the quasi-inversion method. Its main idea is that to the differential equation, we add a summand containing a small parameter ε, so that for the modified equation the initial problem becomes well-posed. In what follows, it is shown that a solution of the ill-posed problem can be approximated by solutions of the well-posed problem. In the present work, we apply the quasi-inversion method for the fractional-order differential equation in a Banach space. Note that the case of Hilbert case was previously considered in [4].

Condition 1.1. The linear operator $B$is closed, and its domain $D(B)$ is dense in E.

Condition 1.2. (i) There exists an operator A with $D(A) \subset D(B)$, being the generator of a C_0-semigroup $T(t)$ that satisfies the estimate

$$\|T(t)\| \leqslant M_1 e^{\omega_A t}, \tag{1.3}$$

and is such that

$$B(\lambda I - A)^{-1}x = (\lambda I - A)^{-1}Bx, \quad x \in D(B), \quad \operatorname{Re}\lambda > \omega_A.$$

(ii) For any $x \in E$, there exist constants $M_2 > 0$ and $\gamma \in [0, 1)$ such that $T(t)x \in D(B)$ (smoothing effect) and

$$\|BT(t)x\| \leqslant M_2 t^{-\gamma} e^{\omega_A t}\|x\|, \quad t \in (0, \infty). \tag{1.4}$$

If the operator $-A$ is strongly positive (the terminology is taken from [10]), i.e., if

$$\|(\lambda I - A)^{-1}\| \leqslant \frac{M_3}{1 + |\lambda|}, \quad \operatorname{Re}\lambda \geqslant 0, \quad M_3 > 0,$$

then inequality (1.4) means that the operator B is subordinated to the fractional power $(-A)^\gamma$ (see [10, p. 298]). Also, we note that condition 1.2(i) implies the relation $BT(t)x = T(t)Bx$, $x \in D(B)$.

2. Quasi-Inversion Method

Let $\epsilon > 0$ and $\eta > 0$. Introduce into consideration the function $v(\epsilon t, \eta t) = T(\epsilon t)u_1(\eta t)$, where $u_1(t)$ is a solution of the problem (1.1), (1.2), such that for $\alpha = 1$, it satisfies the problem

$$\frac{dv(\epsilon t, \eta t)}{dt} = \epsilon A v(\epsilon t, \eta t) + \eta B v(\epsilon t, \eta t), \quad t > 0, \tag{2.1}$$

$$\lim_{t \to 0} v(\epsilon t, \eta t) = u_0, \quad u_0 \in D(A). \tag{2.2}$$

Problem (2.1), (2.2) contains the generator of the semigroup ϵA perturbed by the unbounded operator ηB. Let us show that if Condition 1.1 holds, then this problem has a solution. Moreover, its uniqueness can be proved, but since we are finally interested in the solvability of problem (1.1), (1.2), and, as was said, there is no uniqueness for it, we do not dwell on the proof of the uniqueness of solution to problem (2.1), (2.2).

In semigroup theory [6, 7, 10, 11], it is shown that problem (2.1), (2.2) is equivalent to the integral equation

$$v(\epsilon t, \eta t) = T(\epsilon t)u_0 + \eta \int_0^t BT(\epsilon(t - s))v(\epsilon s, \eta s)\, ds, \tag{2.3}$$

which we solve using the method of successive approximations setting

$$v_0(\epsilon t, \eta t) = 0, \quad v_1(\epsilon t, \eta t) = T(\epsilon t)u_0; \tag{2.4}$$

$$v_{n+1}(\epsilon t, \eta t) = T(\epsilon t)u_0 + \eta \int_0^t BT(\epsilon(t - s))v_n(\epsilon s, \eta s)\, ds, \quad n \in N. \tag{2.5}$$

