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SOLVABILITY OF A FREE-BOUNDARY PROBLEM DESCRIBING THE TRAFFIC FLOWS

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Abstract. We study a mathematical model of the vehicle traffic on straight freeways, which describes the traffic flow by means of equations of one-dimensional motion of the isobaric viscous gas. The corresponding free boundary problem is studied by means of introduction of Lagrangian coordinates, which render the free boundary stationary. It is proved that the equivalent problem posed in a time-independent domain admits unique local and global in time classical solutions. The proof of the local in time existence is performed with standard methods, to prove the global in time existence the system is reduced to a system of two second-order quasilinear parabolic equations.

1. Introduction

This article is devoted to study one of the mathematical models of the vehicle traffic on straight freeways. This is a phenomenological macroscopic model which describes the traffic flow by means of equations of motion of a viscous gas. The first model of this type was proposed in [9, 12] where the flow of vehicles was considered as the one-dimensional flow of a compressible fluid. This model is often called LWR model. The underlying assumptions of this approach are

(1) a bijective relation between the velocity $v(x,t)$ and density $\rho(x,t)$ of the fluid expressed by the condition
$$v(x,t) = V(\rho(x,t)), \tag{1.1}$$
(2) the mass conservation law (the number of vehicles does not change with time).

It is assumed that the function V satisfies the condition $V'(\rho) < 0$.

Let us denote by $Q(\rho) = \rho V(\rho)$ the intensity of the flow of vehicles (the number of vehicles passing through a given cross-section per unit time) and claim that $Q''(\rho) < 0$ for the single-lane traffic. The assumption of mass conservation is expressed by the equality
$$\int_a^b \rho(x,t+\Delta)dx - \int_a^b \rho(x,x)dx = -\int_t^{t+\Delta} Q(\rho(b,\tau))d\tau + \int_t^{t+\Delta} Q(\rho(a,\tau))d\tau.$$

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It follows that for every rectangular contour Γ in the half-plane $t \geq 0, x \in \mathbb{R}$ with the sides parallel to the coordinate axes one has

$$\oint_{\Gamma} \rho(x,t) - Q(\rho(x,t))dt = 0. \tag{1.2}$$

At every point where $\rho(t,x)$ is smooth

$$\frac{\partial \rho}{\partial t} + \frac{\partial (v\rho)}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial (V(\rho)\rho)}{\partial x};$$

that is,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (Q(\rho))}{\partial x} = 0. \tag{1.3}$$

Equation (1.3) is endowed with the initial conditions of Riemann's type:

$$\rho(0,x) = \begin{cases} \rho_-, & x < x_-, \\ \rho_0(x), & x_- \leq x \leq x_+, \\ \rho_+, & x \geq x_+, \end{cases} \tag{1.4}$$

where $\rho^{\pm}$ is a constant.

The Cauchy problem (1.3), (1.4) arises, for example, in the mathematical description of traffic congestion. A number of model problems for the conservation law (1.3), such as the problem of traffic lights or evolution of local congestions, is considered in [8].

It turns out that equation (1.3) always has a solution that satisfies equation (1.2) and the initial condition (1.4) in a suitable weak sense, but this solution need not be unique. In 1963, Tanaka proposed another definition of $V(\rho)$ for the single-lane traffic (see [4]). Let us assume that the velocity of vehicles can not exceed a threshold value v_{max} and represent the density by the formula

$$\rho(v) = \frac{1}{d(v)},$$

where $d(v) = L + c_1 v + c_2 v^2$ is the average (safe) distance between the vehicles at a predetermined velocity of the flow v, L is the average length of the vehicle, c_1 is the time that expresses the driver reaction, c_2 is the factor of proportionality for the stopping distance. From the formulas for $d(v)$ and $\rho(v)$ one can derive the dependence (1.1) for $V(\rho)$, which satisfies the condition $V'(\rho) < 0$. The model of Tanaka is a LWR model with the state equation (1.1) of a special form, this model plays a very important role in the contemporary studies of the traffic flows [4].

It was mentioned yet in 1955 but rigorously formulated only in 1974 by J. Whitham, [13], that the farsightedness of the drivers can be taken into account in the following way:

$$v(t,x) = V(\rho(x,t)) - \frac{D(\rho(x,t))}{\rho(x,t)} \frac{\partial \rho(x,t)}{\partial x} \quad \text{with } D(\rho) > 0.$$

Substituting this expression into the conservation law for the number of vehicles,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0, \tag{1.5}$$

we arrive at the Burgers equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = \frac{\partial}{\partial x}\big(D(\rho)\frac{\partial \rho}{\partial x}\big), \tag{1.6}$$

which expresses the conservation law. The novelty of equation (1.6) consists in the fact that the driver reduces the velocity with the increment of the traffic density in front of his vehicle and increases the velocity otherwise. The hydrodynamic model (1.1), (1.4), (1.6) is called the Whitham model.