Using inequalities (1.3) and (1.4), let us estimate the norm of the differences:

$$\|v_1(\epsilon t,\eta t)-v_0(\epsilon t,\eta t)\|\leqslant M_1\ e^{\epsilon\omega_A t}\|u_0\|;$$

$$\|v_2(\epsilon t,\eta t)-v_1(\epsilon t,\eta t)\|\leqslant \eta\int\limits_0^t \frac{M_2 e^{\epsilon\omega_A(t-s)}}{\epsilon^\gamma (t-s)^\gamma} M_1 e^{\epsilon\omega_A s}\|u_0\|\,ds=\frac{M_1M_2\eta}{(1-\gamma)\epsilon^\gamma}\ t^{1-\gamma}\ e^{\epsilon\omega_A t}\|u_0\|;$$

$$\|v_3(\epsilon t,\eta t)-v_2(\epsilon t,\eta t)\|\leqslant \eta\int\limits_0^t \frac{M_2 e^{\epsilon\omega_A(t-s)}}{\epsilon^\gamma (t-s)^\gamma}\frac{M_1M_2\eta}{(1-\gamma)\epsilon^\gamma} s^{1-\gamma}e^{\epsilon\omega_A s}\|u_0\|\,ds \tag{2.6}$$

$$=\frac{M_1M_2^2\Gamma(1-\gamma)\eta^2}{(1-\gamma)\epsilon^{2\gamma}}e^{\epsilon\omega_A t}I^{1-\gamma}(t^{1-\gamma})\|u_0\|.$$

Taking into account (2.6), by induction, we obtain

$$\|v_{n+1}(\epsilon t,\eta t)-v_n(\epsilon t,\eta t)\|\leqslant \frac{M_1M_2^n\Gamma^{n-1}(1-\gamma)\eta^n}{(1-\gamma)\epsilon^{n\gamma}}e^{\epsilon\omega_A t}I^{(n-1)(1-\gamma)}(t^{1-\gamma})\|u_0\|$$

$$=\frac{M_1M_2^n\Gamma^n(1-\gamma)\eta^n}{\epsilon^{n\gamma}\Gamma(n(1-\gamma)+1)}t^{n(1-\gamma)}e^{\epsilon\omega_A t}\|u_0\|,\quad n\in\mathbb{N}. \tag{2.7}$$

Hence the series

$$\sum_{n=1}^{\infty}(v_n(\epsilon t,\eta t)-v_{n-1}(\epsilon t,\eta t))$$

uniformly converges in any interval of $[0,l]$. Therefore, $v(\varepsilon t,\eta t)$ uniformly converges to the function $v(\epsilon t,\eta t)$, on the same interval, which is continuous on $[0,l]$ and satisfies the integral equation (2.3). By (2.7), the following estimate holds for it:

$$\|v(\epsilon t,\eta t)\|\leqslant\sum_{n=1}^{\infty}\|v_n(\epsilon t,\eta t)-v_{n-1}(\epsilon t,\eta t)\|$$

$$\leqslant\sum_{k=0}^{\infty}\frac{M_1M_2^{n-1}\ \Gamma^{n-1}(1-\gamma)\ \eta^{n-1}}{\epsilon^{(n-1)\gamma}\ \Gamma((n-1)(1-\gamma)+1)}\ t^{(n-1)(1-\gamma)}e^{\epsilon\omega_A t}\|u_0\|$$

$$=M_1e^{\epsilon\omega_A t}E_{1-\gamma,1}\left(\frac{M_2\ \Gamma(1-\gamma)\ \eta}{\epsilon^\gamma}\ t^{1-\gamma}\right)\|u_0\|, \tag{2.8}$$

where $E_{\mu,\rho}(z)=\sum\limits_{k=0}^{\infty}\dfrac{z^k}{\Gamma(\mu k+\rho)}$ is a Mittag-Leffler-type function.

Taking into account the asymptotic behavior of a Mittag-Leffler -type function for $0<\mu<2$ (see, e.g., [8, p. 43])

$$E_{\mu,\rho}(z)=\frac{1}{\mu}z^{(1-\rho)/\mu}\exp\left(z^{1/\mu}\right)-\sum_{j=1}^{n}\frac{1}{\Gamma(\rho-\mu j)\ z^j}+O\left(\frac{1}{|z|^{n+1}}\right),\quad z\in\mathbb{R},\quad z\to+\infty,$$

from (2.8), we obtain the following estimate for the function $v(\epsilon t,\eta t)$:

$$\|v(\epsilon t,\eta t)\|\leqslant M_0\exp\left(\omega\epsilon^{\gamma/(\gamma-1)}\eta^{1/(1-\gamma)}t+\epsilon\omega_A t\right)\|u_0\|,\quad \omega=(M_2\Gamma(1-\gamma))^{1/(1-\gamma)}. \tag{2.9}$$