Another model was proposed by Payne in 1971 [11]. The model relies on the conservation law (1.5) with an independent of ρ velocity v, which means that the desired speed of the vehicle is not attained immediately. The following relation between the desired and the real velocity is accepted:

$$\frac{d}{dt}v = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\tau}\Big(v - \Big(V(\rho) - \frac{D(\rho)}{\rho}\frac{\partial \rho}{\partial x}\Big)\Big),$$

where v is a real speed, while

$$V(\rho) - \frac{D(\rho)}{\rho}\frac{\partial \rho}{\partial x}$$

is the desired speed. The parameter τ is of order 1 sec., it expresses the rate of convergence. The resulting system of equations reads

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ v\end{pmatrix} + \begin{pmatrix} v & \rho \\ D/(\tau\rho) & v\end{pmatrix}\cdot\frac{\partial}{\partial x}\begin{pmatrix}\rho\\ v\end{pmatrix} = \frac{1}{\tau}\begin{pmatrix}0\\ V-v\end{pmatrix}. \tag{1.7}$$

The system is strictly hyperbolic because the matrix of $\frac{\partial}{\partial x}$ has different real eigenvalues.

In 1995, Daganzo [1] pointed out several shortcomings of Payne's model, as well as of some models proposed later. It was shown, in particular, that the strong spatial inhomogeneity of the initial density may lead to negative velocities. These drawbacks were corrected in the recent modifications of the model.

In conclusion, let us mention the Helbing-Euler-Navier-Stokes third-order model proposed in 1995, [5, 6]. In this model, the Payne system is complemented by the energy conservation law, which is represented by an equation for a new unknown θ that describes the dispersion of velocity about some mean value. The second equation of system (1.7), understood as an equation for the mean velocity, includes an additional term which depends on θ.

2. Formulation of the problem

2.1. **Euler variables.** Our model of the traffic of vehicles relies on the hypotheses of continuum mechanics, that is, it is assumed that the traffic flow is continuous and possesses the principal characteristics of continuous media such as density, pressure and velocity. It is to be noted here that if the initial velocity of a gas equals zero, the motion may be caused by the inhomogenuity of density. Unlike gas dynamics, the initially motionless vehicle can start moving only if an exterior force is applied. Since in the system of equations of a viscous gas the gradient of the pressure is the only component that makes the gas moving, to avoid the vehicle motion in the absence of exterior forces one has to assume that the pressure is constant, i.e., the gas is isobaric. For this reason we regard the traffic flow as the one-dimensional flow of an isobaric viscous gas. The flow is described by the system of two differential equations for the velocity $u(x,t)$ and density $\rho(x,t)$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2)=\frac{\partial}{\partial x}\big(\mu\rho\frac{\partial u}{\partial x}\big)+\rho F \tag{2.2}$$

on the interval $-L<x<L$ for $t>0$. Here $\mu=const>0$ is the viscosity of the flow, $F(u)$ denotes a given external force (acceleration), which is assumed to satisfy the following conditions:

$$\begin{gathered} F\in C^2(-\infty,\infty),\quad F(u)\geqslant 0,\\ F(u)=F_0=\text{positive const. for }-\infty<u\leqslant u_*-\delta,\\ F'(u)\leqslant 0\text{ for }u_*-\delta\leqslant u\leqslant u_*,\text{ and }F(u)=0\text{ for }u>u_*=\text{positive const..}\end{gathered}$$

System (2.1)–(2.2) is complemented by the initial and boundary conditions

$$u(\pm L,t)=0, \tag{2.3}$$

$$u(x,0)=u_0(x),\quad \rho(x,0)=\rho_0(x), \tag{2.4}$$

where

$$0\leqslant u_0(x)\leqslant u_*,\quad 0\leqslant\rho_0(x)\leqslant\rho^+=\text{const.} \tag{2.5}$$

The study will be confined to the special situation when the initial density ρ_0 has the form

$$\begin{gathered} \rho_0(x)\equiv 0\quad for-L\leqslant x<0,\\ 0<\rho^-\leqslant\rho_0(x)\leqslant\rho^+,\quad \rho^-=\text{const. for }0\leqslant x\leqslant 1,\\ \rho_0(x)\equiv 0\quad for 1<x\leqslant L. \end{gathered} \tag{2.6}$$

The aim of the work is to find a weak solution of system (2.1)–(2.6) such that

$$\begin{gathered} \rho(x,t)\equiv 0\quad\text{for }-L\leqslant x<X_0(t),\\ 0<\rho(x,t)<\infty\quad\text{for }X_0(t)\leqslant x\leqslant X_1(t),\\ \rho(x,t)\equiv 0\quad\text{for }X_1(t)<x\leqslant L\text{ for all }0<t<T, \end{gathered} \tag{2.7}$$

where $x=X_0(t)$, $x=X_1(t)$ are a priori unknown boundaries of $\operatorname{supp}\rho(x,t)$.

Conditions (2.7) imply discontinuity of the density $\rho(x,t)$ across the boundaries $x=X_i(t)$, for this reason the motion of the medium should be understood in a generalized sense, see, e.g., [10]. Since equations (2.1), (2.2) have the form of conservation laws, the Rankine-Hugoniot jump conditions on the surfaces of discontinuity $x=X_0(t)$, $x=X_1(t)$ read

$$[\rho(u-\dot X_i)]=0,\quad i=0,1, \tag{2.8}$$

$$[\rho u(u-\dot X_i)-\mu\rho\frac{\partial u}{\partial x}]=0,\quad i=0,1. \tag{2.9}$$

Applying (2.7) we find that

$$u(X_i(t),t)=\dot X_i(t),\quad i=0,1, \tag{2.10}$$

$$\frac{\partial u}{\partial x}(X_i(t),t)=0,\quad i=0,1. \tag{2.11}$$

The problem consists in finding a solution of system (2.1)-(2.2) satisfying the initial conditions (2.4), (2.6) and the boundary conditions (2.10)-(2.11). This is a free boundary problem that describes the motion of a group of vehicles initially located within the interval $(0,1)$.

2.2. **Lagrangian variables.** Let the pair ρ, u be a classical solution of problem (2.1)-(2.11). Let us introduce the new independent space variable

$$y = \int_{X_0(t)}^{x} \rho(s,t)ds. \tag{2.12}$$

On the plane of variables (y,t) the unknown domain $\Omega(t) = \{x : X_0(t) < x < X_1(t)\}$ for $t > 0$ transforms into the time-independent domain $Q = \{y : 0 < y < y_*\}$ with

$$y_* = \int_{X_0(t)}^{X_1(t)} \rho(x,t)dx = \int_0^1 \rho_0(x)dx.$$

Indeed: by (2.1),

$$\begin{aligned}
\frac{dy_*}{dt} &= \int_{X_0(t)}^{X_1(t)} \frac{\partial \rho}{\partial t} dx + \dot X_1(t)\rho\big(X_1(t),t\big) - \dot X_0(t)\rho\big(X_0(t),t\big) \\
&= -\int_{X_0(t)}^{X_1(t)} \frac{\partial}{\partial x}(\rho\, u)dx + \dot X_1(t)\rho\big(X_1(t),t\big) - \dot X_0(t)\rho\big(X_0(t),t\big) \\
&= (\rho\, u)\big(X_1(t),t\big) - (\rho\, u)\big(X_0(t),t\big) + \dot X_1(t)\rho\big(X_1(t),t\big) - \dot X_0(t)\rho\big(X_0(t),t\big) = 0.
\end{aligned}$$

The function x is considered as a function of the variables (y,t). Differentiating the second equality of (2.12) with respect to y we find that $x_y(y,t)\rho(x(y,t)) = 1$. This equality defines the first sought function $J = x_y(y,t)$. The second unknown is the velocity of the flow:

$$v(y,t) = u(x,t), \quad J(y,t) = \frac{1}{\rho(x(y,t),t)}.$$

It is easy to calculate that

$$\partial_x = y_x \partial_y = \rho\, \partial_y.$$

Let us take two arbitrary points $y_1, y_2 \in (0,1)$. By (2.12)

$$y_2 - y_1 = \int_{x(y_1,t)}^{x(y_2,t)} \rho(s,t)ds = \int_{y_1}^{y_2} \rho(x(y,t),t)J(y,t)\,dy = \text{const}.$$

Differentiating this equality in t and using (2.1) we find that

$$\begin{aligned}
0 &= \frac{d}{dt}\Big(\int_{y_1}^{y_2} \rho(x(y,t),t)J\Big)\,dy \\
&= \int_{y_1}^{y_2} \big((\rho_t + \rho_x x_t + \rho u_x)J + (\rho J_t - \rho u_x J)\big)\,dy \\
&= \int_{y_1}^{y_2} \big((\rho_t + (\rho u)_x)J + (\rho J_t - \rho u_x J)\big)\,dy \\
&= \int_{y_1}^{y_2} \rho(J_t - v_y)\,dy.
\end{aligned}$$

Since $y_1, y_2 \in (0,1)$ are arbitrary, it is necessary that $J_t = v_y$. Now we may compose the system of equations for defining J and v as functions of Lagrangian coordinates (y,t):

$$\frac{\partial J}{\partial t} = \frac{\partial v}{\partial y}, \tag{2.13}$$

$$\frac{\partial v}{\partial t} = F + \frac{\partial}{\partial y}\Big(\frac{\mu}{J^2}\frac{\partial v}{\partial y}\Big) \quad \text{in } Q_T = Q \times (0,T). \tag{2.14}$$

The initial and boundary conditions transform into

$$\frac{\partial v}{\partial y}(0,t) = \frac{\partial v}{\partial y}(y_*,t) = 0, \tag{2.15}$$

$$v(y,0) = v_0(y) \equiv u_0(x), \tag{2.16}$$

$$J(y,0) = J_0(y) \equiv \frac{1}{\rho_0(x)}. \tag{2.17}$$

It is clear that every classical solution of problem (2.13)-(2.17) generates a classical solution to the problem (2.1)-(2.11) and vice versa, which means the equivalence of the problems in Euler and Lagrange formulations.

3. Main results

Let us denote

$$\Omega_T = \cup_{0<t<T}\Omega(t).$$

Throughout the text we use the traditional notation from [8] for the function spaces and the norms.