For $\eta>t$, $\epsilon=\varepsilon/\eta$, and $\varepsilon>0$, let us introduce into consideration one more function

$$w=T\left(\frac{\varepsilon}{\eta}(\eta-t)\right)v\left(\frac{\varepsilon}{\eta}t,\eta t\right). \tag{2.10}$$

From relations (2.4) and (2.5), it follows that

$$w_1(\varepsilon) = T\left(\frac{\varepsilon}{\eta}(\eta - t)\right)v_1\left(\frac{\varepsilon}{\eta}t, \eta t\right) = T(\varepsilon)u_0;$$

$$w_2(\varepsilon, \eta t) = T\left(\frac{\varepsilon}{\eta}(\eta - t)\right)v_2\left(\frac{\varepsilon}{\eta}t, \eta t\right) =$$

$$= T\left(\frac{\varepsilon}{\eta}(\eta - t)\right)\left(T\left(\frac{\varepsilon}{\eta}t\right)u_0 + \eta\int\limits_0^t BT\left(\frac{\varepsilon}{\eta}(t-s)\right)T\left(\frac{\varepsilon}{\eta}s\right)u_0\,ds\right) = T(\varepsilon)u_0 + \eta tBT(\varepsilon)u_0; \ \dots$$

$$w_{n+1}(\varepsilon, \eta t) = T(\varepsilon)u_0 + \int\limits_0^{\eta t} Bw_n(\varepsilon, \tau)\,d\tau, \quad n \in \mathbb{N}.$$

Therefore, the function w defined by (2.10) continuously depends on ε and ηt, i.e., $w = w(\varepsilon, \eta t)$, and can be defined for $\eta \geqslant 0$. Let us preserve the same notation for the extended function. Since the function $v(\epsilon t, \eta t)$ is a solution of problem (2.1), (2.2), it follows that $w(\varepsilon, \eta t)$ is a solution of the problem

$$\frac{dw(\varepsilon, \eta t)}{dt} = \eta Bw(\varepsilon, \eta t), \quad t > 0, \tag{2.11}$$

$$\lim_{t\to 0} w(\varepsilon, \eta t) = T(\varepsilon)u_0, \quad u_0 \in D(A), \tag{2.12}$$

and for $t = s/\eta$ and $s > 0$, by (2.9), it satisfies the estimate

$$\|w(\varepsilon, s)\| \leqslant M_1 e^{\varepsilon - \varepsilon s/\eta^2}\left\|v\left(\frac{\varepsilon s}{\eta^2}, s\right)\right\| \leqslant M_0M_1\exp\left(\omega\left(\frac{\varepsilon}{\eta^2}\right)^{\gamma/(\gamma-1)} s + \varepsilon\omega_A\right)\|u_0\|. \tag{2.13}$$

As a result of the arguments carried out, we arrive at the following theorem.

Theorem 2.1. *Let $\alpha = 1$, $\varepsilon > 0$, and $u_0 \in D(A)$, and let Conditions* 1.1 *and* 1.2 *hold. Then the function*

$$U_\varepsilon(t) = T\left(\frac{\varepsilon}{b}\left(b - \frac{t}{b}\right)\right)v\left(\frac{\varepsilon}{b^2}t, t\right)$$

is a solution of the problem

$$u'(t) = Bu(t), \quad 0 < t \leqslant b, \tag{2.14}$$

$$\lim_{t\to 0} u(t) = T(\varepsilon)u_0, \tag{2.15}$$

and, moreover,

$$\|U_\varepsilon(t)\| \leqslant M_0M_1\exp\left(\omega\left(\frac{\varepsilon}{b^2}\right)^{\gamma/(\gamma-1)} t + \varepsilon\omega_A\right)\|u_0\|.$$

Remark 2.1. The theorem on the solvability of problem (2.14), (2.15) is also proved in [3] for a certain class of variable operators $B = B(t)$ if, in inequalities (1.3) and (1.4), the constant satisfies $\omega_A = 0$.