Theorem 3.1. *Let $u_0, \rho_0 \in H^{2+\alpha}[0,1]$, $0<\alpha<1$. Assume that u_0, ρ_0 satisfy conditions (2.5) and (2.6). Then there exists a maximal time interval $[0,t_*)$ such that problem (2.1)- (2.11) has a unique solution X_0, $X_1 \in H^{1+\frac{\alpha}{2}}[0,t_*]$, $u, \rho \in H^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}_{t_*})$ which possesses the properties*

$$0 \leqslant u(x,t) \leqslant u_*, \quad 0 < \rho(x,t) < \infty \quad \text{for } x \in \Omega(t),\ 0 \leqslant t < t_*.$$

Theorem 3.2. *Under the conditions of Theorem 3.1, for every $T>0$ there exists a unique solution $X_0, X_1 \in H^{1+\frac{\alpha}{2}}[0,T]$ and $u,\rho \in H^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}_T)$ of problem (2.1)-(2.11) such that*

$$0 \leqslant u(x,t) \leqslant u_*, \quad 0 < \rho_*^{-1} < \rho(x,t) < \rho_* \quad \text{for } x \in \Omega(t),\ 0 \leqslant t < T.$$

4. Proof of Theorem 3.1

To prove Theorem 3.1 we rewrite equations (2.13), (2.14) as a system of two parabolic equations and apply the fixed point theorem, [7]. We establish the existence of a classical solution on some small interval $(0,t_1)$, where

$$0 \leqslant u(x,t) \leqslant u_*, \quad \frac{1}{2}\rho^- \leqslant \rho(x,t) \leqslant 2\rho^+.$$

Starting at the initial moment t_1 we prove the existence of a classical solution on some interval (t_1,t_2) wherein

$$0 \leqslant u(x,t) \leqslant u_*,\ \frac{1}{4}\rho^- \leqslant \rho(x,t) \leqslant 4\rho^+.$$

Continuing this process we obtain a sequence $0 < t_1 < t_2 < \ldots < t_n < \ldots$, such that

$$0 \leqslant u(x,t) \leqslant u_*, \quad \frac{1}{2^n}\rho^- \leqslant \rho(x,t) \leqslant 2^n\rho^+$$

for $x \in \Omega(t), t_{n-1} \leqslant t \leqslant t_n$. Without loss of generality we may assume that $t_1 < 1$ and $t_n - t_{n-1} < 1$. There are two possibilities:

(1) $\lim_{n\to\infty} t_n = \infty$,

(2) $\lim_{n\to\infty} t_n = t_\infty < \infty$.

In the first case t_* might be any positive number, and in the second case $t_* = t_\infty$.

4.1. **Auxiliary equations.** We begin by deriving some auxiliary equations for the solutions of problem (2.13)-(2.17). To derive the first auxiliary equation we substitute $\frac{\partial v}{\partial y}$ into equation (2.14) for v and make use of equation (2.13):

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial y}\left(\frac{\mu}{J^2}\frac{\partial J}{\partial t}\right) = F,$$

or

$$\frac{\partial v}{\partial t} + \frac{\partial^2}{\partial y \partial t}\left(\frac{\mu}{J}\right) = F.$$

Integration in time gives the equality

$$v + \mu\frac{\partial}{\partial y}\big(\frac{1}{J}\big) = \Phi \equiv \int_0^t F\big(v(y,\tau)\big)d\tau + v_0 + \mu\frac{\partial}{\partial y}\big(\frac{1}{J_0}\big). \tag{4.1}$$

To get the second auxiliary equation we differentiate (4.1) with respect to y and once again apply (2.13):

$$\frac{\partial J}{\partial t} + \mu\frac{\partial^2}{\partial y^2}\big(\frac{1}{J}\big) = \frac{\partial \Phi}{\partial y}. \tag{4.2}$$

The boundary conditions (2.15) for v and equation (2.13) yield the boundary conditions for J:

$$\frac{\partial J}{\partial t}(0,t) = \frac{\partial J}{\partial t}(y_*,t) = 0,$$

or

$$J(0,t) = J_0(0), \quad J(y_*,t) = J_0(y_*). \tag{4.3}$$

4.2. **Reduction to a fixed point theorem.** Now we rewrite equations (2.14) and (4.2) in the form

$$\frac{\partial v}{\partial t} - \mu\rho^{2}\frac{\partial^2 v}{\partial y^2} = F_1(\rho, v),$$
$$\frac{\partial \rho}{\partial t} - \mu\rho^{2}\frac{\partial^2 \rho}{\partial y^2} = F_2(\rho, v),$$

where $\rho = 1/J$,

$$F_1(\rho,v) = F + 2\mu\,\rho\,\frac{\partial \rho}{\partial y}\,\frac{\partial v}{\partial y}, \quad F_2(\rho,v) = -\rho^2\,\frac{\partial \Phi}{\partial y}.$$

Let $Q = (0,y_*)$, $Q_{t_1} = Q\times(0,t_1)$, $\rho_0(y) = \frac{1}{J_0(y)}$, $a = \max\{|v_0|_Q^{(1+\alpha)},\ |\rho_0|_Q^{(1+\alpha)}\}$ and

$$\begin{aligned}\mathfrak{M} = \Big\{(\widetilde{v},\,\widetilde{\rho}) : \frac{\partial \widetilde{v}}{\partial y}, \frac{\partial \widetilde{\rho}}{\partial y} \in H^{\alpha,\frac{\alpha}{2}}(\overline{Q}_{t_1}), \quad 0 \leqslant \widetilde{v}(y,t) \leqslant u_*,\\ \frac{1}{2}\rho^- \leqslant \widetilde{\rho}(y,t) \leqslant 2\rho^+, \max\{|\frac{\partial \widetilde{v}}{\partial y}|^{(\alpha)}_{Q_{t_1}}, |\frac{\partial \widetilde{\rho}}{\partial y}|^{(\alpha)}_{Q_{t_1}}\} \leqslant\ 2a\Big\}.\end{aligned}$$

For every $(\widetilde{v},\widetilde{\rho}) \in \mathfrak{M}$, the linear problem constituted by the equations

$$\frac{\partial v}{\partial t} - \mu\widetilde{\rho}^{\,2}\frac{\partial^2 v}{\partial y^2} = F_1(\widetilde{\rho},\,\widetilde{v}), \tag{4.4}$$

$$\frac{\partial \rho}{\partial t} - \mu\widetilde{\rho}^{\,2}\frac{\partial^2 \rho}{\partial y^2} = F_2(\widetilde{\rho},\,\widetilde{v}), \tag{4.5}$$

the initial and boundary conditions (2.15), (2.16), and the conditions

$$\rho(0,t)=\rho_0(0),\quad \rho(y_*,t)=\rho_0(y_*),\quad \rho(y,0)=\rho_0(y) \tag{4.6}$$

defines the operator

$$(\rho,\, v)=\mathbf{\Psi}(\widetilde{\rho},\, \widetilde{v})=\big(\Psi_1(\widetilde{\rho},\, \widetilde{v}),\, \Psi_2(\widetilde{\rho},\, \widetilde{v})\big).$$

Every fixed point of the operator $\mathbf{\Psi}$ in $\mathfrak{M}$ is the sought solution of the problem (2.13)-(2.17) on an interval $(0,t_1)$.

4.3. **Correctness of the linear problem.** A straightforward calculations show that for every $(\widetilde{\rho},\, \widetilde{v})\in\mathfrak{M}$ there is the inclusion $F_i(\widetilde{\rho},\widetilde{v})\in H^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}_{t_1})$, $i=1,2$,

$$|F_i(\widetilde{\rho},\, \widetilde{v})|^{(\alpha)}_{Q_{t_1}}\leqslant C\max\big\{|\frac{\partial\widetilde{v}}{\partial y}|^{(\alpha)}_{Q_{t_1}},|\frac{\partial\widetilde{\rho}}{\partial y}|^{(\alpha)}_{Q_{t_1}}\big\},$$

and for every $(\widetilde{\rho}_1,\widetilde{v}_1),(\widetilde{\rho}_2,\widetilde{v}_2)\in\mathfrak{M}$

$$|F_i(\widetilde{\rho}_1,\, \widetilde{v}_1)-Fi(\widetilde{\rho}_2,\, \widetilde{v}_2)|^{(\alpha)}_{Q_{t_1}}\leqslant C\max\big\{|\frac{\partial\widetilde{v}_1}{\partial y}-\frac{\partial\widetilde{v}_2}{\partial y}|^{(\alpha)}_{Q_{t_1}},\,|\frac{\partial\widetilde{\rho}_1}{\partial y}-\frac{\partial\widetilde{\rho}_2}{\partial y}|^{(\alpha)}_{Q_{t_1}}\big\},\quad i=1,2,$$

where the constant C is independent of $(\widetilde{\rho}_1,\widetilde{v}_1),(\widetilde{\rho}_2,\, \widetilde{v}_2)\in\mathfrak{M}$. It follows from the well-known results in [8] that the linear problem (2.15), (2.16), (4.4)-(4.6) has a unique solution $(\rho,v)=\mathbf{\Psi}(\widetilde{\rho},\widetilde{v})\in H^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}_{t_1})$ and

$$\max\big\{|v|^{(2+\alpha)}_{Q_{t_1}},|\rho|^{(2+\alpha)}_{Q_{t_1}}\big\}\leqslant C\max\big\{|\frac{\partial\widetilde{v}}{\partial y}|^{(\alpha)}_{Q_{t_1}},|\frac{\partial\widetilde{\rho}}{\partial y}|^{(\alpha)}_{Q_{t_1}}\big\}. \tag{4.7}$$

Moreover, for every $(\widetilde{\rho}_1,\, \widetilde{v}_1)$, $(\widetilde{\rho}_2,\, \widetilde{v}_2)\in\mathfrak{M}$,

$$\begin{aligned}&\max\big\{|v_1-v_2|^{(2+\alpha)}_{Q_{t_1}},|\rho_1-\rho_2|^{(2+\alpha)}_{Q_{t_1}}\big\}\\&\leqslant C\max\big\{|\frac{\partial\widetilde{v}_1}{\partial y}-\frac{\partial\widetilde{v}_2}{\partial y}|^{(\alpha)}_{Q_{t_1}},\,|\frac{\partial\widetilde{\rho}_1}{\partial y}-\frac{\partial\widetilde{\rho}_2}{\partial y}|^{(\alpha)}_{Q_{t_1}}\big\}\end{aligned} \tag{4.8}$$

with C depending only on the constant a.