Now let us pass to the case $0 < \alpha < 1$. In what follows, we use the function (see [7, p. 357])

$$f_{\tau,\alpha}(t) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int\limits_{\sigma - i\infty}^{\sigma + i\infty} \exp\left(tz - \tau z^\alpha\right)dz, & t \geqslant 0, \\ 0, & t < 0, \end{cases} \tag{2.16}$$

where $\sigma > 0$, $\tau > 0$, $0 < \alpha < 1$, and the branch of the function z^α is chosen such that $\operatorname{Re} z^\alpha > 0$ for $\operatorname{Re} z > 0$. This function is an univalent function on the complex z-plane with cut along the negative part of the real line. The convergence of integral (2.16) is ensured by the factor $\exp(-\tau z^\nu)$.

Let us mention some properties of the function $f_{\tau,\alpha}(t)$, which are also proved in [7, Propositions 1–3, pp. 358–361].

If, in the integral defining the function $f_{\tau,\alpha}(t)$, we pass from the integration over the line $\operatorname{Re} z = \sigma > 0$ to the contour consisting of two rays $z = r\exp(-i\theta)$ and $z = r\exp(i\theta)$, where $0 < r < \infty$, $\pi/2 \leqslant \theta \leqslant \pi$, then for $t > 0$, the following representation is obtained for the function $f_{\tau,\alpha}(t)$:

$$f_{\tau,\alpha}(t) = \frac{1}{\pi}\int\limits_0^\infty \exp\big(tr\cos\theta - \tau r^\alpha \cos\alpha\theta\big)\sin\big(tr\sin\theta - \tau r^\alpha \sin\alpha\theta + \theta\big)\,dr. \tag{2.17}$$

The function $f_{\tau,\alpha}(t)$ is nonnegative, and the following relation holds:

$$\exp\big(-\tau\lambda^\alpha\big) = \int\limits_0^\infty \exp(-\lambda t) f_{\tau,\alpha}(t)\,dt, \quad \tau > 0, \quad \lambda > 0, \quad 0 < \alpha < 1. \tag{2.18}$$

Also, we note that for $t > 0$, the function $f_{\tau,\alpha}(t)$ can be expressed through the Wright-type function (see [13, Chapter 1]):

$$f_{\tau,\alpha}(t) = t^{-1} e^{1,0}_{1,\alpha}\big(-\tau t^{-\alpha}\big), \qquad e^{\mu,\delta}_{\nu,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\nu k + \mu)\,\Gamma(\delta - \beta k)}, \tag{2.19}$$

where $\alpha > \max\{0;\beta\}$, $\mu, z \in C$. For the Wright-type function, the following estimate holds (see [13, Lemma 1.2.7]):

$$e^{1,0}_{1,\alpha}(-x) \leqslant m\big(1 + x^n\big)\exp\big(-\rho x^{1/(1-\alpha)}\big), \tag{2.20}$$
$$n \in \mathbb{N}, \quad n \geqslant \frac{1}{1-\alpha}, \quad \rho = (1-\alpha)\alpha^{\alpha/(1-\alpha)}, \quad x \geqslant 0.$$

Theorem 2.2. *let $0 < \alpha < 1$, $\varepsilon > 0$, $\eta > 0$, $u_o \in D(A)$, and let Conditions 1.1 and 1.2 hold. Then the function*

$$u_{\varepsilon,\eta}(t) = \int\limits_0^\infty f_{\tau,\alpha}(t) w(\varepsilon, \eta\tau)\,d\tau, \tag{2.21}$$

where $f_{\tau,\alpha}(t)$ is defined by relation (2.16) and $w(\varepsilon,\eta\tau)$ is defined by relation (2.10), is a solution of the problem

$$D^\alpha u(t) = \eta B u(t), \quad t > 0, \tag{2.22}$$
$$\lim_{t\to 0} I^{1-\alpha} u(t) = T(\varepsilon) u_0. \tag{2.23}$$

Proof. Note that the convergence of the integral in (2.21) (after the change $\eta\tau = s$) is ensured by estimates (2.13) and (2.20).