4.4. **Existence of the fixed point.** The existence of at least one fixed point of the operator $\mathbf{\Psi}$ follows from we the Schauder fixed point theorem [7]. To apply this theorem one has to prove that

(a) the operator $\mathbf{\Psi}$ is completely continuous on $\mathfrak{M}$,
(b) $\mathbf{\Psi}$ transforms the set $\mathfrak{M}$ into itself.

Assertion (a) follows from estimates (4.7), (4.8). The former estimate and the imbedding $H^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}_{t_1})\subset H^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega}_{t_1})$ (see [8]) yield compactness of the operator $\mathbf{\Psi}$. The latter estimate implies continuity of $\mathbf{\Psi}$.

Assertion (b) follows from the maximum principle (5.3) proved in Subsection 5.1 below and the properties of the norms in $H^{k+\alpha,\frac{k+\alpha}{2}}(\overline{\Omega}_{t_1})$, $k=0,1,2,\dots$, [8]. For example,

$$\begin{gathered}|\rho(y,t)-\rho_0(y)|\leqslant|\rho|^{(2+\alpha)}_{Q_{t_1}}t_1,\\ \big|\frac{\partial\rho}{\partial y}(y,t)-\frac{\partial\rho_0}{\partial y}(y)\big|\leqslant|\rho|^{(2+\alpha)}_{Q_{t_1}}t_1^{\frac{1+\alpha}{2}},\\ \Big|\frac{\frac{\partial\rho}{\partial y}(y,t+\tau)-\frac{\partial\rho}{\partial y}(y,t)}{\tau^{\frac{\alpha}{2}}}\Big|\leqslant|\rho|^{(2+\alpha)}_{Q_{t_1}}t_1^{1/2},\end{gathered}$$

$$
\Big|\frac{\frac{\partial\rho}{\partial y}(y+h,t)-\frac{\partial\rho}{\partial y}(y,t)}{h^\alpha}\Big| \leqslant \max_{Q_{t_1}}\Big|\frac{\partial^2\rho}{\partial y^2}(y,t)\Big|,
$$

$$
\Big|\frac{\partial^2\rho}{\partial y^2}(y,t)-\frac{\partial^2\rho_0}{\partial y^2}(y)\Big| \leqslant |\rho|^{(2+\alpha)}_{Q_{t_1}} t_1^{\frac{\alpha}{2}}.
$$

These relations entail the inequalities

$$
\frac{1}{2}\rho^- \leqslant \rho^- - |\rho|^{(2+\alpha)}_{Q_{t_1}} t_1 \leqslant \rho(y,t) \leqslant \rho^+ + |\rho|^{(2+\alpha)}_{Q_{t_1}} t_1 \leqslant 2\rho^+,
$$

$$
\Big|\frac{\partial\rho}{\partial y}(y,t)\Big| \leqslant \Big|\frac{\partial\rho_0}{\partial y}(y)\Big| + |\rho|^{(2+\alpha)}_{Q_{t_1}} t_1^{\frac{1+\alpha}{2}} \leqslant a + Cat_1^{\frac{1+\alpha}{2}},
$$

$$
\Big|\frac{\frac{\partial\rho}{\partial y}(y,t+\tau)-\frac{\partial\rho}{\partial y}(y,t)}{\tau^{\frac{\alpha}{2}}}\Big| \leqslant Cat_1^{\frac{\alpha}{2}},
$$

$$
\Big|\frac{\frac{\partial\rho}{\partial y}(y+h,t)-\frac{\partial\rho}{\partial y}(y,t)}{h^\alpha}\Big| \leqslant \max_{Q_{t_1}}\Big|\frac{\partial^2\rho_0}{\partial y^2}(y)\Big| + |\rho|^{(2+\alpha)}_{Q_{t_1}} t_1^{\frac{\alpha}{2}} \leqslant a + Cat_1^{\frac{\alpha}{2}}.
$$

It follows that for the sufficiently small t_1 operator $\mathbf{\Psi}$ transforms the convex set $\mathfrak{M}$ into itself.

5. Proof of Theorem 3.2

We will rely on the already established existence of a classical solution to problem (2.13)-(2.17) on the interval $[0,t_*)$. The estimates

$$
0 \leqslant v(y,t) \leqslant u_*, \quad 0 < \rho_*^{-1} < \rho(y,t) < \rho_*
$$

for $x \in Q_{t_*}$ with $\rho_* = \xi(t_*)$, and $0 < \xi(t_*) < \infty$ for $t_* < \infty$, are the main ingredients of the proof of the global in time existence. The proof is split into several steps.

5.1. **The maximum principle.** The proof of the estimate

$$
0 \leqslant v(y,t) \leqslant u_* \tag{5.1}
$$

is quite standard. Let us introduce the function w by the relation $v = we^{\alpha t}$, $\alpha > 0$. The function w satisfies in Q_T the equation

$$
\frac{\partial w}{\partial t} + \alpha\, w - \frac{\partial}{\partial y}\Big(\frac{\mu}{J^2}\frac{\partial w}{\partial y}\Big) = F\, e^{-\alpha t}. \tag{5.2}
$$

Let us assume that w attains its negative minimum at a point $(y_0,t_0) \in Q_T$. Then the left-hand side of equation (5.2) is strictly negative because

$$
\frac{\partial w}{\partial t}(y_0,t_0) \leqslant 0, \quad \frac{\partial w}{\partial y}(y_0,t_0) = 0, \quad \frac{\partial^2 w}{\partial y^2}(y_0,t_0) \geqslant 0, \quad \alpha\, w(y_0,t_0) < 0,
$$

while the right-hand side remains strictly positive. This contradiction means that w is nonnegative in Q_T. If w attains its local positive maximum at a point (y_0,t_0), it is necessary that at this point

$$
\frac{\partial w}{\partial t}(y_0,t_0) \geqslant 0, \quad \frac{\partial w}{\partial y}(y_0,t_0) = 0, \quad \frac{\partial^2 w}{\partial y^2}(y_0,t_0) \leqslant 0
$$

and by equation (5.2),

$$
\alpha\, w(y_0,t_0)\, e^{\alpha t_0} \leqslant F\big(w(y_0,t_0)\, e^{\alpha t_0}\big).
$$

Since $F(v) = 0$ for $v > u_*$ by assumption, we have

$$
w(y_0,t_0)\, e^{\alpha t_0} \leqslant u_*,
$$

whence

$$v(y,t) = w(y,t)\,e^{\alpha t} \leqslant w(y_0,t_0)\,e^{\alpha t} \leqslant u_*\,e^{\alpha\,(t-t_0)} \quad \text{in } Q_T.$$

It remains to notice that by Hopf's principle [2, 3] and the boundary conditions (2.16) v cannot attain its maximal and minimal values on the lateral boundaries of Q_T. Since $\alpha > 0$ is arbitrary, estimate (5.1) follows.

5.2. **Corollaries of the maximum principle.** In what follows we choose $T < t_*$. The first corollary is the estimate

$$|\frac{\partial \rho}{\partial y}| = |\frac{\partial}{\partial y}\big(\frac{1}{J}\big)| = \frac{1}{J^2}|\frac{\partial J}{\partial y}| \leqslant C, \quad (y,t) \in Q_T, \tag{5.3}$$

which follows after applying estimate (5.1) to (4.1).

The second corollary is the estimate

$$\rho = \frac{1}{J} \leqslant C, \quad (y,t) \in Q_T, \tag{5.4}$$

which follows from (5.3) and the boundary condition (4.3).

5.3. **The basic integral estimate.** Let us multiply equation (4.2) by $\frac{\partial J}{\partial t}$ and integrate by parts over the domain Q:

$$\int_Q |\frac{\partial J}{\partial t}|^2 dy + \mu \int_Q \frac{1}{J^2}\frac{\partial J}{\partial y}\frac{\partial^2 J}{\partial t \partial y} dy = \int_Q \frac{\partial \Phi}{\partial y}\frac{\partial J}{\partial t} dy.$$

The integrals over the boundaries $y = 0$ and $y = y_*$ equal zero due to the boundary conditions (4.3). The second term on the left-hand side of the last equality can be written in the form

$$\mu \int_Q \frac{1}{J^2}\,\frac{\partial J}{\partial y}\,\frac{\partial^2 J}{\partial t \partial y} dy = \frac{\mu}{2}\frac{d}{dt}\int_Q \frac{1}{J^2}|\frac{\partial J}{\partial y}|^2 dy + \mu \int_Q \frac{1}{J^3}\,\frac{\partial J}{\partial t}|\frac{\partial J}{\partial y}|^2 dy.$$

Thus,

$$\int_Q |\frac{\partial J}{\partial t}|^2 dy + \frac{\mu}{2}\frac{d}{dt}\int_Q \frac{1}{J^2}|\frac{\partial J}{\partial y}|^2 dy = J_1 + J_2,$$

where

$$J_1 = -\mu \int_Q \frac{1}{J^3}\,\frac{\partial J}{\partial t}|\frac{\partial J}{\partial y}|^2 dy, \quad J_2 = \int_Q \frac{\partial \Phi}{\partial y}\frac{\partial J}{\partial t} dy,$$

$$\frac{\partial \Phi}{\partial y} = \int_0^t F'\big(v(y,\tau)\big)\frac{\partial v}{\partial y}(y,\tau)d\tau + v_0'(y) + \big(\frac{1}{J_0}\big)''(y).$$