Let us obtain the necessary expressions for $I^{1-\alpha}u_{\varepsilon,\eta}(t)$, $D^\alpha u_{\varepsilon,\eta}(t) = \dfrac{d}{dt} I^{1-\alpha} u_{\varepsilon,\eta}(t)$, and $Bu_{\varepsilon,\eta}(t)$. Using (2.16), we have

$$I^{1-\alpha}u_{\varepsilon,\eta}(t) = I^{1-\alpha}\int\limits_0^\infty \left(\frac{1}{2\pi i}\int\limits_{\sigma - i\infty}^{\sigma + i\infty} \exp(tz - \tau z^\alpha)\,dz\right) w(\varepsilon,\eta\tau)\,d\tau$$
$$= I^{1-\alpha}\int\limits_0^\infty \left(\frac{1}{2\pi i}\int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \exp(\xi - \rho\xi^\alpha)\,d\xi\right) t^{\alpha-1}\, w(\varepsilon, \eta\rho t^\alpha)\,d\rho$$
$$= \frac{1}{\Gamma(1-\alpha)}\int\limits_0^\infty f_{\rho,\alpha}(1)\int\limits_0^1 (1-s)^{-\alpha} s^{\alpha-1}\, w(\varepsilon, \eta\rho s^\alpha t^\alpha)\,ds\,d\rho. \tag{2.24}$$

In (2.24), let us pass to the limit as $t \to 0$. Taking into account (2.12), we obtain

$$\lim_{t\to 0} I^{1-\alpha} u_{\varepsilon,\eta}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty f_{\rho,\alpha}(1) \int_0^1 (1-s)^{-\alpha} s^{\alpha-1}\, ds\, d\rho\, T(\varepsilon) u_0$$

$$= \Gamma(\alpha) \int_0^\infty f_{\rho,\alpha}(1)\, d\rho\, T(\varepsilon)u_0 = \Gamma(\alpha) \int_0^\infty \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \exp(\xi - \rho \xi^\alpha)\, d\xi\, d\rho\, T(\varepsilon) u_0$$

$$= \frac{\Gamma(\alpha)}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} e^{\xi} \xi^{-\alpha}\, d\xi\, T(\varepsilon) u_0 = T(\varepsilon) u_0,$$

and the fulfillment of the initial condition (2.23) is proved.

Furthermore, let us calculate $D^\alpha u_{\varepsilon,\eta}(t)$. Analogously to (2.24), we have

$$D^\alpha u_{\varepsilon,\eta}(t) = \frac{d}{dt} I^{1-\alpha} u_{\varepsilon,\eta}(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^\infty \int_0^1 t^{1-\alpha} f_{\tau,\alpha}(ts)(1-s)^{-\alpha}\, w(\varepsilon, \eta\tau)\; ds\, d\tau. \tag{2.25}$$

Finally, taking into account (2.11), we find that

$$B u_{\varepsilon,\eta}(t) = \int_0^\infty f_{\tau,\alpha}(t) B w(\varepsilon, \eta\tau)\, d\tau = \frac{1}{\eta} \int_0^\infty f_{\tau,\alpha}(t) \frac{d}{d\tau} w(\varepsilon, \eta\tau)\; d\tau =$$

$$= \left(\frac{1}{\eta}\; f_{\tau,\alpha}(t) w(\varepsilon, \eta\tau) \right)\Big|_{\tau=0}^{\tau=\infty} - \frac{1}{\eta} \int_0^\infty \left(\frac{d}{d\tau}\; f_{\tau,\alpha}(t) \right) w(\varepsilon, \eta\tau)\, d\tau. \tag{2.26}$$

Using representation (2.17), let us calculate the summands standing outside the integral in (2.26). Setting $\theta = \pi$, we obtain

$$\lim_{\tau \to 0+} f_{\tau,\alpha}(t) = \lim_{\tau\to 0+} \frac{1}{\pi} \int_0^\infty \exp(-\xi t - \tau \xi^\alpha \cos \pi\alpha) \sin(\tau \xi^\alpha \sin \pi\alpha)\, d\xi = 0. \tag{2.27}$$

Setting $\theta = \pi/2$, we have

$$\lim_{\tau\to+\infty} f_{\tau,\alpha}(t) = \lim_{\tau\to+\infty} \frac{1}{\pi} \int_0^\infty \exp\left(-\tau\xi^\alpha \cos\frac{\pi\alpha}{2} \right) \cos\left(\xi t - \tau \xi^\alpha \sin \frac{\pi\alpha}{2} \right) d\xi = 0. \tag{2.28}$$