Let us estimate J_1 and J_2:

$$\begin{aligned}
|J_1(y,t)| &\leqslant \mu \int_Q |\frac{\partial J}{\partial t}||\frac{1}{J}||\frac{\partial J}{\partial y}||\frac{1}{J^2}||\frac{\partial J}{\partial y}|dy \\
&\leq C\mu \int_Q |\frac{\partial J}{\partial t}||\frac{1}{J}||\frac{\partial J}{\partial y}|dy \\
&\leq C\Big(\int_Q |\frac{\partial J}{\partial t}|^2 dy\Big)^{1/2}\Big(\int_Q \frac{1}{J^2}|\frac{\partial J}{\partial y}|^2 dy\Big)^{1/2} \\
&\leq C\frac{1}{4}\int_Q |\frac{\partial J}{\partial t}|^2 dy + C\int_Q \frac{1}{J^2}|\frac{\partial J}{\partial y}|^2 dy,
\end{aligned}$$

$|J_2(y,t)|$

$$
\begin{aligned}
&\leqslant \int_Q |\frac{\partial J}{\partial t}(y,t)| \int_0^t |F'\big(v(y,\tau)\big)|\,|\frac{\partial v}{\partial y}(y,\tau)|d\tau\,dy + C\Big(\int_Q |\frac{\partial J}{\partial t}(y,t)|^2 dy\Big)^{1/2}\\
&\le C\int_Q |\frac{\partial J}{\partial t}(y,t)|\Big(\int_0^t |\frac{\partial v}{\partial y}(y,\tau)|^2 d\tau\Big)^{1/2} dy + C\Big(\int_Q |\frac{\partial J}{\partial t}(y,t)|^2 dy\Big)^{1/2}\\
&\le C\Big(\int_Q |\frac{\partial J}{\partial t}(y,t)|^2 dy\Big)^{1/2}\Big(\int_0^t\int_Q |\frac{\partial v}{\partial y}(y,\tau)|^2 dy\,d\tau\Big)^{1/2} + C\Big(\int_Q |\frac{\partial J}{\partial t}(y,t)|^2 dy\Big)^{1/2}\\
&\le \frac{1}{4}\int_Q |\frac{\partial J}{\partial t}(y,t)|^2 dy + C\int_0^t\int_Q |\frac{\partial J}{\partial t}(y,\tau)t|^2\,dy\,d\tau + C.
\end{aligned}
$$

Here we have used (2.13) and expressed $\frac{\partial v}{\partial y}$ through $\frac{\partial J}{\partial t}$. Finally we have

$$
\begin{aligned}
&\frac{1}{4}\int_Q |\frac{\partial J}{\partial t}(y,t)|^2 dy + \frac{\mu}{2}\frac{d}{dt}\int_Q \frac{1}{J^2(y,t)}|\frac{\partial J}{\partial y}(y,t)|^2 dy\\
&\le C\Big(\int_Q \frac{1}{J^2(y,t)}|\frac{\partial J}{\partial y}(y,t)|^2 dy + \int_0^t\int_Q |\frac{\partial J}{\partial t}(y,\tau)|^2\,dy\,d\tau + 1\Big).
\end{aligned}
\tag{5.5}
$$

Set

$$
z(t) = \int_0^t\int_Q |\frac{\partial J}{\partial t}(y,\tau)|^2\,dy\,d\tau + \int_Q \frac{1}{J^2(y,t)}|\frac{\partial J}{\partial y}(y,t)|^2 dy.
$$

Then (5.5) is equivalent to

$$
\frac{dz}{dy} \leqslant C(z+1), \quad z(0) = z_0.
$$

By the Gronwall inequality, the last inequality entails the estimate

$$
\int_0^T\int_Q |\frac{\partial J}{\partial t}(y,t)|^2\,dy\,dt + \max_{0\leqslant t\leqslant T}\int_Q \frac{1}{J^2(y,t)}|\frac{\partial J}{\partial y}(y,t)|^2 dy \leqslant C.
$$

5.4. **Consequences of the basic integral estimate.** Let us consider the function $w(y,t) = \ln J(y,t)$. By (5.4) we have

$$
\max_{0\leqslant t\leqslant T}\int_0^{y_*} |\frac{\partial w}{\partial y}(y,t)|^2 dy \leqslant C,
$$

whence

$$
\begin{aligned}
|w(y,t)|^2 &= |w(0,t)|^2 + 2\int_0^y w(s,t)\frac{\partial w}{\partial s}(s,t)|ds\\
&\le |\ln J_0(0)|^2 + \Big(\int_Q |w(y,t)|^2 dy\Big)^{1/2}\Big(\int_Q |\frac{\partial w}{\partial y}(y,t)|^2 dy\Big)^{1/2}\\
&\le C\Big(1 + \Big(\int_Q |w(y,t)|^2 dy\Big)^{1/2}\Big).
\end{aligned}
\tag{5.6}
$$

Integrating over Q and using Hölder's inequality we obtain the estimate

$$
\max_{0\leqslant t\leqslant T}\int_Q |w(y,t)|^2 dy \leqslant C.
$$

Reverting to (5.6) we conclude that

$$
\max_{0\leqslant t\leqslant T}|w(y,t)| \leqslant C,
$$

and, therefore,

$$
J(y,t) = e^{w(y,t)}, \quad 0 < e^{-C} \leqslant J(y,t) \leqslant e^{C} < \infty.
$$

This estimate means that $t_* = \infty$ and T might be any bounded number.

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