Therefore, from (2.26)–(2.28), we deduce that

$$\eta B u_{\varepsilon,\eta}(t) = -\int_0^\infty \left(\frac{d}{d\tau}\; f_{\tau,\alpha}(t) \right) w(\varepsilon, \eta\tau)\, d\tau, \tag{2.29}$$

and, as follows from (2.25) and (2.29), to prove the theorem, it remains to verify the fulfillment of the relation

$$\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^1 (1-s)^{-\alpha} t^{1-\alpha} f_{\tau,\alpha}(ts)\, ds + \frac{d}{d\tau}\; f_{\tau,\alpha}(t) = 0. \tag{2.30}$$

Expressing the function $f_{\tau,\alpha}(t)$ through the Wright-type function by formula (2.18) and calculating the obtained integrals, we have

$$\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^1 (1-s)^{-\alpha}t^{1-\alpha}\, f_{\tau,\alpha}(ts)\,ds + \frac{d}{d\tau}f_{\tau,\alpha}(t)$$

$$= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^1 (1-s)^{-\alpha}t^{1-\alpha}(ts)^{-1}e^{1,0}_{1,\alpha}\Big(-\tau(ts)^{-\alpha}\Big)\,ds + \frac{d}{d\tau}\big(t^{-1}e^{1,0}_{1,\alpha}(-\tau t^{-\alpha})\big)$$

$$= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^1 (1-s)^{-\alpha}t^{1-\alpha}(ts)^{-1}\sum_{k=0}^{\infty}\frac{(-\tau(ts)^{-\alpha})^k}{\Gamma(k+1)\,\Gamma(-\alpha k)}\,ds + \frac{d}{d\tau}\sum_{k=1}^{\infty}\frac{t^{-1}(-\tau t^{-\alpha})^k}{\Gamma(k+1)\,\Gamma(-\alpha k)}$$

$$= \frac{d}{dt}\sum_{k=0}^{\infty}\frac{t^{-\alpha}(-\tau t^{-\alpha})^k}{\Gamma(k+1)\,\Gamma(1-\alpha-\alpha k)} - \sum_{k=1}^{\infty}\frac{kt^{-\alpha k-1}(-\tau)^{k-1}}{\Gamma(k+1)\,\Gamma(-\alpha k)}$$

$$= -\sum_{k=0}^{\infty}\frac{\alpha(k+1)(-\tau)^k\,t^{-\alpha(k+1)-1}}{\Gamma(k+1)\,\Gamma(1-\alpha-\alpha k)} - \sum_{m=0}^{\infty}\frac{(m+1)(-\tau)^m\,t^{-\alpha(m+1)-1}}{\Gamma(m+2)\,\Gamma(-\alpha-\alpha m)}$$

$$= \sum_{k=0}^{\infty}\frac{(-\tau)^k\,t^{-\alpha(k+1)-1}}{\Gamma(k+1)\,\Gamma(-\alpha-\alpha k)} - \sum_{m=0}^{\infty}\frac{(-\tau)^m\,t^{-\alpha(m+1)-1}}{\Gamma(m+1)\,\Gamma(-\alpha-\alpha m)} = 0.$$

Hence the function $u_{\varepsilon,\eta}(t)$ satisfies Eq. (2.22).

Corollary 2.1. *The function* $u_\varepsilon(t) = \lim\limits_{\eta\to 1} u_{\varepsilon,\eta}(t)$ *satisfies problem* (2.22), (2.23) *written for the parameter value* $\eta = 1$*.*

Example 2.1. As was already noted after Condition 1.2, the operator B in problem (2.22), (2.23) can be a fractional power of the strongly positive operator A, i.e., $B = (-A)^\gamma$, $0 < \gamma < 1$.

Let us present one more example taken from [2].

Example 2.2. Let $E = L_2(\mathbb{R}^n)$. On the set $D(A) = W_2^{2m}(\mathbb{R}^n)$, let us define the operator A as follows:

$$Au(t,x) = \sum_{|p|=2m} a_p(x)\frac{\partial^{p_1+\dots+p_n}u(t,x)}{\partial x_1^{p_1}\cdots\partial x_n^{p_n}},$$

where, for any x, $\xi \in \mathbb{R}^n$,

$$\sum_{|p|=2m} a_p(x)\xi^p \geqslant (-1)^{m+1}M_0|\xi|^{2m},$$

and the coefficients $a_p(x)$ for $|p| = 2m$ satisfy the uniform Hölder condition in $\mathbb{R}^n$.

Define the operator B on $D(B) = W_2^{2m-1}(\mathbb{R}^n) \supset D(A)$ by the relation

$$Bu(t,x) = \sum_{|p|\leqslant 2m-1} a_p(x)\frac{\partial^{p_1+\dots+p_n}u(t,x)}{\partial x_1^{p_1}\cdots\partial x_n^{p_n}} + \int_\Omega \sum_{|p|\leqslant 2m-1} b_p(x,\xi)\frac{\partial^{p_1+\dots+p_n}u(t,\xi)}{\partial \xi_1^{p_1}\cdots\partial \xi_n^{p_n}}\,d\xi,$$

where $\Omega \subset \mathbb{R}^n$; the coefficients $a_p(x)$ for $|p| \leqslant 2m-1$ are continuous and bounded in $x \in \mathbb{R}^n$; the coefficients $b_p(x,\xi)$ are continuous and

$$\int_{\mathbb{R}^n}\int_\Omega |b_p(x,\xi)|^2\,d\xi dx < +\infty.$$

As was noted in [2], the operators A and B satisfy Conditions 1.1 and 1.2 for $\omega_A = 0$ and for a certain $\gamma \in (0,1)$. If $u_0(x) \in W_2^{2m}(\mathbb{R}^n)$ and $\alpha < 1$, then, by Theorem 2.2, problem (2.22), (2.23) (Cauchy-type problem for the integro-differential equation) has a solution.

In conclusion, let us present the corresponding results for the Cauchy problem with Caputo fractional derivative $\partial^\alpha u(t) = D^\alpha\,(u(t) - u(0))$.

Theorem 2.3. *Let $u(t)$ be a solution of problem* (2.22), (2.23). *Then the function $U(t) = I^{1-\alpha}u(t)$ is a solution of the problem*

$$\partial^\alpha U(t) = \eta B U(t), \quad t > 0, \tag{2.31}$$

$$\lim_{t\to 0} U(t) = T(\varepsilon)u_0. \tag{2.32}$$

Proof. The fulfillment of the initial condition (2.32) is obvious. Let us verify that the function $U(t)$ satisfies Eq. (2.31). We have

$$\begin{aligned}
\partial^\alpha U(t) &= D^\alpha\big(U(t) - U(0)\big) = D^\alpha U(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}T(\varepsilon)u_0 \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}U(s)\,ds - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}T(\varepsilon)u_0 \\
&= \frac{-1}{(1-\alpha)\,\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{d}{ds}(t-s)^{1-\alpha}U(s)\,ds - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}T(\varepsilon)u_0 \\
&= \frac{-1}{(1-\alpha)\,\Gamma(1-\alpha)}\frac{d}{dt}\left(-t^{1-\alpha}T(\varepsilon)u_0 - \int_0^t (t-s)^{1-\alpha}U'(s)\,ds\right) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}T(\varepsilon)u_0 \\
&= \frac{1}{(1-\alpha)\,\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{1-\alpha}U'(s)\,ds = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}U'(s)\,ds \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{d}{ds}I^{1-\alpha}u(s)\,ds = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\eta B u(s)\,ds \\
&= \eta B I^{1-\alpha}u(t) = \eta B U(t).
\end{aligned}$$

Corollary 2.2. *Let the conditions of Theorem* 2.2 *hold. Then the function*

$$U_{\varepsilon,\eta}(t) = \int_0^\infty g_{\tau,\alpha}(t)w(\varepsilon,\eta\tau)\,d\tau,$$

where $g_{\tau,\alpha}(t) = I^{1-\alpha}f_{\tau,\alpha}(t) = t^{-\alpha}e_{1,\alpha}^{1,1-\alpha}(-\tau t^{-\alpha})$, is a solution of problem (2.31), (2.32).

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